Basel II and the Risk Management of Basket Options with Time-Varying Correlations*

Amy S. K. Wong
Tinbergen Institute
Erasmus University Rotterdam

The impact of jumps, regime switches, and linearly changing correlation term structures on the risk management of basket options has been examined in this paper. First, the results show that there is an asymmetric correlation effect on the value-at-risk of basket options. Second, the time at which a correlation shock occurs during the life of an option is particularly important for hedged basket options. Finally, the square-root-of-time rule can lead to severe underestimation of value-at-risk for basket options with time-varying correlations—for some cases, even by a factor exceeding the minimum regulatory stress factor.

JEL Codes: G15, G21.

1. Introduction

An important aspect of the "International Convergence of Capital Measurement and Capital Standards: A Revised Framework," also known as Basel II, is the management of the risk of financial products that cannot be entirely captured by value-at-risk (VaR); for example, nonlinear products or sudden correlation shifts (see Bank for International Settlements 2005). Basket options are derivatives that belong to the class of products that are subject to nonlinear and correlation risk. A basket option is an option on a portfolio of underlying assets, and the option price is highly dependent on the correlations between the underlying assets. This study examines

*I would like to thank the co-editor and anonymous referee for helpful comments. The usual disclaimer applies. Author contact: Tinbergen Institute, Erasmus University Rotterdam, The Netherlands; Home page: www.askwong.com.

basket options, because they are widely used across many financial markets, such as foreign exchange markets (see, e.g., Bennett and Kennedy 2004, Dammers and McCauley 2006), credit derivatives markets (e.g., Duffie and Singleton 2003), and equity markets (e.g., Pellizzari 2005). Many studies, such as Margrabe (1978), Curran (1994), Milevsky and Posner (1998), Brigo et al. (2004), and Deelstra, Liinev, and Vanmaele (2004), value basket options under the assumption of constant correlations between the processes of the underlying assets. However, recent empirical studies (e.g., Goetzmann, Li, and Rouwenhorst 2005), have shown that correlations of stock returns are considerably time varying.

Therefore, this paper examines the impact of time-varying correlation term structures on pricing and hedging of basket options as well as the implications for risk management. Empirical examination of the correlations between equity indices S&P 500, FTSE 100, and the Merrill Lynch government bond index shows that empirical features such as jumps, regime switches, and (nearly) linearly changing correlations can occur in practice. The main contribution of this study is to take these features into account in the correlation term structures of the basket options. To my knowledge, the impact of correlation jumps, regime switches, and linear correlation changes on the value-at-risk of basket options and the performance of the square-root-of-time rule have not yet been examined in the literature. Studies such as Skintzi, Skiadopoulos, and Refenes (2005) have analyzed the impact of estimation errors of constant correlation on value-at-risk of a portfolio of standard European options. Pellizzari (2005) examines a linearly increasing volatility structure for hedged basket options and the corresponding risk measures but does not look at time variation of correlations. Kupiec (1998) performs stress tests taking time-varying correlations into account for portfolios with linear exposure to underlying assets.

A second contribution of this study is an analysis on VaR of hedged basket options and the performance of the square-root-of-time rule for these positions when the correlation term structure contains the above-mentioned time variation. In practice, financial institutions often hedge their outstanding option positions to reduce the exposure to risk of the position and apply the square-root-of-time rule to the one-day VaR in order to obtain an estimate of the regulatory ten-day VaR. The performance of the square-root-of-time rule has been studied, for example, for GARCH processes by Diebold et al. (1997) and for jump diffusion processes by Danielsson and Zigrand (2005).

Moreover, this paper discusses the differences between the risk measure VaR and the coherent risk measure CVaR (conditional value at risk) for basket options, where the latter (more robust) measure can give additional information needed in some cases to give an adequate risk assessment of the derivatives position.

In this paper, a Monte Carlo simulation study has been performed for basket options with time-varying correlation term structures against the benchmark of constant correlations. The results are as follows.

First, there is an asymmetric correlation effect on the VaR of the basket option, where a change in negative (constant) correlations between the underlying assets has a greater impact on the VaR than a change in positive correlations of the same magnitude. This result is surprising: it is widely known that well-diversified asset portfolios (i.e., negative correlations) are less risky than portfolios with highly correlated assets, so one might expect that a basket option on a well-diversified portfolio is also relatively less risky. The results show that the potential loss *values* given by (C)VaR are indeed lower for basket options on negatively correlated assets, but at the same time the *changes* in (C)VaR are more subject to correlation risk as well. Ignoring this result can lead to serious underestimation of the VaR if sudden changes in market conditions occur. Another implication of this result is that VaR estimation of basket options on well-diversified baskets is relatively more prone to model risk, since in practice correlations have to be estimated under the assumption of a certain correlation model such as RiskMetrics^{TM}. For studies on the impact of estimation errors of constant correlations, see Fengler and Schwendner (2004) for basket options and Skintzi, Skiadopoulos, and Refenes (2005).

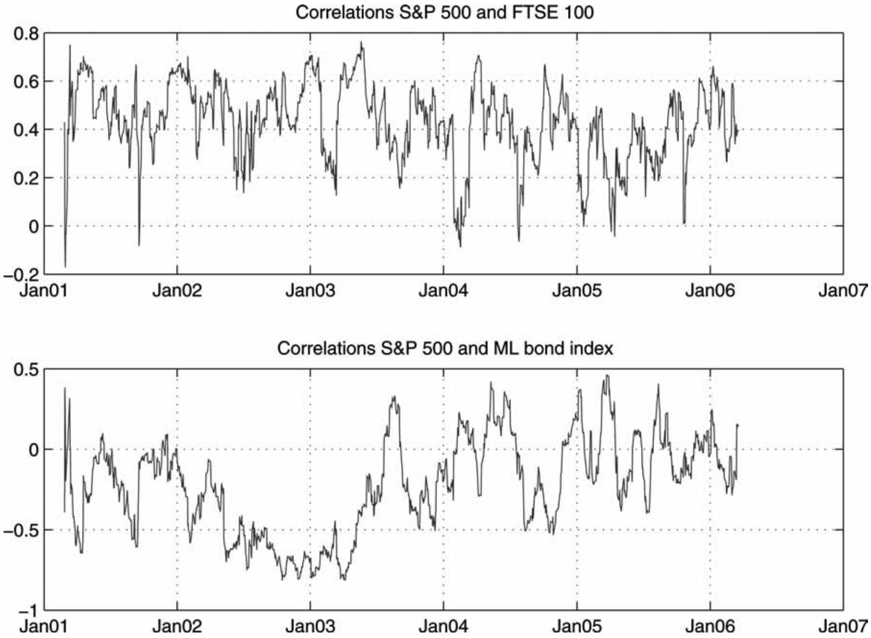
Second, the time at which correlation shocks occur during the life of the option is important for the VaR of basket options (especially if hedging is applied), even though the payoff of the basket option only depends on the value of the underlying assets at maturity. Compared with constant correlations at the average value of the time-varying correlations over the life of the option as a benchmark, the results show that the VaR estimates are dependent on the specific type of correlation time variation, even if the average correlation over the life of the option is the same. The estimates for the risk measures are also highly dependent on the hedge effectiveness for the option. This result is relevant for financial institutions, because option positions are often hedged in practice.

Third, the VaR estimate obtained by the square-root-of-time rule can lead to underestimation of the ten-day VaR for the unhedged option with time-varying correlations. The risk assessment of deep out-of-the-money derivatives plays an important role in the Basel II framework. This study shows that the square-root-of-time rule for the risk measure VaR underestimates the risks when the OTM basket option has a (time-varying) highly negatively correlated portfolio, and this underestimation even exceeds the minimum regulatory stress factor of value 3 for some cases. There is also underestimation of the VaR for basket options with constant correlations, but this underestimation remains below a factor of 3. When the time-varying correlations are relatively low at the start of the option, by using the square-root-of-time rule one implicitly assumes that the correlations remain this low for ten days. As a result, the risk implied by the ten-day VaR can be much larger than the estimate obtained from the square-root-of-time rule for time-varying correlations. For hedged options, the performance of the square-root-of-time rule is highly dependent on the difference in hedge effectiveness over a one-day and ten-day horizon and can lead to large deviations from the ten-day VaR.

Finally, VaR gives information about the potential loss of the option position corresponding to a certain confidence level but does not reveal the size of the loss if the VaR is exceeded. The difference in VaR and CVaR can be more than 40 percent for certain correlation term structures. Therefore, it is advisable to use VaR together with the coherent risk measure CVaR, because the CVaR can provide additional information needed to assess the risk of the basket options.

This paper is structured as follows. Section 2 contains an empirical examination of the correlations between equity and bond indices. Section 3 proceeds with the simulation framework, and in Section 4 the results of the simulation study are discussed. Finally, Section 5 concludes.

Figure 1. Correlations Generated by RiskMetrics^{TM} Model



2. Empirical Motivation

This study examines the impact of time-varying correlations on the basket option and the implications for risk management. In order to see in what way this time variation could occur, the correlations between equity indexes S&P 500 (United States) and FTSE 100 (United Kingdom) as well as the Merrill Lynch U.S. Treasury one- to ten-year bond index (henceforth, ML bond index) are illustrated in figure 1. The data are from October 2000 to March 2006 and are obtained from Datastream. The correlations between the returns of N assets can be estimated in many different ways, but a widely used benchmark model is RiskMetrics^{TM} of J. P. Morgan (1996), specified as follows. Let r_t be the $1 \times N$ vector of asset returns at time t and $H_t = \{\sigma_{ij,t}\}_{i,j=1}^{N}$ be the conditional covariance matrix at time t. Then, RiskMetrics^{TM} estimates the covariance matrix by

$$H_t = \lambda H_{t-1} + (1 - \lambda) r'_{t-1} r_{t-1},$$

where the weighting parameter λ has value 0.94 for daily data.[1] Subsequently, the correlations can be easily calculated as $\rho_{ij,t} = \frac{\sigma_{ij,t}}{\sqrt{\sigma_{ii,t}\sigma_{jj,t}}}$.

