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# Firm-Specific Capital and the New Keynesian Phillips Curve*

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A relation between inflation and the path of average marginal cost (often measured by unit labor cost) implied by the Calvo (1983) model of staggered pricing—sometimes referred to as the "New Keynesian" Phillips curve—has been the subject of extensive econometric estimation and testing. Standard theoretical justifications of this form of aggregate-supply relation, however, either assume (1) the existence of a competitive rental market for capital services, so that the shadow cost of capital services is equated across firms and sectors at all points in time, despite the fact that prices are set at different times, or (2) that the capital stock of each firm is constant, or at any rate exogenously given, and so independent of the firm's pricing decision. But neither assumption is realistic. The present paper examines the extent to which existing empirical specifications and interpretations of parameter estimates are compromised by reliance on either of these assumptions.

The paper derives an aggregate-supply relation for a model with monopolistic competition and Calvo pricing in which capital is firm specific and endogenous, and investment is subject to convex adjustment costs. The aggregate-supply relation is shown to again take the standard New Keynesian form, but with an elasticity of inflation with respect to real marginal cost that is a different function of underlying parameters than in the simpler cases studied earlier. Thus the relations estimated in the empirical literature remain correctly specified under the assumptions proposed here, but the interpretation of the estimated elasticity is different; in particular, the implications of the estimated Phillips-curve slope for the frequency of price

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adjustment is changed. Assuming a rental market for capital results in a substantial exaggeration of the infrequency of price adjustment; assuming exogenous capital instead results in a smaller underestimate.

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A popular specification in recent analyses of alternative monetary policies is the "New Keynesian" Phillips curve,

$$\pi_t = \xi \hat{s}_t + \beta E_t \pi_{t+1}, \tag{1}$$

where $\pi_t$ is the rate of inflation, $\hat{s}_t$ is the departure of the (average) log of real marginal cost from its steady-state value, the coefficient $\xi > 0$ depends on the degree of stickiness of prices, and $0 < \beta < 1$ is a utility discount factor that, under an empirically realistic calibration, must nearly equal 1. As is well known, this relation follows (in a log-linear approximation) from the Calvo model of staggered price setting under certain assumptions.[1] The implications of (1) for the co-movement of the general level of prices and marginal cost have been subject to extensive econometric testing, beginning with the work of Galí and Gertler (1999) and Sbordone (2002).

In standard derivations, (1) follows from the optimal pricing problem of a firm that adjusts the price of its product at random intervals, under the assumption that the marginal cost $S_t(i)$ of supplying a given good $i$ in period $t$ is given by a function of the form

$$S_t(i) = S(y_t(i); X_t), \tag{2}$$

where $y_t(i)$ is the quantity sold of good $i$ in that period, and $X_t$ is a vector of variables that firm $i$ takes to be unaffected by its pricing decision. Under the further assumption of a demand curve of the form $y_t(i) = Y(p_t(i); X_t)$, this implies that marginal cost can be expressed as a function of the price $p_t(i)$ that $i$ chooses to charge in that period, together with variables that are unaffected by its actions.

The specification (2) is in turn correct as long as all factors of production are either purchased on a spot market (at a price that

[1]See, e.g., Woodford (2003, chap. 3, sec. 2.2).

is independent of the quantity used by $i$), or completely fixed. In particular, one can treat the case in which capital is not a variable factor of production (and output is simply a concave function of the variable labor input, as in Woodford 2003, chap. 3), or the case in which capital is variable, but capital services are obtained on a rental market (as in Galí and Gertler [1999] and the baseline case considered in Sbordone [2002]).[2] Matters are more complex, however, under the more realistic assumption that capital is endogenous and *firm specific*. That is, we shall assume that each firm accumulates capital for its own use only, and that (as in standard neoclassical investment theory) there are convex costs of more rapid adjustment of an individual firm's capital stock. In this case, $S_t(i)$ will depend not only on the quantity that firm $i$ produces in period $t$, but also on the firm's capital stock in that period, and this latter variable depends on the firm's decisions in previous periods, including its previous pricing decisions. The dynamic linkages in a firm's optimal price-setting decision are therefore more complex in this case than is assumed in standard derivations of the New Keynesian Phillips curve.

Here I treat the optimal price-setting problem in a model with firm-specific capital, and show that once again a relation of the form (1) can be derived.[3] Hence the econometric estimates reported by authors such as Galí and Gertler (1999) and Sbordone (2002) can be interpreted without making assumptions as restrictive as those papers had appeared to rely upon. However, the coefficient $\xi$ is a more complex function of underlying model parameters, such as the frequency with which prices are reoptimized, in the case that capital is firm specific.

This is potentially of considerable importance for the interpretation of econometric estimates of the coefficient $\xi$. Estimates of $\xi$ are often interpreted in terms of the frequency of price of adjustment that they imply, given estimated or calibrated values for other model parameters. (Indeed, in many papers in the literature, beginning with

[2] Both assumptions lead to a relation of the form (1). However, the interpretation of the coefficient $\xi$ in terms of underlying model parameters is different in the two cases, as discussed in Sbordone (2002).

[3] The derivation here corrects the analysis given in Woodford (2003, chap. 5, sec. 3), to take account of an error in the original calculations noted by Sveen and Weinke (2004a).

Galí and Gertler [1999], equation [1] is estimated in a form that results directly in an estimate of the frequency of price adjustment rather than of the elasticity $\xi$.) Furthermore, it is often argued that estimated values of $\xi$ are so small as to imply that prices are sticky for an implausibly long length of time; this is taken to cast doubt on the realism of the Calvo pricing model and hence of the aggregate-supply specification (1). But the mapping between the frequency of price adjustment and the value of $\xi$ is different in the case of firm-specific capital than under the more common assumption of a rental market for capital services.[4] The assumption of a rental market for capital substantially weakens the degree of strategic complementarity among the pricing decisions of different firms—or alternatively, it reduces the importance of real rigidities in the sense of Ball and Romer (1990)—with the consequence that $\xi$ is larger for any given frequency of price adjustment. It then follows that a small estimated value of $\xi$ will be taken to imply very infrequent price adjustment. But allowing for firm-specific capital can make the implied frequency of price adjustment much greater, as shown in section 3.4 below.

The fact that an assumption that capital is firm specific will lead to a lower estimate of the degree of price stickiness was first demonstrated by Sbordone (1998) and also illustrated by Galí, Gertler, and Lopez-Salido (2001). However, in these papers, the treatment of capital as firm specific is accompanied (at least implicitly) by an assumption that the capital stock of each firm is exogenously given, as in the analysis in Woodford (2003, chap. 3), rather than responding endogenously to the firm's incentives to invest. This is because it is only in this case that a specification of the form (2) remains consistent with the assumption of firm-specific capital. The analysis here instead presents an analysis of aggregate supply in the case that capital is both firm specific and endogenous.[5] This case is a

[4]It is also different under the assumption of a fixed quantity of capital for each firm, as noted above. However, that simple model is disconfirmed by the observation that capital varies over time, and that investment spending is substantially affected by monetary disturbances.

[5]Subsequent to the first circulation of these notes, Eichenbaum and Fisher (2004), Altig et al. (2005), and Matheron (2005) have built on the analysis here to examine the consequences of endogenous firm-specific capital for the estimated frequency of price adjustment in empirical versions of the New Keynesian Phillips curve. These authors extend the present analysis to more complicated versions of (1) that allow a closer fit to aggregate U.S. time series.

good deal more complicated to analyze, but it turns out still to be possible to derive an aggregate-supply relation that (in a log-linear approximation) takes the simple form (1).

The paper proceeds as follows. In section 1, I introduce a model of firm-specific investment demand with convex costs of adjustment of an individual firm's capital stock, with particular attention to the way in which standard neoclassical investment theory must be modified when the firm is not a price-taker in its product market, but instead fixes its price for a period of time and fills whatever orders it may receive. In section 2, I then consider the price-setting problem of such a firm, under the assumption that the price remains fixed for a random interval of time, and characterize the joint dynamics of the firm's price and its capital stock. Finally, in section 3, I derive the model's implications for the form of the aggregate-supply relation that connects the overall inflation rate with the overall level of real activity, and discuss the consequences for the inference about the frequency of price adjustment that can be drawn from an estimate of the elasticity $\xi$ in (1).

## 1. Investment Demand when Prices Are Sticky

I wish to analyze the relation between inflation and aggregate output in a model with staggered pricing (modeled after the fashion of Calvo [1983] and Yun [1996]) and endogenous capital accumulation. The main source of complication in this analysis is the assumption that the producers of individual differentiated goods (that adjust their prices at different dates) invest in firm-specific capital that is relatively durable, so that the distribution of capital stocks across different firms (as a result of differing histories of price adjustment) matters, and not simply the economy's aggregate capital stock. Nonetheless, I shall show that (in the same kind of log-linear approximation that is used in standard derivations of the New Keynesian Phillips curve) it is possible to derive structural relations that constitute the "aggregate supply block" of a macro model, which involve only the economy's aggregate capital stock, aggregate output, and overall index of prices.

A first task is to develop a model of optimizing investment demand by suppliers with sticky prices, and that are demand constrained as a result. As in the sticky-price models with exogenous capital presented in Woodford (2003, chap. 3), there is a continuum of differentiated goods, each supplied by a single (monopolistically competitive) firm. The production function for good $i$ is assumed to be of the form

$$y_t(i) = k_t(i) f(A_t h_t(i)/k_t(i)), \tag{3}$$

where $f$ is an increasing, concave function, with $f(0) = 0$. I assume that each monopoly supplier makes an independent investment decision each period; there is a separate capital stock $k_t(i)$ for each good, which can be used only in the production of good $i$.