Figure 1 shows that the correlations between S&P 500 and FTSE 100 as well as the correlations between S&P 500 and the ML bond index, are considerably time varying. The correlations between S&P 500 and FTSE 100 remain positive during most of the sample period, but there are several large jumps downward—for example, in 2005. The correlations between S&P 500 and the ML bond index are, during some periods, positive (the value stays below 0.5) but remain negative most of the time. One can also observe the following correlation patterns in figure 1: starting from the second quarter of 2003, the correlation appears to increase almost linearly. Moreover, one can see from the figure that the correlations from the second quarter of 2002 until the first quarter of 2003 are on average much more negative than the correlations before and after this period, which resembles a regime switch. Following these observations, the correlation term structures in the subsequent simulation study will contain jumps and regime switches as well as linear increase and decrease in correlations during the life of the option.

3. Simulation Methods

For simplicity, we will use a basket option on two underlying assets to analyze the impact of the correlation changes on basket options. The asset price dynamics will be simulated using the differential equations

$$\begin{aligned} dS_{1,t} &= \mu_1 S_{1,t}dt + \sigma_1 S_{1,t}dW_{1,t}, \\ dS_{2,t} &= \mu_2 S_{2,t}dt + \sigma_2 S_{2,t}dW_{2,t}, \\ dW_{1,t}dW_{2,t} &= \rho_t dt, \qquad \text{for } t = 1, \ldots, T, \end{aligned} \tag{1}$$

where $W_{1,t}$ and $W_{2,t}$ are correlated Brownian motions with correlation ρ_t at time t. The value V_t of the basket option with strike price K

[1]The first 100 daily observations of the data sample are used to estimate the initial covariance matrix.

at time t is given by

$$C_T = (\omega_1 S_{1,T} + \omega_2 S_{2,T} - K)^+$$
$$= \max(\omega_1 S_{1,T} + \omega_2 S_{2,T} - K, 0), \quad (2)$$

$$V_t = e^{-r(T-t)} \mathbb{E}_Q \left[\sum_{m=1}^{M} C_T^{(m)} / M \right], \quad (3)$$

where C_T is the claim of the option at maturity T, ω_i are the portfolio weights of the underlying assets ($i = 1, 2$), Q is the risk-neutral martingale measure, and M is the number of simulations. The delta, Δ_i, is the sensitivity of the option value with respect to the price of asset i (used for delta hedging), and it is computed by the central difference method (for more details, see, e.g., Glasserman 2004):

$$\Delta_i(S_i) = \frac{\partial V}{\partial S_i} \approx \frac{V(S_i + h) - V(S_i - h)}{2h}, \quad i = 1, 2. \quad (4)$$

We will analyze the different correlation specifications for in-the-money (ITM), at-the-money (ATM), and out-of-the-money (OTM) basket options. Delta hedging will be done in the standard way (see, e.g., Hull 2006) with daily rebalancing. Each basket option contract is written on 100,000 underlying shares and has a maturity of three months in trading days ($T = \frac{63}{252}$). The simulated asset price paths and option values are computed using the parameters given in table 1 and the following correlation term structures for $t = \frac{1}{252}, \frac{2}{252}, \ldots, T$.

Table 1. Option Parameters

VaR horizon	10/252	$[\mu_1, \mu_2]$	$[0.1, 0.1]$
Maturity option T	63/252	r	0.05
dt	1/252	K_{ITM}	95
Nr. of assets	2	K_{ATM}	100
$[S_{1,0}, S_{2,0}]$	$[100, 100]$	K_{OTM}	105
$[\sigma_1, \sigma_2]$	$[0.35, 0.35]$	h in (4)	0.01
$[\omega_1, \omega_2]$	$[1/2, 1/2]$	M	5000

Constant correlations:

$$\text{C1 to C9:} \quad \rho_t = \rho,$$
$$\text{where } \rho \in \{-0.9, -0.7, -0.5, -0.2, 0, 0.2, 0.5, 0.7, 0.9\}$$

Negative correlations jump upward:

$$\text{T1:} \quad \rho_t = -0.9 + \left[0.9\left(t - \frac{1}{63}T\right)252\right]\mathbf{I}_{\left(\frac{1}{63}T < t \leq \frac{3}{63}T\right)} + \left[1.8 - 0.9\left(t - \frac{3}{63}T\right)252\right]\mathbf{I}_{\left(\frac{3}{63}T < t \leq \frac{5}{63}T\right)}$$

$$\text{T2:} \quad \rho_t = -0.9 + \left[0.9\left(t - \frac{59}{63}T\right)252\right]\mathbf{I}_{\left(\frac{59}{63}T < t \leq \frac{61}{63}T\right)} + \left[1.8 - 0.9\left(t - \frac{61}{63}T\right)252\right]\mathbf{I}_{\left(\frac{61}{63}T < t \leq T\right)}$$

Positive correlations jump downward:

$$\text{T3:} \quad \rho_t = 0.9 - 0.9\left[\left(t - \frac{1}{63}T\right)252\right]\mathbf{I}_{\left(\frac{1}{63}T < t \leq \frac{3}{63}T\right)} - \left[1.8 - 0.9\left(t - \frac{3}{63}T\right)252\right]\mathbf{I}_{\left(\frac{3}{63}T < t \leq \frac{5}{63}T\right)}$$

$$\text{T4:} \quad \rho_t = 0.9 - 0.9\left[\left(t - \frac{59}{63}T\right)252\right]\mathbf{I}_{\left(\frac{59}{63}T < t \leq \frac{61}{63}T\right)} - \left[1.8 - 0.9\left(t - \frac{61}{63}T\right)252\right]\mathbf{I}_{\left(\frac{61}{63}T < t \leq T\right)}$$

Correlation regime shifts:

$$\text{T5:} \quad \rho_t = -0.9\mathbf{I}_{\left(t \leq \frac{30}{63}T\right)} + -0.5\mathbf{I}_{\left(\frac{30}{63}T < t \leq T\right)}$$

$$\text{T6:} \quad \rho_t = 0.9\mathbf{I}_{\left(t \leq \frac{30}{63}T\right)} + 0.5\mathbf{I}_{\left(\frac{30}{63}T < t \leq T\right)}$$

Affine correlation term structure:

$$\text{T7:} \quad \rho_t = -0.9 + \frac{1.8}{62}(252t - 1)$$

$$\text{T8:} \quad \rho_t = 0.9 - \frac{1.8}{62}(252t - 1)$$

Figure 2. True Time-Varying Correlation Term Structures

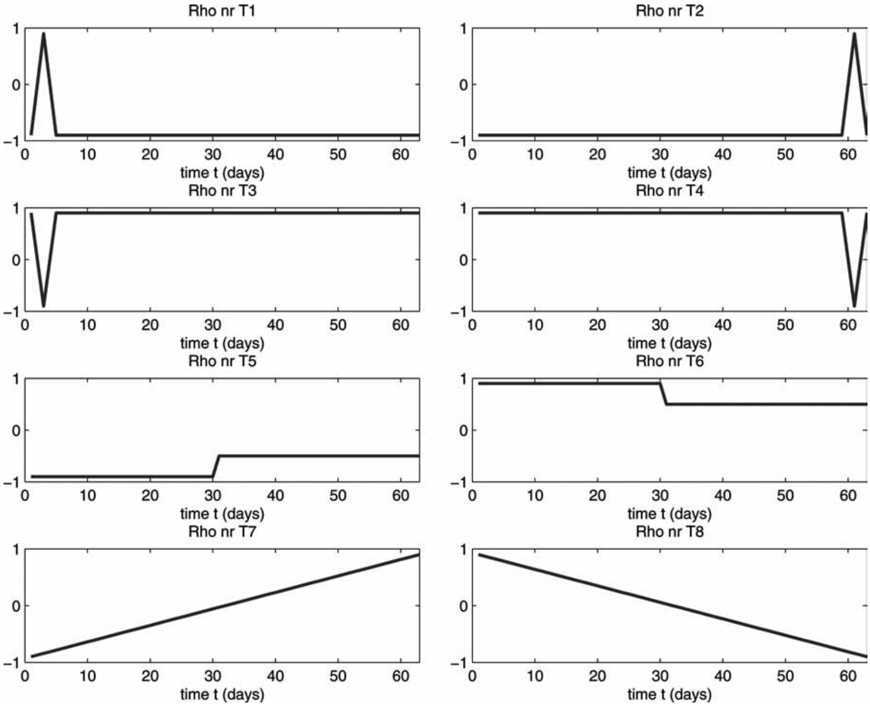


Figure 2 illustrates the time-varying correlation term structures, which can be interpreted as follows. The correlation term structures T1 and T2 correspond to a situation where the assets in the basket are usually well diversified (i.e., negative correlations), but suddenly the correlations jump upward at, respectively, the start and the end of the option life. This could happen, for example, during a financial crisis, when all stocks plummet simultaneously and the correlations between these stocks suddenly jump upward. Correlation specifications T3 and T4 correspond to a situation where the correlation is highly positive, but they suddenly jump downward at, respectively, the start and the end of the option life. T5 and T6 are regime-switching types of specifications, where the correlations shift, respectively, upward or downward, but are locally constant before and after the shift. Finally, T7 corresponds to a gradually linear increase of the correlations from −0.9 to +0.9, often associated with a bear market in which stocks become more correlated over time. T8 represents a linear decrease in correlations, which can be interpreted as a period in a bull market.