I also assume convex adjustment costs for investment by each firm, of the usual kind assumed in neoclassical investment theory. Increasing the capital stock to the level $k_{t+1}(i)$ in period $t+1$ requires investment spending in the amount $I_t(i) = I(k_{t+1}(i)/k_t(i))k_t(i)$ in period $t$. Here $I_t(i)$ represents purchases by firm $i$ of the composite good, defined as the usual Dixit-Stiglitz aggregate over purchases of each of the continuum of goods (with the same constant elasticity of substitution $\theta > 1$ as for consumption purchases).[6] In this way, the allocation of investment expenditure across the various goods is in exactly the same proportion as consumption expenditure, resulting in a demand curve for each producer that is again of the form

$$y_t(i) = Y_t \left( \frac{p_t(i)}{P_t} \right)^{-\theta}, \tag{4}$$

but where now aggregate demand is given by $Y_t = C_t + I_t + G_t$, in which expression $C_t$ is the representative household's demand for the composite good for consumption purposes, $G_t$ is the government's demand for the composite good (treated as an exogenous random variable), and $I_t$ denotes the integral of $I_t(i)$ over the various firms $i$.

I assume as usual that the function $I(\cdot)$ is increasing and convex; the convexity implies the existence of costs of adjustment. I further assume that near a zero growth rate of the capital stock, this function satisfies $I(1) = \delta$, $I'(1) = 1$, and $I''(1) = \epsilon_\psi$, where $0 < \delta < 1$ and $\epsilon_\psi > 0$ are parameters. This implies that in the steady state to which the economy converges in the absence of shocks (which here

[6]See Woodford (2003, chap. 3) for discussion of this aggregator and its consequences for the optimal allocation of demand across alternative differentiated goods.

involves a constant capital stock, as I abstract from trend growth), the steady rate of investment spending required to maintain the capital stock is equal to $\delta$ times the steady-state capital stock (so that $\delta$ can be interpreted as the rate of depreciation). It also implies that near the steady state, a marginal unit of investment spending increases the capital stock by an equal amount (as there are locally no adjustment costs). Finally, in my log-linear approximation to the equilibrium dynamics, $\epsilon_\psi$ is the parameter that indexes the degree of adjustment costs. A central goal of the analysis is consideration of the consequences of alternative values for $\epsilon_\psi$; the model with exogenous firm-specific capital presented in Woodford (2003, chaps. 3, 4) is recovered as the limiting case of the present model in which $\epsilon_\psi$ is made unboundedly large.

Profit-maximization by firm $i$ then implies that the capital stock for period $t+1$ will be chosen in period $t$ to satisfy the first-order condition

$$I'(g_t(i)) = E_t Q_{t,t+1}\Pi_{t+1}\left\{\rho_{t+1}(i) + g_{t+1}(i)I'(g_{t+1}(i)) - I(g_{t+1}(i))\right\}, \quad (5)$$

where $g_t(i) \equiv k_{t+1}(i)/k_t(i)$, $\rho_{t+1}(i)$ is the (real) shadow value of a marginal unit of additional capital for use by firm $i$ in period $t+1$ production, and $Q_{t,t+1}\Pi_{t+1}$ is the stochastic discount factor for evaluating real income streams received in period $t+1$. Expressing the real stochastic discount factor as $\beta\lambda_{t+1}/\lambda_t$, where $\lambda_t$ is the representative household's marginal utility of real income in period $t$ and $0 < \beta < 1$ is the utility discount factor, and then log-linearizing (5) around the steady-state values of all state variables, we obtain

$$\hat{\lambda}_t + \epsilon_\psi(\hat{k}_{t+1}(i) - \hat{k}_t(i)) = E_t\hat{\lambda}_{t+1} + [1-\beta(1-\delta)]E_t\hat{\rho}_{t+1}(i) + \beta\epsilon_\psi E_t(\hat{k}_{t+2}(i) - \hat{k}_{t+1}(i)), \quad (6)$$

where $\hat{\lambda}_t \equiv \log(\lambda_t/\bar{\lambda})$, $\hat{k}_t(i) \equiv \log(k_t(i)/\bar{K})$, $\hat{\rho}_t(i) \equiv \log(\rho_t(i)/\bar{\rho})$, and variables with bars denote steady-state values.