Eydeland and Wolyniec (2003) distinguish the instantaneous correlation ρ_t in (1) from the cumulative correlation defined by them as

$$\rho^*_{T,t} = \rho_{\ln S_{1,T} \ln S_{2,T}} = \frac{\mathbb{E}_t[\ln S_{1,T} \ln S_{2,T}] - \mathbb{E}_t[\ln S_{1,T}]\mathbb{E}_t[\ln S_{2,T}]}{\sqrt{\mathrm{var}_t[\ln S_{1,T}]}\sqrt{\mathrm{var}_t[\ln S_{2,T}]}} \quad (5)$$

$$= \frac{\int_t^T \sigma_{1,s}\sigma_{2,s}\rho_s ds}{\sqrt{\int_t^T \sigma^2_{1,s} ds}\sqrt{\int_t^T \sigma^2_{2,s} ds}}, \quad (6)$$

with the following properties

$$\lim_{t \to T} \rho^*_{T,t} = \rho_T \quad (7)$$

$$\text{if} \quad \sigma_{1,t} = \sigma_{2,t} = \sigma, \quad \forall t = 1, \cdots, T \quad \text{then} \quad \rho^*_{T,t} = \frac{\int_t^T \rho_s ds}{T - t}. \quad (8)$$

According to Eydeland and Wolyniec (2003), the cumulative correlation $\rho^*_{T,t}$ is more important than the instantaneous correlation ρ_t for option pricing and hedging as well as for VaR computation. When the VaR horizon is very short, by (7) the cumulative correlation becomes close to the instantaneous correlation. By (8) the cumulative correlation boils down to the average correlation over time period $T - t$, if the volatilities of both assets are the same and constant over time. So, in this special case one needs only an estimate of the average value of the correlation term structure instead of information on the entire term structure to compute the VaR of the option. This is a great simplification, and the results in the next section show to what extent this property holds for different time-varying correlation term structures.

Empirical correlations of asset returns are unobserved and have to be estimated using variance-covariance models, so it is very likely that the resulting correlations contain estimation errors. In this paper the focus is on the effects of changes in the true correlation values instead of correlation estimation errors. For this purpose, the same parameter values will be used both to generate the asset price data and to value the basket options. Hence, the effects of changes in the true correlations can be isolated without bothering about estimation errors of correlations (for more details on correlation estimation errors, see Fengler and Schwendner 2004).

3.1 Value-at-Risk

In the current Basel II framework, banks should develop a more forward-looking approach with respect to risk management. The risks of the basket option position are assessed using the risk measure value-at-risk. Value-at-risk (VaR) is a widely used measure for quantifying potential losses of asset portfolios at a certain confidence level α (conventionally, 95 percent or 99 percent). Let X be the profit and loss realizations of the simulations; then VaR is defined by

$$\mathbb{P}(X \leq VaR^{\alpha}) = 1 - \alpha. \tag{9}$$

The main methods of VaR computation are delta-normal, historical simulation, and Monte Carlo simulation (see, e.g., Jorion 2001). To determine which method is suitable, it is important to look at the characteristics of the portfolio for which the VaR needs to be computed. The asset portfolio here consists of a basket option with (time-dependent) correlations between the underlying assets; therefore, the risks involved in this asset position are highly nonlinear as well as time dependent. The Monte Carlo simulation method is used to compute the VaR in this study. The motivation for the choice of the Monte Carlo simulation method is that it can take time dependency and nonlinear risk into account. Moreover, with respect to the requirements of Basel II, the computation of VaR can be easily adapted to reflect current market conditions by changing the underlying parameters such that these coincide with market parameters.

The alternative methods are more often used by market practitioners, since they are less computationally intensive than Monte Carlo simulation. However, the delta-normal method and the historical simulation method are less adequate for computing the VaR of the basket options for the following reasons. The delta-normal method is based on the assumption of a normally distributed portfolio, thus it is only suitable for portfolios involving linear risk in the underlyings. The historical simulation method is often used by market practitioners, since this method does not involve any distributional assumptions of the portfolio and is relatively fast to compute. Historical returns are used to represent potential future losses, and this implies that all information of the historical data is retained. However, the resulting VaR estimate can only reflect the type of losses that already have occurred in the historical data sample used for estimation. Hence, this approach is not in line with the forward-looking approach required in the Basel II framework. Moreover, due to the use of a relatively large number of historical observations to avoid small-sample bias, the most recent market movements cannot be easily captured in the VaR computation. For more details on the historical simulation method, see Pritsker (2006) and the references therein.

3.2 CVaR and the Square-Root-of-Time Rule

This study will assess the risks of the option portfolio using VaR, because this risk measure is widely used for regulatory purposes as specified by the Basel Committee on Banking Supervision. However, it is well known that VaR is only an indication of the loss corresponding to a confidence level α and over a certain time horizon (usually ten days); it does not give an indication of the size of that loss if the VaR is exceeded. Moreover, it is not a coherent risk measure (for more details, see Artzner et al. 1999). Therefore, the conditional value-at-risk (CVaR) will also be computed, which satisfies the coherence properties. Let X be the profit and loss realizations of the simulations. The CVaR (also called expected shortfall) is defined for a given confidence level α as

$$CVaR^{\alpha} = \mathbb{E}[X|X \leq VaR^{\alpha}]. \tag{10}$$

The difference between the CVaR and the VaR can be expressed by the CVaR-to-VaR ratio. McNeil, Frey, and Embrechts (2005) describe that for the CVaR-to-VaR ratio of the normal distribution, it holds that $\lim_{\alpha\to 1} \frac{\mathrm{CVaR}^{\alpha}}{\mathrm{VaR}^{\alpha}} = 1$, whereas for the t-distribution, this ratio will go to a value greater than 1 in the limit. Hence, for a heavy-tailed distribution, the difference between the CVaR and the VaR will be larger than for a normal distribution. The 1996 Amendment of the 1988 Basel Capital Accord requires a capital charge for market risk (see, e.g., Jorion 2001 and Hull 2006 for more details). The market risk capital charge by the internal models approach can be computed as the maximum of the VaR of the previous day and the average VaR over the previous sixty days times a stress factor k. The stress factor k is determined by the local regulators and has at least an absolute value of 3. In practice, the square-root-of-time rule is often applied to compute the ten-day VaR and CVaR by multiplying the one-day VaR with $\sqrt{10}$, since the direct estimation of the ten-day VaR often requires too much historical data. This approach is valid for i.i.d. normally distributed observations; otherwise, the square-root-of-time rule gives an approximation of the true VaR (see Jorion 2001).

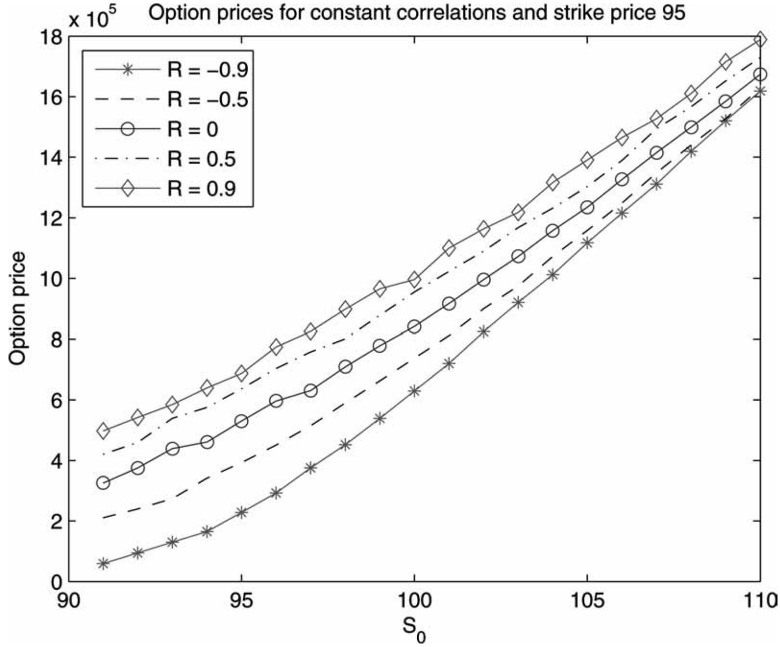
4. Results

In this section, the results obtained for basket options with time-varying correlation term structures will be discussed using the benchmark of constant correlation term structures. To gain a better understanding of the results of time-varying correlations, the results obtained from constant correlations are first discussed.

4.1 Constant Correlations

To illustrate how basket option prices of different moneyness levels change with respect to different constant correlation values between the underlyings, the option prices are plotted in figure 3 and are calculated with the parameters in table 1. Figure 3 shows the basket option prices that are computed for different moneyness levels using constant correlation values 0, ± 0.5, and ± 0.9 for the underlying assets. From this figure one can see that the differences between the option prices across correlation specifications are relatively pronounced for (near) ATM option prices. Moreover, the most striking observation from figure 3 is that there is an asymmetric correlation effect on the ATM option prices: changes in negative correlations have a greater impact on the option prices than changes of the same magnitude in positive correlations (for example, the difference between the option prices for ρ equal to -0.9 and zero is greater than the difference between the prices of ρ equal to $+0.9$ and zero). This asymmetric correlation effect can also be seen for the OTM

Figure 3. Basket Option Prices for Different Correlations for S_0 from 90 to 110



option prices, but to a lesser extent than for the ATM option prices. In figure 3 the option prices converge to the same value beyond a certain moneyness level regardless of the correlation value for the underlying (see, e.g., beyond S_0 of 100). However, the subsequent discussion of the simulation results shows that the risk of potential loss on the far ITM option over a certain time horizon does depend substantially on the correlation values of the underlying assets.

4.1.1 Asymmetric Correlation Effects

In this section the asymmetric effects of negative and positive correlation values on the option price as well as the risk measures VaR and CVaR of the basket option position will be discussed. The results of the simulation study for constant correlations are given in table 2 and turn out to be highly dependent on the initial moneyness level of the basket option. To discuss these results in more detail, let $\bar{V}_0$

Table 2. VaR and CVaR Results for ITM Options with Constant Correlations

	$\mathrm{VaR}^{0.99}$	$\mathrm{VaR}^{0.95}$	$\mathrm{CVaR}^{0.99}$	$\mathrm{CVaR}^{0.95}$	$\frac{\mathrm{CVaR}^{0.99}}{\mathrm{VaR}^{0.99}}$ [a]
ITM C1 NH	$-3.66 \cdot 10^5$	$-2.68 \cdot 10^5$	$-4.24 \cdot 10^5$	$-3.29 \cdot 10^5$	1.16
ITM C1 HE	$-1.54 \cdot 10^4$	$-1.08 \cdot 10^4$	$-1.67 \cdot 10^4$	$-1.35 \cdot 10^4$	1.09
ITM C1 $\sqrt{10}$NH1	$-3.51 \cdot 10^5$	$-2.46 \cdot 10^5$	$-3.93 \cdot 10^5$	$-3.11 \cdot 10^5$	1.12
ITM C1 $\sqrt{10}$HE1	$-4.35 \cdot 10^4$	$-3.08 \cdot 10^4$	$-4.95 \cdot 10^4$	$-3.90 \cdot 10^4$	1.14
ITM C2 NH	$-5.60 \cdot 10^5$	$-4.14 \cdot 10^5$	$-6.63 \cdot 10^5$	$-5.10 \cdot 10^5$	1.19
ITM C2 HE	$-2.52 \cdot 10^4$	$-1.69 \cdot 10^4$	$-3.10 \cdot 10^4$	$-2.21 \cdot 10^4$	1.23
ITM C2 $\sqrt{10}$NH1	$-5.40 \cdot 10^5$	$-3.68 \cdot 10^5$	$-6.23 \cdot 10^5$	$-4.75 \cdot 10^5$	1.15
ITM C2 $\sqrt{10}$HE1	$-5.10 \cdot 10^4$	$-3.53 \cdot 10^4$	$-5.71 \cdot 10^4$	$-4.45 \cdot 10^4$	1.12
ITM C3 NH	$-6.98 \cdot 10^5$	$-5.06 \cdot 10^5$	$-8.11 \cdot 10^5$	$-6.28 \cdot 10^5$	1.16
ITM C3 HE	$-3.50 \cdot 10^4$	$-2.26 \cdot 10^4$	$-4.38 \cdot 10^4$	$-3.04 \cdot 10^4$	1.25
ITM C3 $\sqrt{10}$NH1	$-6.77 \cdot 10^5$	$-4.44 \cdot 10^5$	$-7.74 \cdot 10^5$	$-5.76 \cdot 10^5$	1.14
ITM C3 $\sqrt{10}$HE1	$-6.30 \cdot 10^4$	$-4.37 \cdot 10^4$	$-7.22 \cdot 10^4$	$-5.54 \cdot 10^4$	1.15
ITM C4 NH	$-8.48 \cdot 10^5$	$-6.07 \cdot 10^5$	$-9.86 \cdot 10^5$	$-7.67 \cdot 10^5$	1.16
ITM C4 HE	$-4.84 \cdot 10^4$	$-2.87 \cdot 10^4$	$-5.85 \cdot 10^4$	$-4.04 \cdot 10^4$	1.21
ITM C4 $\sqrt{10}$NH1	$-7.87 \cdot 10^5$	$-5.34 \cdot 10^5$	$-9.40 \cdot 10^5$	$-6.95 \cdot 10^5$	1.20
ITM C4 $\sqrt{10}$HE1	$-8.46 \cdot 10^4$	$-5.45 \cdot 10^4$	$-9.49 \cdot 10^4$	$-7.04 \cdot 10^4$	1.12
ITM C5 NH	$-9.42 \cdot 10^5$	$-6.60 \cdot 10^5$	$-1.10 \cdot 10^6$	$-8.43 \cdot 10^5$	1.17
ITM C5 HE	$-5.45 \cdot 10^4$	$-3.37 \cdot 10^4$	$-6.70 \cdot 10^4$	$-4.60 \cdot 10^4$	1.23
ITM C5 $\sqrt{10}$NH1	$-8.45 \cdot 10^5$	$-5.90 \cdot 10^5$	$-1.03 \cdot 10^6$	$-7.61 \cdot 10^5$	1.22
ITM C5 $\sqrt{10}$HE1	$-9.50 \cdot 10^4$	$-6.15 \cdot 10^4$	$-1.09 \cdot 10^5$	$-8.04 \cdot 10^4$	1.15
ITM C6 NH	$-1.05 \cdot 10^6$	$-7.05 \cdot 10^5$	$-1.20 \cdot 10^6$	$-9.12 \cdot 10^5$	1.15
ITM C6 HE	$-5.96 \cdot 10^4$	$-3.72 \cdot 10^4$	$-7.25 \cdot 10^4$	$-5.03 \cdot 10^4$	1.22
ITM C6 $\sqrt{10}$NH1	$-9.04 \cdot 10^5$	$-6.53 \cdot 10^5$	$-1.11 \cdot 10^6$	$-8.21 \cdot 10^5$	1.23
ITM C6 $\sqrt{10}$HE1	$-1.04 \cdot 10^5$	$-6.87 \cdot 10^4$	$-1.21 \cdot 10^5$	$-8.98 \cdot 10^4$	1.17
ITM C7 NH	$-1.19 \cdot 10^6$	$-7.60 \cdot 10^5$	$-1.35 \cdot 10^6$	$-1.01 \cdot 10^6$	1.13
ITM C7 HE	$-6.36 \cdot 10^4$	$-4.21 \cdot 10^4$	$-7.83 \cdot 10^4$	$-5.57 \cdot 10^4$	1.23
ITM C7 $\sqrt{10}$NH1	$-1.04 \cdot 10^6$	$-7.33 \cdot 10^5$	$-1.22 \cdot 10^6$	$-9.08 \cdot 10^5$	1.17
ITM C7 $\sqrt{10}$HE1	$-1.19 \cdot 10^5$	$-8.09 \cdot 10^4$	$-1.37 \cdot 10^5$	$-1.03 \cdot 10^5$	1.15
ITM C8 NH	$-1.21 \cdot 10^6$	$-8.12 \cdot 10^5$	$-1.43 \cdot 10^6$	$-1.07 \cdot 10^6$	1.19
ITM C8 HE	$-6.80 \cdot 10^4$	$-4.44 \cdot 10^4$	$-8.16 \cdot 10^4$	$-5.92 \cdot 10^4$	1.20
ITM C8 $\sqrt{10}$NH1	$-1.09 \cdot 10^6$	$-7.67 \cdot 10^5$	$-1.28 \cdot 10^6$	$-9.71 \cdot 10^5$	1.17
ITM C8 $\sqrt{10}$HE1	$-1.25 \cdot 10^5$	$-8.57 \cdot 10^4$	$-1.49 \cdot 10^5$	$-1.11 \cdot 10^5$	1.19
ITM C9 NH	$-1.30 \cdot 10^6$	$-8.67 \cdot 10^5$	$-1.51 \cdot 10^6$	$-1.14 \cdot 10^6$	1.16
ITM C9 HE	$-7.16 \cdot 10^4$	$-4.68 \cdot 10^4$	$-8.43 \cdot 10^4$	$-6.23 \cdot 10^4$	1.18
ITM C9 $\sqrt{10}$NH1	$-1.16 \cdot 10^6$	$-8.10 \cdot 10^5$	$-1.32 \cdot 10^6$	$-1.04 \cdot 10^6$	1.14
ITM C9 $\sqrt{10}$HE1	$-1.40 \cdot 10^5$	$-9.25 \cdot 10^4$	$-1.61 \cdot 10^5$	$-1.20 \cdot 10^5$	1.15

[a]NH = No hedging, HE = daily delta hedging, $\sqrt{10}$NH1 = $\sqrt{10}\mathrm{VaR}_{1-day}$ (no hedging), $\sqrt{10}$HE1 = $\sqrt{10}\mathrm{VaR}_{1-day}$(daily delta hedging)

denote the average basket option price of the simulation sample at time 0 and define[2]

$$\begin{aligned}\delta\mathrm{VaR}^{\alpha}_{-} &= \mathrm{VaR}^{\alpha}(\rho = 0) - \mathrm{VaR}^{\alpha}(\rho = -0.9)\\ \delta\mathrm{VaR}^{\alpha}_{+} &= \mathrm{VaR}^{\alpha}(\rho = 0.9) - \mathrm{VaR}^{\alpha}(\rho = 0)\\ \delta\mathrm{VaR}^{\alpha}_{\mathrm{Total}} &= \delta\mathrm{VaR}^{\alpha}_{-} + \delta\mathrm{VaR}^{\alpha}_{+}.\end{aligned}$$

The results for the unhedged basket option are as follows. First, as the constant correlation ρ increases from -0.9 to 0.9 (respectively, C1 and C9), the average initial option price $\bar{V}_0$ as well as the potential loss measured by VaR and CVaR increase for each moneyness level. For ITM options, this increase in the potential loss is unproportionally high relative to the increase in the average initial option price. For example, $\bar{V}_0^{ITM}$ of C1 and C9 are, respectively, $6.30 \cdot 10^5$ and $1.01 \cdot 10^6$, which is an increase with factor 1.61. The corresponding unhedged ten-day $\mathrm{VaR}^{0.95}$ estimates for C1 and C9 as given in table 2 are $-2.68 \cdot 10^5$ to $-8.67 \cdot 10^5$, respectively, so the potential losses increase with a factor of 3.23 (this factor is even larger for $\mathrm{VaR}^{0.99}$ and CVaR). As the correlations change from -0.9 to $+0.9$, the $(1 - \alpha)$-quantiles of the option position's profits and losses increase to more than three times larger than the $\bar{V}_0$ for ITM options. In contrast to this result, higher correlation values affect $\bar{V}_0^{ATM}$ option prices and the corresponding VaR and CVaR estimates more proportionally, where $\bar{V}_0^{ATM}$ increases with factor 3.08 and (C)VaR estimates increase with about factor 4 when going from correlation value -0.9 to $+0.9$.