Note that $\rho_{t+1}(i)$ would correspond to the real "rental price" for capital services if a market existed for such services, though I do not assume one here.[7] It is *not* possible in the present model to equate

[7]The case in which there is a rental market for capital services is instead considered in section 3.2 below.

this quantity with the marginal product, or even the marginal revenue product of capital (using the demand curve [4] to compute marginal revenue), for suppliers are demand constrained in their sales, given the prices that they have posted; it is not possible to increase sales by moving down the demand curve. Thus the shadow value of additional capital must instead be computed as the reduction in labor costs through substitution of capital inputs for labor, while still supplying the quantity of output that happens to be demanded. In this way I obtain

$$\rho_t(i) = w_t(i)\left(\frac{f(\tilde{h}_t(i)) - \tilde{h}_t(i)f'(\tilde{h}_t(i))}{A_t f'(\tilde{h}_t(i))}\right),$$

where $w_t(i)$ is the real wage for labor of the kind hired by firm $i$ and $\tilde{h}_t(i) \equiv A_t h_t(i)/k_t(i)$ is firm $i$'s effective labor-capital input ratio.[8] I can alternatively express this in terms of the output-capital ratio for firm $i$ (in order to derive an "accelerator" model of investment demand), by substituting (3) to obtain

$$\rho_t(i) = \frac{w_t(i)}{A_t} f^{-1}(y_t(i)/k_t(i))[\phi(y_t(i)/k_t(i)) - 1], \tag{7}$$

where $\phi(y/k)$ is the reciprocal of the elasticity of the function $f$, evaluated at the argument $f^{-1}(y/k)$.

As in the baseline model treated in Woodford (2003, chap. 3), I shall assume a sector-specific labor market. In this case, the first-order condition for optimizing labor supply can be written in the form

$$w_t(i) = \frac{v_h(f^{-1}(y_t(i)/k_t(i))k_t(i)/A_t; \xi_t)}{\lambda_t}, \tag{8}$$

where labor demand has been expressed as a function of the demand for good $i$. This can be log-linearized as

$$\hat{w}_t(i) = \nu(\hat{h}_t(i) - \bar{h}_t) - \hat{\lambda}_t,$$

[8]Note that in the case of a flexible-price model, the ratio of $w_t(i)$ to the denominator would always equal marginal revenue, and so this expression would equal the marginal revenue product of capital, though it would be a relatively cumbersome way of writing it.

where $\nu > 0$ is the elasticity of the marginal disutility of labor with respect to labor supply, and $\bar{h}_t$ is an exogenous disturbance to preferences, indicating the percentage increase in labor supply needed to maintain a constant marginal disutility of working. Substituting (8) into (7) and log-linearizing, I obtain

$$\hat{\rho}_t(i) = \left(\nu\phi_h + \frac{\phi_h}{\phi_h - 1}\omega_p\right)(\hat{y}_t(i) - \hat{k}_t(i)) + \nu\hat{k}_t(i) - \hat{\lambda}_t - \omega q_t, \quad (9)$$

where $\phi_h > 1$ is the steady-state value of $\phi(y/k)$ (i.e., the reciprocal of the elasticity of the production function with respect to the labor input), and $\omega_p > 0$ is the negative of the elasticity of the marginal product $f'(f^{-1}(y/k))$ with respect to $y/k$. The composite exogenous disturbance $q_t$ is defined as

$$q_t \equiv \omega^{-1}[\nu\bar{h}_t + (1+\nu)a_t],$$

where $a_t \equiv \log A_t$; it indicates the percentage change in output required to maintain a constant marginal disutility of output supply, in the case that the firm's capital remains at its steady-state level.[9] Substituting (9) into (6), I then have an equation to solve for the dynamics of firm $i$'s capital stock, given the evolution of demand $\hat{y}_t(i)$ for its product, the marginal utility of income $\hat{\lambda}_t$, and the exogenous disturbance $q_t$.

As the coefficients of these equations are the same for each firm, an equation of the same form holds for the dynamics of the aggregate capital stock (in our log-linear approximation). The equilibrium condition for the dynamics of the capital stock is thus of the form

$$\hat{\lambda}_t + \epsilon_\psi(\hat{K}_{t+1} - \hat{K}_t) = \beta(1-\delta)E_t\hat{\lambda}_{t+1} + [1-\beta(1-\delta)][\rho_y E_t\hat{Y}_{t+1} - \rho_k\hat{K}_{t+1} - \omega E_t q_{t+1}] + \beta\epsilon_\psi E_t(\hat{K}_{t+2} - \hat{K}_{t+1}), \quad (10)$$

where the elasticities of the marginal valuation of capital are given by

$$\rho_y \equiv \nu\phi_h + \frac{\phi_h}{\phi_h - 1}\omega_p > \rho_k \equiv \rho_y - \nu > 0.$$

[9] That is, $q_t$ measures the output change that would be required to maintain a fixed marginal disutility of supply given possible fluctuations in preferences and technology, but not taking account of the effect of possible fluctuations in the firm's capital stock. With this modification of the definition given in Woodford (2003, chap. 3) for the model with exogenous capital, $q_t$ is again an exogenous disturbance term.

The implied dynamics of investment spending are then given by

$$\hat{I}_t = k[\hat{K}_{t+1} - (1-\delta)\hat{K}_t], \tag{11}$$

where $\hat{I}_t$ is defined as the percentage deviation of investment from its steady-state level, as a share of steady-state output, and $k \equiv \bar{K}/\bar{Y}$ is the steady-state capital-output ratio.

Thus far I have derived investment dynamics as a function of the evolution of the marginal utility of real income of the representative household. This is in turn related to aggregate spending through the relation $\lambda_t = u_c(Y_t - I_t - G_t; \xi_t)$, which we may log-linearize as

$$\hat{\lambda}_t = -\sigma^{-1}(\hat{Y}_t - \hat{I}_t - g_t), \tag{12}$$

where the composite disturbance $g_t$ reflects the effects both of government purchases and of shifts in private impatience to consume.[10] Finally, because of the relation between the marginal utility of income process and the stochastic discount factor that prices bonds,[11] the nominal interest rate must satisfy

$$1 + i_t = \{\beta E_t[\lambda_{t+1}/(\lambda_t \Pi_{t+1})]\}^{-1},$$

which one may log-linearize as

$$\hat{\imath}_t = E_t \pi_{t+1} + \hat{\lambda}_t - E_t \hat{\lambda}_{t+1}. \tag{13}$$

The system of equations (10)–(13) then comprises the “IS block” of the model. These jointly suffice to determine the paths of the variables $\{\hat{Y}_t, \hat{I}_t, \hat{K}_t, \lambda_t\}$, given an initial capital stock and the evolution of short-term real interest rates $\{\hat{\imath}_t - E_t\pi_{t+1}\}$. The nature of the effects of real interest-rate expectations on these variables is discussed further in Woodford (2004).

[10] Note that the parameter $\sigma$ in this equation is not precisely the intertemporal elasticity of substitution in consumption, but rather $\bar{C}/\bar{Y}$ times that elasticity. In a model with investment, these quantities are not exactly the same, even in the absence of government purchases.

[11] See Woodford (2003, chaps. 2, 4) for further discussion of the stochastic discount factor and the Fisher relation between the nominal interest rate and expected inflation.

## 2. Optimal Price Setting with Endogenous Capital

I turn next to the implications of an endogenous capital stock for the price-setting decisions of firms. The capital stock affects a firm's marginal cost, of course; but more subtly, a firm considering how its future profits will be affected by the price it sets must also consider how its capital stock will evolve over the time that its price remains fixed.

I begin with the consequences for the relation between marginal cost and output. Real marginal cost can be expressed as the ratio of the real wage to the marginal product of labor,

$$s_t(i) = \frac{w_t(i)}{A_t f'(f^{-1}(y_t(i)/k_t(i)))}. \tag{14}$$

Again writing the factor input ratio as a function of the capital-output ratio, and using (8) for the real wage, we obtain

$$s_t(i) = \frac{v_h(f^{-1}(y_t(i)/k_t(i))k_t(i)/A_t;\xi_t)}{\lambda_t A_t f'(f^{-1}(y_t(i)/k_t(i)))} \tag{15}$$

for the real marginal cost of supplying good $i$. This can be log-linearized to yield

$$\hat{s}_t(i) = \omega(\hat{y}_t(i) - \hat{k}_t(i) - q_t) + \nu\hat{k}_t(i) - \hat{\lambda}_t, \tag{16}$$

where $\hat{s}_t(i) \equiv \log(s_t(i)/\bar{s})$, and $\omega \equiv \omega_w + \omega_p \equiv \nu\phi_h + \omega_p > 0$ is the elasticity of marginal cost with respect to a firm's own output.

Letting $\hat{s}_t$ without the index $i$ denote the average level of real marginal cost in the economy as a whole, I note that (16) implies that

$$\hat{s}_t(i) = \hat{s}_t + \omega(\hat{y}_t(i) - \hat{Y}_t) - (\omega - \nu)(\hat{k}_t(i) - \hat{K}_t). \tag{17}$$

Then using (4) to substitute for the relative output of firm $i$ in (17), one obtains

$$\hat{s}_t(i) = \hat{s}_t - (\omega - \nu)\tilde{k}_t(i) - \omega\theta\tilde{p}_t(i), \tag{18}$$

where $\tilde{p}_t(i) \equiv \log(p_t(i)/P_t)$ is the firm's log relative price, and $\tilde{k}_t(i) \equiv \hat{k}_t(i) - \hat{K}_t$ is its log relative capital stock. Note also that the average level of real marginal cost satisfies

$$\hat{s}_t = \omega(\hat{Y}_t - \hat{K}_t - q_t) + \nu\hat{K}_t - \hat{\lambda}_t. \tag{19}$$