Secondly, the largest part of the increase in potential losses for ITM and ATM options can be found by increasing the negative correlations. For example, from table 2 the ITM $\delta\mathrm{VaR}^{0.95}_{-}$ has a value of $-3.92 \cdot 10^5$, which constitutes 65 percent of the total change $\delta\mathrm{VaR}^{0.95}_{\mathrm{Total}}$ of $-5.98 \cdot 10^5$. For the ATM and OTM option, this percentage can be obtained similarly and is 63 percent of the $\delta\mathrm{VaR}^{0.95}_{\mathrm{Total}}$. The relatively high sensitivity of basket options on well-diversified baskets (negative correlations) with respect to correlation changes seems counterintuitive. It is well known that well-diversified portfolios are

[2]The VaR here is the ten-day VaR^{α} of the unhedged option position given in table 2.

less risky than portfolios with highly correlated assets; therefore, investors might expect that options on a well-diversified portfolio are also less risky than options on highly correlated underlying assets. Although the results in table 2 show that the absolute *values* of the (C)VaR of the basket option increase with higher correlations, the *changes* in these (C)VaR values are relatively more subject to correlation risk for a negatively correlated portfolio. This observation is important for investors, because they may not be aware of this increased correlation risk when buying basket options on well-diversified ("safe") baskets of assets. Another implication of this result is that VaR estimation of basket options on well-diversified baskets is relatively more prone to model risk, since in practice correlations have to be estimated under the assumption of a correlation model (e.g., RiskMetrics^{TM}), which is in line with the results of Skintzi, Skiadopoulos, and Refenes (2005). They have found that the VaR measure becomes relatively more sensitive to correlation estimation errors with decreasing true correlations for linear portfolios as well as option portfolios containing plain-vanilla European options written on correlated underlying assets.

Finally, the option prices and risk measures of OTM options vary even more with the correlation of the underlying assets than those of ITM and ATM options. The $\bar{V}_0$ prices for ρ of -0.9 and $+0.9$ are, respectively, $5.45 \cdot 10^4$ and $5.23 \cdot 10^5$, and the corresponding $\mathrm{VaR}^{0.95}$ estimates are $-7.47 \cdot 10^4$ and $-6.11 \cdot 10^5$. So, the difference in option price and risk is extremely large across different correlation values for the OTM basket option position.

4.1.2 Delta-Hedged Portfolios

In practice, option positions of financial institutions are usually hedged to decrease the exposure to the full risks of the option position. Therefore, it is perhaps even more important to consider the effects of correlations on daily delta-hedged option positions. A delta-hedged option position accounts for the risk of changes in the price of the underlying assets. The estimates of the risk measures are given in table 2. The effectiveness of the delta hedge in reducing the risk of the option position will be measured by the no-hedge-to-hedge ratio of the risk measures given in table 3. If this ratio is very high, the risk of potential loss is reduced much better by hedging

Table 3. Risk Measure Ratios for ITM Options with Constant Correlations

	$VaR^{0.99}$	$VaR^{0.95}$	$CVaR^{0.99}$	$CVaR^{0.95}$[a]
ITM C1 NH/HE	23.86	24.89	25.33	24.48
ITM C1 $\sqrt{10}$NH1/$\sqrt{10}$HE1	8.08	8.00	7.93	7.96
ITM C1 NH/$\sqrt{10}$NH1	1.04	1.09	1.08	1.06
ITM C1 HE/$\sqrt{10}$HE1	0.35	0.35	0.34	0.35
ITM C2 NH/HE	22.18	24.50	21.38	23.10
ITM C2 $\sqrt{10}$NH1/$\sqrt{10}$HE1	10.59	10.44	10.90	10.68
ITM C2 NH/$\sqrt{10}$NH1	1.04	1.13	1.06	1.07
ITM C2 HE/$\sqrt{10}$HE1	0.50	0.48	0.54	0.50
ITM C3 NH/HE	19.96	22.41	18.53	20.69
ITM C3 $\sqrt{10}$NH1/$\sqrt{10}$HE1	10.75	10.14	10.72	10.40
ITM C3 NH/$\sqrt{10}$NH1	1.03	1.14	1.05	1.09
ITM C3 HE/$\sqrt{10}$HE1	0.56	0.52	0.61	0.55
ITM C4 NH/HE	17.51	21.12	16.86	18.96
ITM C4 $\sqrt{10}$NH1/$\sqrt{10}$HE1	9.30	9.80	9.91	9.88
ITM C4 NH/$\sqrt{10}$NH1	1.08	1.14	1.05	1.10
ITM C4 HE/$\sqrt{10}$HE1	0.57	0.53	0.62	0.58
ITM C5 NH/HE	17.28	19.60	16.41	18.35
ITM C5 $\sqrt{10}$NH1/$\sqrt{10}$HE1	8.89	9.59	9.41	9.46
ITM C5 NH/$\sqrt{10}$NH1	1.12	1.12	1.07	1.11
ITM C5 HE/$\sqrt{10}$HE1	0.57	0.55	0.61	0.57
ITM C6 NH/HE	17.57	18.93	16.60	18.14
ITM C6 $\sqrt{10}$NH1/$\sqrt{10}$HE1	8.73	9.51	9.20	9.15
ITM C6 NH/$\sqrt{10}$NH1	1.16	1.08	1.08	1.11
ITM C6 HE/$\sqrt{10}$HE1	0.58	0.54	0.60	0.56
ITM C7 NH/HE	18.74	18.07	17.24	18.14
ITM C7 $\sqrt{10}$NH1/$\sqrt{10}$HE1	8.72	9.06	8.90	8.83
ITM C7 NH/$\sqrt{10}$NH1	1.15	1.04	1.11	1.11
ITM C7 HE/$\sqrt{10}$HE1	0.53	0.52	0.57	0.54
ITM C8 NH/HE	17.72	18.28	17.59	18.15
ITM C8 $\sqrt{10}$NH1/$\sqrt{10}$HE1	8.73	8.96	8.59	8.75
ITM C8 NH/$\sqrt{10}$NH1	1.10	1.06	1.12	1.11
ITM C8 HE/$\sqrt{10}$HE1	0.54	0.52	0.55	0.53
ITM C9 NH/HE	18.13	18.52	17.87	18.25
ITM C9 $\sqrt{10}$NH1/$\sqrt{10}$HE1	8.31	8.75	8.23	8.67
ITM C9 NH/$\sqrt{10}$NH1	1.12	1.07	1.14	1.10
ITM C9 HE/$\sqrt{10}$HE1	0.51	0.51	0.53	0.52

[a]NH = No hedging, HE = daily delta hedging, $\sqrt{10}$NH1 = $\sqrt{10}VaR_{1-day}$ (no hedging), $\sqrt{10}$HE1 = $\sqrt{10}VaR_{1-day}$(daily delta hedging)

than in the case of a low ratio. The results are very different across moneyness levels.

For ITM options, daily delta hedging reduces the risk by a factor of 23 or more for the highly negative correlation specification C1 for all risk measures. The hedge effectiveness reduces for higher correlations, but it is still considerable for positive correlations, as the no-hedge-to-hedge ratio of the risk measures has a value of more than 16 for different confidence levels.

The delta-hedge effectiveness deteriorates for ATM and OTM options for all values of correlations. Still, the risk reduction is substantial compared with the unhedged ATM and OTM option position in most cases, and the no-hedge-to-hedge ratios vary from around 7 to 15. In contrast to the result for ITM options, the reduction in risk achieved by delta hedging is the largest for highly positive correlations (especially for OTM options). Overall, the delta hedge is most effective for ITM options but still substantial for ATM and OTM options.

4.1.3 Risk Measures and the Square-Root-of-Time Rule

This section discusses the difference between the VaR and CVaR risk measures in this simulation experiment as well as the performance of the widely used square-root-of-time rule. Overall, the risk of potential loss increases for a higher value of the underlying constant correlation (see the VaR and CVaR estimates in table 2). From table 3 one can see that the $\mathrm{CVaR}^{0.99}$-to-$\mathrm{VaR}^{0.99}$ ratios of the ITM option are larger for the hedged option position than for the unhedged position in most cases, thus implying that the hedged profit and loss distribution has relatively heavier tails. However, for the ATM and OTM option, the profit and loss distribution of the unhedged option position exhibits heavier tails compared with the hedged option position. Define γ as

$$\gamma = \frac{\text{ten-day VaR}^{0.99}}{\sqrt{10} \cdot \text{one-day VaR}^{0.99}}, \tag{11}$$

where γ is used to measure the performance of the square-root-of-time rule. If γ is larger than one, the unhedged ten-day VaR is underestimated when applying the square-root-of-time rule. One would expect that the square-root-of-time rule performs worse for cases with high $\text{CVaR}^{0.99}$-to-$\text{VaR}^{0.99}$ ratios, since the validity of this rule critically depends on the assumption of i.i.d.-normal observations. This expectation is indeed confirmed by the results, since the $\text{CVaR}^{0.99}$-to-$\text{VaR}^{0.99}$ ratios for the unhedged OTM option are relatively high compared with the corresponding values for the unhedged ATM option, and this also holds for the underestimation of the ten-day $\text{VaR}^{0.99}$. A notable exception is the unhedged ITM option position, where the square-root-of-time rule performs quite well. For the hedged option position, there is overestimation of the ten-day $\text{VaR}^{0.99}$ by the square-root-of-time rule, since daily delta hedging has an offsetting effect on the profit and loss changes of the option position. The no-hedge-to-hedge ratios are quite high, with a minimum value of about 7. This overestimation declines as the difference of the hedge effectiveness between the one-day and ten-day horizon diminishes, i.e., as the difference in the no-hedge-to-hedge ratios decreases.

So, in most cases, the square-root-of-time rule leads to either underestimation or overestimation of the ten-day $\text{VaR}^{0.99}$. However, none of the γ ratios exceed the minimum absolute stress factor value of 3. Hence, for the computation of market risk capital charge, the square-root-of-time rule used in combination with the stress factor k of at least 3 is reasonable here for basket options in case of constant correlations.

4.2 Time-Varying Correlations

In this section the results of time-varying correlation term structures containing jumps, regime switches, and affine term structures will be discussed for the basket option. By (8) the computation of the VaR for basket options with time-varying correlation term structures can be greatly simplified: one only needs to know an estimate of the average correlation over the option life instead of information on the entire future correlation term structure. The results in table 4 show that property (8) does not hold for the basket option with different correlation term structures. If the correlations are relatively high during the ten-day horizon over which the VaR is computed, then the potential loss will be higher than indicated by the VaR estimated using the average correlation over the life of the option

Table 4. VaR and CVaR Results for ITM Options with Time-Varying Correlations

	$\mathrm{VaR}^{0.99}$	$\mathrm{VaR}^{0.95}$	$\mathrm{CVaR}^{0.99}$	$\mathrm{CVaR}^{0.95}$	$\frac{\mathrm{CVaR}^{0.99}}{\mathrm{VaR}^{0.99}}$ [a]
ITM T1 NH	$-8.39\cdot 10^5$	$-5.55\cdot 10^5$	$-9.27\cdot 10^5$	$-7.17\cdot 10^5$	1.11
ITM T1 HE	$-5.96\cdot 10^4$	$-3.19\cdot 10^4$	$-8.58\cdot 10^4$	$-5.12\cdot 10^4$	1.44
ITM T1 $\sqrt{10}$NH1	$-3.40\cdot 10^5$	$-2.35\cdot 10^5$	$-3.78\cdot 10^5$	$-2.97\cdot 10^5$	1.11
ITM T1 $\sqrt{10}$HE1	$-4.29\cdot 10^4$	$-3.24\cdot 10^4$	$-4.84\cdot 10^4$	$-3.89\cdot 10^4$	1.13
ITM T2 NH	$-3.55\cdot 10^5$	$-2.59\cdot 10^5$	$-4.12\cdot 10^5$	$-3.18\cdot 10^5$	1.16
ITM T2 HE	$-1.49\cdot 10^4$	$-1.10\cdot 10^4$	$-1.68\cdot 10^4$	$-1.35\cdot 10^4$	1.12
ITM T2 $\sqrt{10}$NH1	$-3.37\cdot 10^5$	$-2.36\cdot 10^5$	$-3.78\cdot 10^5$	$-2.98\cdot 10^5$	1.12
ITM T2 $\sqrt{10}$HE1	$-4.52\cdot 10^4$	$-3.08\cdot 10^4$	$-5.02\cdot 10^4$	$-3.96\cdot 10^4$	1.11
ITM T3 NH	$-1.15\cdot 10^6$	$-7.36\cdot 10^5$	$-1.47\cdot 10^6$	$-9.98\cdot 10^5$	1.28
ITM T3 HE	$-6.82\cdot 10^4$	$-4.24\cdot 10^4$	$-8.08\cdot 10^4$	$-5.80\cdot 10^4$	1.19
ITM T3 $\sqrt{10}$NH1	$-1.16\cdot 10^6$	$-8.11\cdot 10^5$	$-1.33\cdot 10^6$	$-1.04\cdot 10^6$	1.14
ITM T3 $\sqrt{10}$HE1	$-1.38\cdot 10^5$	$-9.13\cdot 10^4$	$-1.60\cdot 10^5$	$-1.19\cdot 10^5$	1.16
ITM T4 NH	$-1.30\cdot 10^6$	$-8.72\cdot 10^5$	$-1.51\cdot 10^6$	$-1.14\cdot 10^6$	1.16
ITM T4 HE	$-7.08\cdot 10^4$	$-4.72\cdot 10^4$	$-8.50\cdot 10^4$	$-6.19\cdot 10^4$	1.20
ITM T4 $\sqrt{10}$NH1	$-1.16\cdot 10^6$	$-8.07\cdot 10^5$	$-1.32\cdot 10^6$	$-1.04\cdot 10^6$	1.14
ITM T4 $\sqrt{10}$HE1	$-1.32\cdot 10^5$	$-8.98\cdot 10^4$	$-1.55\cdot 10^5$	$-1.18\cdot 10^5$	1.18
ITM T5 NH	$-3.30\cdot 10^5$	$-2.40\cdot 10^5$	$-3.86\cdot 10^5$	$-2.97\cdot 10^5$	1.17
ITM T5 HE	$-1.66\cdot 10^4$	$-1.22\cdot 10^4$	$-1.83\cdot 10^4$	$-1.49\cdot 10^4$	1.10
ITM T5 $\sqrt{10}$NH1	$-3.15\cdot 10^5$	$-2.15\cdot 10^5$	$-3.52\cdot 10^5$	$-2.73\cdot 10^5$	1.12
ITM T5 $\sqrt{10}$HE1	$-4.82\cdot 10^4$	$-3.24\cdot 10^4$	$-5.42\cdot 10^4$	$-4.16\cdot 10^4$	1.12
ITM T6 NH	$-1.32\cdot 10^6$	$-8.79\cdot 10^5$	$-1.53\cdot 10^6$	$-1.15\cdot 10^6$	1.16
ITM T6 HE	$-7.36\cdot 10^4$	$-4.78\cdot 10^4$	$-8.60\cdot 10^4$	$-6.32\cdot 10^4$	1.17
ITM T6 $\sqrt{10}$NH1	$-1.18\cdot 10^6$	$-8.17\cdot 10^5$	$-1.34\cdot 10^6$	$-1.05\cdot 10^6$	1.14
ITM T6 $\sqrt{10}$HE1	$-1.29\cdot 10^5$	$-8.60\cdot 10^4$	$-1.50\cdot 10^5$	$-1.13\cdot 10^5$	1.17
ITM T7 NH	$-4.25\cdot 10^5$	$-3.15\cdot 10^5$	$-4.97\cdot 10^5$	$-3.79\cdot 10^5$	1.17
ITM T7 HE	$-2.89\cdot 10^4$	$-2.06\cdot 10^4$	$-3.31\cdot 10^4$	$-2.55\cdot 10^4$	1.15
ITM T7 $\sqrt{10}$NH1	$-2.86\cdot 10^5$	$-1.97\cdot 10^5$	$-3.24\cdot 10^5$	$-2.47\cdot 10^5$	1.13
ITM T7 $\sqrt{10}$HE1	$-8.14\cdot 10^4$	$-5.63\cdot 10^4$	$-9.15\cdot 10^4$	$-7.12\cdot 10^4$	1.12
ITM T8 NH	$-1.32\cdot 10^6$	$-9.24\cdot 10^5$	$-1.57\cdot 10^6$	$-1.18\cdot 10^6$	1.19
ITM T8 HE	$-7.90\cdot 10^4$	$-5.09\cdot 10^4$	$-1.02\cdot 10^5$	$-7.00\cdot 10^4$	1.29
ITM T8 $\sqrt{10}$NH1	$-1.25\cdot 10^6$	$-8.54\cdot 10^5$	$-1.40\cdot 10^6$	$-1.10\cdot 10^6$	1.12
ITM T8 $\sqrt{10}$HE1	$-1.24\cdot 10^5$	$-7.45\cdot 10^4$	$-1.55\cdot 10^5$	$-1.04\cdot 10^5$	1.25

[a]NH = No hedging, HE = daily delta hedging, $\sqrt{10}$NH1 = $\sqrt{10}\mathrm{VaR}_{1-day}$ (no hedging), $\sqrt{10}$HE1 = $\sqrt{10}\mathrm{VaR}_{1-day}$(daily delta hedging)

(and vice versa). This result is in line with Eydeland and Wolyniec (2003), where they explain that for a short VaR horizon of ten days, the instantaneous correlation becomes more important. Hence, there are differences in the results depending on the specific type of time variation, even though the average correlation over the option life is the same, and this will be discussed in more detail later. This result implies that, in practice, the cumulative correlation might not be as relevant for VaR computations as expected. The reduction in risks achieved by daily delta hedging is often less for time-varying correlation term structures than for the constant (cumulative) correlations, but the risk reduction is still substantial for ITM or ATM options and, to a lesser extent, for OTM options. The results for the hedged option are highly dependent on the hedge effectiveness. The following discussion deals with the differences in the specific time variation of the correlation term structures T1 to T8.

4.2.1 Correlation Jumps

The correlation term structures T1 and T2 are initially highly negative and jump upward, respectively, at the start and the end of the option life. First, the time at which the correlation jump occurs is important for the VaR and CVaR estimates of the unhedged option but has an even larger impact on the effectiveness of the delta hedge. When no hedging is applied, a jump at the start of the option life is more than twice as risky as a jump near expiration for negative correlations; see table 4. For the delta-hedged option position this result is even stronger, since the risk measures indicate that the potential loss for a jump at the start is more than four times larger than for a jump near expiration. These results are robust for different moneyness levels. The average correlation for T1 and T2 is -0.84. The risk measures of only T2 give results that are comparable to those of C1 (constant correlation value of -0.9), but the risks for T1 are much higher than for C1. The no-hedge-to-hedge-ratios of the risk measures of T2 in table 5 are twice as large as for T1. So, the time of the temporary jump upward in the correlation has a large impact on the effectiveness of the delta hedge across all moneyness levels.