Following the same logic as in Woodford (2003, chap. 3), the Calvo price-setting framework implies that if a firm $i$ resets its price in period $t$, it chooses a price that satisfies the (log-linear approximate) first-order condition

$$\sum_{k=0}^{\infty}(\alpha\beta)^k \hat{E}_t^i[\tilde{p}_{t+k}(i) - \hat{s}_{t+k}(i)] = 0, \tag{20}$$

where $0 < \alpha < 1$ is the fraction of prices that are not reset in any period. Here I introduce the notation $\hat{E}_t^i$ for an expectation conditional on the state of the world at date $t$, but integrating *only over those future states in which $i$ has not reset its price since period $t$*. Note that in the case of any *aggregate-state* variable $x_t$ (i.e., a variable the value of which depends only on the history of aggregate disturbances, and not on the individual circumstances of firm $i$), $\hat{E}_t^i x_T = E_t x_T$, for any date $T \geq t$. However, the two conditional expectations differ in the case of variables that depend on the relative price or relative capital stock of firm $i$. For example,

$$\hat{E}_t^i \tilde{p}_{t+k}(i) = \tilde{p}_t(i) - \sum_{j=1}^{k} E_t \pi_{t+j} \tag{21}$$

for any $k \geq 1$, since firm $i$'s price remains unchanged along all of the histories that are integrated over in this case. Instead, the expectation when one integrates over all possible future states conditional upon the state of the world at date $t$ is given by

$$E_t \tilde{p}_{t+1}(i) = \alpha[\tilde{p}_t(i) - E_t \pi_{t+1}] + (1-\alpha) E_t \hat{p}_{t+1}^*(i), \tag{22}$$

where $\hat{p}_t^*(i)$ is the (log) relative price chosen when $i$ reconsiders its price at date $t$. (Similar expressions can be given for horizons $k > 1$.)

Substituting (18) for $s_{t+k}(i)$ and (21) for $\hat{E}_t^i \tilde{p}_{t+k}(i)$ in (20), one obtains

$$(1+\omega\theta)\hat{p}_t^*(i) = (1-\alpha\beta)\sum_{k=0}^{\infty}(\alpha\beta)^k \hat{E}_t^i\left[\hat{s}_{t+k} + (1+\omega\theta)\sum_{j=1}^{k}\pi_{t+j} - (\omega-\nu)\tilde{k}_{t+k}(i)\right] \tag{23}$$

for the optimal relative price that should be chosen by a firm that resets its price at date $t$. This relation differs from the result obtained in Woodford (2003, chap. 3) for a model with exogenous capital only in the presence of the $\hat{E}_t^i \tilde{k}_{t+k}(i)$ terms.

The additional terms complicate the analysis in several respects. Note that the first two terms inside the square brackets are aggregate-state variables, so that the distinction between $\hat{E}_t^i$ and $E_t$ would not matter in this expression, were it not for the dependence of marginal cost on $i$'s relative capital stock; it is for this reason that the alternative form of conditional expectation did not have to be introduced in Woodford (2003, chap. 3). However, in the model with endogenous capital, it is important to make this distinction when evaluating the $\hat{E}_t^i \tilde{k}_{t+k}(i)$ terms.[12] Furthermore, these new terms will not have the same value for all firms $i$ that reset their prices at date $t$, for they will depend on $i$'s relative capital stock $\tilde{k}_t(i)$ at the time that prices are reconsidered; hence $p_t^*(i)$ is no longer independent of $i$, as in the model with exogenous capital (or a model with an economy-wide rental market for capital). And finally, (23) is not yet a complete solution for the optimal price-setting rule, since the value of the right-hand side still depends on the expected evolution of $i$'s relative capital stock; this in turn depends on the expected evolution of $i$'s relative price, which depends on the choice of $\hat{p}_t^*(i)$. A complete solution for this decision rule requires that one consider the effect of a firm's relative price on the evolution of its relative capital stock.

## *2.1 Dynamics of the Relative Capital Stock*

Equation (10) implies that $i$'s relative capital stock must evolve in accordance with the relation

$$\epsilon_\psi(\tilde{k}_{t+1}(i) - \tilde{k}_t(i)) = [1 - \beta(1-\delta)][\rho_y E_t(\hat{y}_{t+1}(i) - \hat{Y}_t) - \rho_k \tilde{k}_{t+1}(i)] + \beta\epsilon_\psi E_t(\tilde{k}_{t+2}(i) - \tilde{k}_{t+1}(i)).$$

Again using $i$'s demand curve to express relative output as a function of the firm's relative price, this can be written as

$$E_t[Q(L)\tilde{k}_{t+2}(i)] = \Xi E_t \tilde{p}_{t+1}(i), \tag{24}$$

[12] It is the failure to distinguish between $\hat{E}_t^i$ and $E_t$ in evaluating these terms that results in the incorrect calculations in the treatment of the present model in Woodford (2003, chap. 5) noted by Sveen and Weinke (2004a).

where the lag polynomial is

$$Q(L) \equiv \beta - [1 + \beta + (1 - \beta(1 - \delta))\rho_k \epsilon_\psi^{-1}]L + L^2,$$

and

$$\Xi \equiv (1 - \beta(1 - \delta))\rho_y \theta \epsilon_\psi^{-1} > 0.$$

I note for later reference that the lag polynomial can be factored as

$$Q(L) = \beta(1 - \mu_1 L)(1 - \mu_2 L).$$

Given that $Q(0) = \beta > 0$, $Q(\beta) < 0$, $Q(1) < 0$, and that $Q(z) > 0$ for all large enough $z > 0$, one sees that $\mu_1, \mu_2$ must be two real roots that satisfy $0 < \mu_1 < 1 < \beta^{-1} < \mu_2$.