When the correlations are initially positive, a jump downward at the start (T3) is less risky than a jump near expiration (T4) for the unhedged basket option, regardless of the moneyness level. The hedge effectiveness of a jump near expiration is greater than a jump at the start for positive correlations. So, T2 and T4 have a relatively better hedging performance compared with, respectively, T1 and T3,

Table 5. Risk Measure Ratios for ITM Options with Time-Varying Correlations

	$VaR^{0.99}$	$VaR^{0.95}$	$CVaR^{0.99}$	$CVaR^{0.95}$[a]
ITM T1 NH/HE	14.08	17.43	10.81	13.99
ITM T1 $\sqrt{10}$NH1/$\sqrt{10}$HE1	7.93	7.27	7.80	7.62
ITM T1 NH/$\sqrt{10}$NH1	2.47	2.36	2.46	2.42
ITM T1 HE/$\sqrt{10}$HE1	1.39	0.99	1.77	1.32
ITM T2 NH/HE	23.75	23.44	24.53	23.57
ITM T2 $\sqrt{10}$NH1/$\sqrt{10}$HE1	7.47	7.64	7.52	7.52
ITM T2 NH/$\sqrt{10}$NH1	1.05	1.10	1.09	1.07
ITM T2 HE/$\sqrt{10}$HE1	0.33	0.36	0.34	0.34
ITM T3 NH/HE	16.85	17.36	18.16	17.23
ITM T3 $\sqrt{10}$NH1/$\sqrt{10}$HE1	8.46	8.88	8.30	8.78
ITM T3 NH/$\sqrt{10}$NH1	0.99	0.91	1.11	0.96
ITM T3 HE/$\sqrt{10}$HE1	0.50	0.47	0.51	0.49
ITM T4 NH/HE	18.39	18.46	17.81	18.46
ITM T4 $\sqrt{10}$NH1/$\sqrt{10}$HE1	8.80	8.98	8.52	8.83
ITM T4 NH/$\sqrt{10}$NH1	1.12	1.08	1.15	1.10
ITM T4 HE/$\sqrt{10}$HE1	0.54	0.53	0.55	0.53
ITM T5 NH/HE	19.89	19.68	21.11	19.96
ITM T5 $\sqrt{10}$NH1/$\sqrt{10}$HE1	6.53	6.63	6.49	6.57
ITM T5 NH/$\sqrt{10}$NH1	1.05	1.12	1.10	1.09
ITM T5 HE/$\sqrt{10}$HE1	0.34	0.38	0.34	0.36
ITM T6 NH/HE	17.95	18.38	17.77	18.25
ITM T6 $\sqrt{10}$NH1/$\sqrt{10}$HE1	9.12	9.49	8.88	9.25
ITM T6 NH/$\sqrt{10}$NH1	1.12	1.08	1.14	1.10
ITM T6 HE/$\sqrt{10}$HE1	0.57	0.56	0.57	0.56
ITM T7 NH/HE	14.71	15.26	15.01	14.87
ITM T7 $\sqrt{10}$NH1/$\sqrt{10}$HE1	3.51	3.50	3.54	3.47
ITM T7 NH/$\sqrt{10}$NH1	1.49	1.60	1.54	1.54
ITM T7 HE/$\sqrt{10}$HE1	0.36	0.37	0.36	0.36
ITM T8 NH/HE	16.76	18.15	15.44	16.93
ITM T8 $\sqrt{10}$NH1/$\sqrt{10}$HE1	10.08	11.45	9.01	10.60
ITM T8 NH/$\sqrt{10}$NH1	1.06	1.08	1.12	1.08
ITM T8 HE/$\sqrt{10}$HE1	0.64	0.68	0.65	0.67

[a]NH = No hedging, HE = daily delta hedging, $\sqrt{10}$NH1 = $\sqrt{10}\mathrm{VaR}_{1-day}$ (no hedging), $\sqrt{10}$HE1 = $\sqrt{10}\mathrm{VaR}_{1-day}$(daily delta hedging)

as shown by the no-hedge-to-hedge ratios in table 5. Hence, delta hedging of the basket option is more effective in reducing the risk of potential loss when the correlation jump does not occur within the VaR horizon of ten days. The mean and standard deviation of the initial option prices are nearly the same for T1 and T2, as well as for T3 and T4. Therefore, stress testing is quite important, since the valuation of the option does not show any of these differences in the potential loss of the position.

4.2.2 Correlation Regime Switches

The correlation term structure T5 has initially highly negative (constant) correlations but shifts upward in the second half of the life of the option. For the unhedged ITM and ATM option, a jump near expiration (T2) is riskier than a correlation shift (T5), although the jump only lasts for five days (see the risk measures in table 4).

Due to the relatively high hedge effectiveness of T2 for the ITM option compared with T5 (i.e., high no-hedge-to-hedge ratios), the (C)VaR estimates show that the potential loss of a jump near expiration is lower than a correlation shift for the hedged ITM option. For the hedged ATM option, the no-hedge-to-hedge ratios of T2 and T5 are comparable, and a jump near expiration still has a higher potential loss than a correlation shift for negative correlations. Regarding the OTM option, a correlation shift is riskier than a jump near expiration when no hedging is applied, and this also holds for the hedged option according to the (C)VaR estimates at the 5 percent significance level.

The risk measures do not differ largely between T4 and T6, which are correlation term structures with initially positive correlations and, respectively, a correlation jump and shift downward. The hedge effectiveness is much larger for T6 than for T5 in case of ATM and OTM options in terms of higher no-hedge-to-hedge ratios of risk measures.

4.2.3 Affine Correlation Term Structures

The linear increase and decrease of, respectively, correlations T7 and T8 have cumulative correlation of zero, which corresponds to C5. For C5, T7, and T8, the initial option prices have very similar mean and standard deviations. The unhedged (C)VaR results of T7 are much lower than those of C5, but T8 shows relatively higher risk of potential loss. However, the hedge effectiveness of T8 is relatively higher than for T7, as shown by the higher no-hedge-to-hedge ratios for T8. During bull markets, the correlations could decline; during bear markets or crisis, correlations often increase. Although the unhedged basket option is riskier for declining correlations (e.g., bull markets) in terms of higher potential loss estimates, the daily delta hedge is much more effective in reducing risk than for increasing correlations (e.g., bear markets).

4.2.4 Risk Measures and the Square-Root-of-Time Rule

When comparing the ten-day VaR to the ten-day CVaR, the results in table 4 show that the difference in the risk measures can be quite large. For example, the potential loss indicated by the $\text{CVaR}^{0.99}$ is more than 40 percent larger than by the $\text{VaR}^{0.99}$ for the hedged ITM option with correlation term structure T1. For the other term structures the difference is less pronounced but can still be substantial, especially at the 5 percent significance level. Therefore, it is advisable to look at both the VaR and CVaR estimates, since the CVaR could provide additional information on the risk exposure.

Many market practitioners use the square-root-of-time rule. Previously, the VaR and CVaR results for constant correlations have shown that the square-root-of-time rule provides reasonable estimates for the unhedged basket option. To my knowledge, there is not much evidence in the literature on how the square-root-of-time rule for VaR and CVaR performs for basket options if the correlation term structure is time varying. In this paper, simulation results show that the square-root-of-time rule (C)VaR estimates substantially underestimate the ten-day (C)VaR for the unhedged option with time-varying correlation term structures as shown in table 4. This underestimation is relatively severe for the cases T1 and T7, where the negative correlation between the underlying assets suddenly increases (as is often observed in financial crises). If the correlations at the first day of the VaR horizon are highly negative, then the VaR given by the square-root-of-time rule is computed based on this negative value and thus neglects the fact that the correlations increase over time.

The ratio γ of the ten-day $\mathrm{VaR}^{0.99}$ to the square-root-of-time rule estimate is given in table 3 for the constant correlations and in table 5 for time-varying correlation term structures. The minimum of the stress factor k is 3, and in table 3 it is shown that the γ from our VaR results for the constant correlation specification do not exceed 3 for all constant correlation values and moneyness levels. The underestimation of the unhedged ten-day VaR by the square-root-of-time rule stays well below a factor of 1.5 for constant correlations. The results in table 5 show that the underestimation is much more severe for time-varying correlations, where the unhedged ten-day $\mathrm{VaR}^{0.99}$ for T1 is underestimated with a factor of 2.47 for the unhedged ITM option and increases to 4.08 for the OTM option, thus even violating the minimum stress factor k of 3.

The performance of the square-root-of-time rule is worse for the hedged option, as can be seen in table 5. Depending on the difference between the hedge effectiveness for the one-day and ten-day horizon, the square-root-of-time rule could lead to either severe underestimation or overestimation of the ten-day (C)VaR. For example, the no-hedge-to-hedge ratios for the ten-day $\mathrm{VaR}^{0.99}$ and the scaled one-day $\mathrm{VaR}^{0.99}$ are, respectively, 14.08 and 7.93 for T1 in table 5. The T1 results for the unhedged ITM option show an underestimation of the ten-day $\mathrm{VaR}^{0.99}$ by the square-root-of-time rule, and this also holds for the ten-day $\mathrm{VaR}^{0.99}$ of the hedged option. For T7, the no-hedge-to-hedge ratios of the ten-day $\mathrm{VaR}^{0.99}$ and the scaled one-day $\mathrm{VaR}^{0.99}$ are, respectively, 14.71 and 3.51. Hence, the hedge effectiveness for the T7 scaled one-day $\mathrm{VaR}^{0.99}$ is relatively much lower than for the ten-day $\mathrm{VaR}^{0.99}$ in the case of T1. The hedged ten-day $\mathrm{VaR}^{0.99}$ shows that the risk of potential loss for T1 is much larger than for T7, but the relatively low hedge effectiveness for the one-day horizon of T7 leads to the result that the scaled one-day $\mathrm{VaR}^{0.99}$ of T7 is twice as large as the same risk measure for T1. Thus, the square-root-of-time rule leads to an overestimation of the T7 ten-day (C)VaR for the hedged option due to this large difference in hedge effectiveness between the one-day and ten-day horizon. The ten-day $\mathrm{VaR}^{0.99}$ is also overestimated by the square-root-of-time rule for many other time-varying correlation term structures for the hedged option.

Since the performance of the square-root-of-time rule for hedged options is very sensitive to the hedge effectiveness, the minimum stress factor is likely to be violated in situations where hedging becomes relatively difficult—for example, for options that are out-of-the-money or near expiration. Hence, the square-root-of-time rule applied to a one-day (C)VaR to obtain a ten-day (C)VaR appears to be inadequate for time-varying correlation term structures, even in this simplified and ideal simulation environment.

4.3 Time Scaling for Time-Varying Correlations

So, the square-root-of-time rule gives reasonable results for unhedged VaR estimates in case of constant correlations but does not perform well for the hedged option and is highly dependent on the hedge effectiveness over different horizons. For time-varying correlations, the square-root-of-time rule can lead to serious under- and over-estimation of the ten-day $\mathrm{VaR}^{0.99}$. Therefore, this section examines whether there is another scaling horizon adequate enough to account for the time-varying correlation term structure such that the under- or overestimation by the square-root-of-time rule remains within 10 percent of the ten-day $\mathrm{VaR}^{0.99}$. The square-root-of-time rule has been examined for the horizon of ten days with the following scalings of $\sqrt{10/j}\times\mathrm{VaR}_{j-day}$ for $j = 1, \ldots, 10$ to see whether any type of scaling is valid. The results show that for correlation term structures with initially positive correlations (T3, T4, T6, and T8), the square-root-of-time rule by scaling the one-day VaR is still reasonable for j of 7 or higher for both hedged and unhedged options.[3] However, the results for the correlation term structures with initially negative correlations (T1, T2, T5, and T7) show that the γ's become close to a value of 1 only for a scaling of j of 9 or more in most cases of the hedged option position. Hence, the scaling of the VaR can only be applied for a time horizon of one to three days in the future for the cases considered here.

4.4 Multivariate Baskets with Time-Varying Correlations

The simulation study in this paper considers a basket option with two underlying assets for simplicity, since there is only one correlation term structure involved in the analysis. However, in practice,

[3]Results available by request.

basket options are often traded on baskets consisting of more than two assets. Longin and Solnik (2001) find that during bear markets, two assets can become increasingly correlated, whereas these assets might have lower correlations in other market conditions. If many assets simultaneously become highly correlated during financial crises, not only do the pairwise correlations have an effect on a basket option on multiple assets, but an interaction effect between different correlation term structures could also exist. According to Fengler and Schwendner (2004), multiple assets in a basket can provide a diversification effect but could also introduce additional risks of unknown correlations, and as correlations between assets rise, the diversification effect will be diminished.

If the purpose of including many assets in a basket is to benefit from a diversification effect, the portfolio of the selected underlying assets should have low correlations in normal market conditions. In that case, the results in this paper for the correlation term structures with initially low correlations (such as T1, T2) are likely to be more relevant to basket options in practice than the other specifications considered. Moreover, the hedging of the basket options with many assets could be much more difficult. Since the performance of the square-root-of-time rule is highly dependent on the hedge effectiveness, this can have a negative impact on the accuracy of the scaled one-day VaR estimates as an approximation for the ten-day VaR. The basket option does not only react to changes in one correlation term structure as in the case of two assets, but will also react to changes in $\frac{1}{2}N(N-1)$ different correlation term structures and possibly to interaction effects between these correlation term structures for N assets ($N > 2$). To what extent opposite correlation movements in the basket will offset each other will also depend on the relative importance (i.e., basket weights) of the corresponding assets in the basket. A more detailed analysis of high-dimensional basket options is beyond the scope of this study and is left for further research.

5. Conclusions

The purpose of this paper is to analyze the effect of time-varying correlation term structures on pricing and hedging of basket options as well as on the risk measures VaR and CVaR. First, the benchmark with constant correlations has been used to examine the effect of the sign and size of correlations, and the results are as follows. A surprising result is that basket options on a well-diversified portfolio of assets (i.e., negatively correlated assets) are relatively much more sensitive to correlation changes than on a positively correlated portfolio of assets. This implies that sudden changes in the financial markets can lead to large losses for a basket option with well-diversified assets, which cannot be captured by the VaR when using the square-root-of-time rule. The ATM and OTM basket option prices react asymmetrically to positive and negative correlation changes, where a change in negative correlations has a higher impact on the option price than a change in positive correlations of the same magnitude. As a result, the corresponding VaR and CVaR estimates are also more sensitive to changes in negative correlations. The option price of a far ITM option does not differ largely for different correlation values. However, the risk measures of the far ITM basket show that the risk of potential loss increases substantially as the correlation increases. Thus, the risk of potential loss of basket options cannot be entirely captured by the value of the option, but extensive stress testing is needed to reveal these risks.

Second, dynamic delta hedging can reduce the VaR substantially. The specific type of time variation of the correlation term structure is essential to the effectiveness of the dynamic delta hedge of the basket option. Hence, one cannot use the average constant correlation for the computation of the VaR of a hedged basket option with time-varying correlations. Thus, the time of occurrence of a jump during the life of the option is very relevant for the risk of potential loss, even though the basket option payoff is based on the value of the underlying assets at the time of the expiration.

Third, the CVaR can differ from the VaR estimates substantially, especially for the time-varying negative correlations with a jump at the start of the option life and for estimates at the 5 percent significance level. Therefore, providing VaR estimates might not be sufficient, and the coherent measure CVaR can give the additional information needed in certain market conditions.

Finally, the square-root-of-time rule leads to underestimation of the unhedged ten-day VaR in most cases considered, but the underestimation ratio still remains below the regulatory stress factor minimum of 3 for constant correlation term structure. The scaled one-day

VaR can deviate heavily from the ten-day VaR for the hedged option when the difference in the hedge effectiveness for the one-day and ten-day horizon is large. The square-root-of-time rule performs relatively well for unhedged ITM basket options. However, applying the square-root-of-time rule can lead to large deviations from the ten-day VaR for (un)hedged ATM and OTM options. In general, the estimates do not improve very much by using different scaling horizons, unless the scaling is applied for a square-root-of-time factor close to 1. In these cases, stress testing is much more important for determining an accurate VaR estimate.

References

Artzner, P., F. Delbaen, J. M. Eber, and D. Heath. 1999. "Coherent Measures of Risk." *Mathematical Finance* 9 (3): 203–28.

Bank for International Settlements. 2005. "International Convergence of Capital Measurement and Capital Standards: A Revised Framework." Technical Report (November).

Bennett, M. N., and J. E. Kennedy. 2004. "Quanto Pricing with Copulas." *Journal of Derivatives* (Fall):26–45.

Brigo, D., F. Mercurio, F. Rapisarda, and R. Scotti. 2004. "Approximated Moment-Matching Dynamics for Basket-Options Simulation. *Quantitative Finance* 4 (1): 1–16.

Curran, M. 1994. "Valuing Asian and Portfolio Options by Conditioning on the Geometric Mean Price." *Management Science* 40 (12): 1705–11.

Dammers, C. R., and R. N. McCauley. 2006. "Basket Weaving: The Euromarket Experience with Basket Currency Bonds." *BIS Quarterly Review* (March).

Danielsson, J., and J. P. Zigrand. 2005. "On Time-Scaling of Risk and the Square-Root-of-Time Rule." Working Paper, available at www.riskresearch.org.

Deelstra, G., J. Liinev, and M. Vanmaele. 2004. "Pricing of Arithmetic Basket Options by Conditioning." *Insurance, Mathematics and Economics* 34 (1): 55–77.

Diebold, F. X., A. Hickman, A. Inoue, and T. Schuermann. 1997. "Converting 1-Day Volatility to h-Day Volatility: Scaling by Root-h Is Worse than You Think." Wharton Working Paper Series, No. 97-34.

Duffie, D., and K. J. Singleton. 2003. *Credit Risk: Pricing, Measurement, and Management.* 1st ed. Princeton, NJ: Princeton University Press.

Eydeland, A., and K. Wolyniec. 2003. *Energy and Power Risk Management.* 1st ed. Hoboken, NJ: John Wiley and Sons, Inc.

Fengler, M. R., and P. Schwendner. 2004. "Quoting Multiasset Equity Options in the Presence of Errors from Estimating Correlations." *Journal of Derivatives* (Summer):43–54.

Glasserman, P. 2004. *Monte Carlo Methods in Financial Engineering.* 1st ed. New York: Springer-Verlag, Inc.

Goetzmann, W. N., L. Li, and K. G. Rouwenhorst. 2005. "Long-Term Global Market Correlations." *Journal of Business* 78 (1): 1–38.

Hull, J. 2006. *Options, Futures and Other Derivatives.* 6th ed. New York: Pearson Prentice Hall.

Jorion, P. 2001. *Value at Risk: The New Benchmark for Managing Financial Risk.* 2nd ed. McGraw Hill.

Kupiec, P. H. 1998. "Stress Testing in a Value at Risk Framework." *Journal of Derivatives* (Fall):7–24.

Longin, F., and B. Solnik. 2001. "Extreme Correlation of International Equity Markets." *Journal of Finance* 56 (2): 649–76.

Margrabe, W. 1978. "The Value of an Option to Exchange One Asset for Another." *Journal of Finance* 33 (1): 177–86.

McNeil, A. J., R. Frey, and P. Embrechts. 2005. *Quantitative Risk Management: Concepts, Techniques, and Tools.* 1st ed. Princeton, NJ: Princeton University Press.

Milevsky, M. A., and S. E. Posner. 1998. "A Closed-Form Approximation for Valuing Basket Options." *Journal of Derivatives* (Summer):54–61.

Morgan, J. P. 1996. "RiskMetrics—Technical Document." J. P. Morgan Bank. 4th ed. New York.

Pellizzari, P. 2005. "Static Hedging of Multivariate Derivatives by Simulation." *European Journal of Operational Research* 166 (2): 507–19.

Pritsker, M. G. 2006. "The Hidden Dangers of Historical Simulation." *Journal of Banking and Finance* 30 (2): 561–82.

Skintzi, V. D., G. Skiadopoulos, and A. P. N. Refenes. 2005. "The Effect of Misestimating Correlation on Value at Risk." *Journal of Alternative Investments* (Spring):66–82.