Equation (24) cannot yet be solved for the expected evolution of the relative capital stock because of the dependence of the expected evolution of $i$'s relative price (the "forcing term" on the right-hand side) on the expected evolution of the relative capital stock itself, for reasons just discussed. However, one may note that insofar as $i$'s decision problem is locally convex, so that the first-order conditions characterize a locally unique optimal plan, the optimal decision for $i$'s relative price in the event that the price is reset at date $t$ must depend *only* on $i$'s relative capital stock at date $t$ and on the economy's aggregate state. Thus a log-linear approximation to $i$'s pricing rule must take the form

$$\hat{p}_t^*(i) = \hat{p}_t^* - \psi \tilde{k}_t(i), \tag{25}$$

where $\hat{p}_t^*$ depends only on the aggregate state (and so is the same for all $i$), and $\psi$ is a coefficient to be determined below.

Note that the assumption that the firms that reset prices at date $t$ are drawn with uniform probability from the entire population implies that the average value of $\tilde{k}_t(i)$ over the set of firms that reset prices is zero (just as it is over the entire population of firms). Hence $\hat{p}_t^*$ is also the average relative price chosen by firms that reset prices at date $t$, and the overall rate of price inflation will be given (in our log-linear approximation) by

$$\pi_t = \frac{1 - \alpha}{\alpha} \hat{p}_t^*. \tag{26}$$

Substitution of this, along with (25), into (22) then yields

$$E_t \tilde{p}_{t+1}(i) = \alpha \tilde{p}_t(i) - (1 - \alpha)\psi \tilde{k}_{t+1}(i). \tag{27}$$

Similarly, the optimal quantity of investment in any period $t$ must depend only on $i$'s relative capital stock in that period, its relative price (which matters as a separate argument of the decision rule in the event that the price is *not* reset in period $t$), and the economy's aggregate state. Thus a log-linear approximation to $i$'s investment rule must imply an expression of the form

$$\tilde{k}_{t+1}(i) = \lambda \tilde{k}_t(i) - \tau \tilde{p}_t(i), \tag{28}$$

where the coefficients $\lambda$ and $\tau$ remain to be determined. This in turn implies that

$$\begin{aligned} E_t \tilde{k}_{t+2}(i) &= \lambda \tilde{k}_{t+1}(i) - \tau E_t \tilde{p}_{t+1}(i) \\ &= [\lambda + (1-\alpha)\tau\psi]\tilde{k}_{t+1}(i) - \alpha\tau\tilde{p}_t(i), \end{aligned}$$

using (27) to substitute for $E_t\tilde{p}_{t+1}(i)$ in the second line. Using this to substitute for $E_t\tilde{k}_{t+2}(i)$ in (24), and again using (27) to substitute for $E_t\tilde{p}_{t+1}(i)$, we obtain a linear relation that can be solved for $\tilde{k}_{t+1}(i)$ as a linear function of $\tilde{k}_t(i)$ and $\tilde{p}_t(i)$. The conjectured solution (28) satisfies this equation, so that the first-order condition (24) is satisfied, if and only if the coefficients $\lambda$ and $\tau$ satisfy

$$R(\lambda; \psi) = 0, \tag{29}$$

$$(1 - \alpha\beta\lambda)\tau = \Xi\alpha\lambda, \tag{30}$$

where

$$R(\lambda; \psi) \equiv (\beta^{-1} - \alpha\lambda)Q(\beta\lambda) + (1-\alpha)\Xi\psi\lambda$$

is a cubic polynomial in $\lambda$, with a coefficient on the linear term that depends on the value of the (as yet unknown) coefficient $\psi$. Condition (29) involves only $\lambda$ (given the value of $\psi$); given a solution for $\lambda$, (30) then yields a unique solution for $\tau$, as long as $\lambda \neq (\alpha\beta)^{-1}$.[13]

The dynamics of the relative capital stock given by (28), together with (27), imply an expected joint evolution of $i$'s relative price and relative capital stock satisfying

[13]It is obvious from (30) that no solution with $\lambda = (\alpha\beta)^{-1}$ is possible, as long as $\Xi > 0$, as we assume here (i.e., there exists some cost of adjusting capital). Even in the case that $\Xi = 0$, such a solution would violate condition (32) below, so one can exclude this possibility.

$$\begin{bmatrix} E_t\tilde{p}_{t+1}(i) \\ \tilde{k}_{t+1}(i) \end{bmatrix} = \begin{bmatrix} \alpha + (1-\alpha)\tau\psi & -(1-\alpha)\psi\lambda \\ -\tau & \lambda \end{bmatrix} \begin{bmatrix} \tilde{p}_t(i) \\ \tilde{k}_t(i) \end{bmatrix}. \tag{31}$$

This implies convergent dynamics—so that both the means and variances of the distribution of possible future values for $i$'s relative price and relative capital stock remain bounded no matter how far in the future one looks, as long as the fluctuations in the average desired relative price $\hat{p}_t^*$ are bounded—if and only if both eigenvalues of the matrix in this equation are inside the unit circle. This stability condition is satisfied if and only if

$$\lambda < \alpha^{-1}, \tag{32}$$

$$\lambda < 1 - \tau\psi, \tag{33}$$

and

$$\lambda > -1 - \frac{1-\alpha}{1+\alpha}\tau\psi. \tag{34}$$

These conditions must be satisfied if the implied dynamics of firm $i$'s capital stock and relative price are to remain forever near enough to the steady-state values around which I have log-linearized the first-order conditions for the solution to the linearized equations to accurately approximate a solution to the exact first-order conditions. Hence the firm's decision problem has a solution that can be characterized using the local methods employed above only if equations (29)–(30) have a solution $(\lambda, \tau)$ satisfying (32)–(34). I show below that a unique solution consistent with these bounds exists, in the case of large enough adjustment costs.

### *2.2 The Optimal Pricing Rule*

I return now to an analysis of the first-order condition for optimal price setting (23). The term that depends on firm $i$'s own intended future behavior is proportional to

$$\sum_{k=0}^{\infty} (\alpha\beta)^k \hat{E}_t^i \tilde{k}_{t+k}(i).$$

It is now possible to write this term as a function of $i$'s relative capital stock at the time of the pricing decision and of the expected evolution of aggregate variables, allowing me to obtain an expression of the form (25) for the optimal pricing rule.

Equation (28) for the dynamics of the relative capital stock implies that

$$\hat{E}_t^i \tilde{k}_{t+k+1}(i) = \lambda \hat{E}_t^i \tilde{k}_{t+k}(i) - \tau[\tilde{p}_t(i) - E_t \sum_{j=1}^{k} \pi_{t+j}]$$

for each $k \geq 0$, using (21) to substitute for $\hat{E}_t^i \tilde{p}_{t+k}(i)$. This can be integrated forward (given that[14] $|\lambda| < (\alpha\beta)^{-1}$), to obtain

$$\sum_{k=0}^{\infty} (\alpha\beta)^k \hat{E}_t^i \tilde{k}_{t+k}(i) = (1 - \alpha\beta\lambda)^{-1} \tilde{k}_t(i) - \tau \frac{\alpha\beta}{(1-\alpha\beta)(1-\alpha\beta\lambda)} \left[ \tilde{p}_t(i) - \sum_{k=1}^{\infty} (\alpha\beta)^k E_t \pi_{t+k} \right]. \quad (35)$$

Substitution of this into (23) then yields

$$\phi \hat{p}_t^*(i) = (1-\alpha\beta) \sum_{k=0}^{\infty} (\alpha\beta)^k E_t \hat{s}_{t+k} + \phi \sum_{k=1}^{\infty} (\alpha\beta)^k E_t \pi_{t+k} - (\omega - \nu) \frac{1-\alpha\beta}{1-\alpha\beta\lambda} \tilde{k}_t(i),$$

where

$$\phi \equiv 1 + \omega\theta - (\omega - \nu)\tau \frac{\alpha\beta}{1 - \alpha\beta\lambda}. \quad (36)$$

The solution to this equation is a pricing rule of the conjectured form (25) if and only if the process $\hat{p}_t^*$ satisfies

$$\phi \hat{p}_t^* = (1-\alpha\beta) \sum_{k=0}^{\infty} (\alpha\beta)^k E_t \hat{s}_{t+k} + \phi \sum_{k=1}^{\infty} (\alpha\beta)^k E_t \pi_{t+k}, \quad (37)$$

where $\hat{s}_t$ is defined by (19), and the coefficient $\psi$ satisfies

$$\phi\psi = (\omega - \nu) \frac{1-\alpha\beta}{1-\alpha\beta\lambda}. \quad (38)$$

[14]Note that (33)–(34) jointly imply that $\lambda > -\alpha^{-1}$. Hence any solution consistent with the stability conditions derived in the previous section must imply convergence of the infinite sum in (35).

Note that this last equation can be solved for $\psi$, given the values of $\lambda$ and $\tau$; however, the equations given earlier to determine $\lambda$ and $\tau$ depend on the value of $\psi$. Hence equations (29), (30), and (38) comprise a system of three equations that jointly determine the coefficients $\lambda$, $\tau$, and $\psi$ of the firm's optimal decision rules.

This system of equations can be reduced to a single equation for $\lambda$ in the following manner. First, note that for any conjectured value of $\lambda \neq 0$, (29) can be solved for $\psi$. This defines a function[15]

$$\psi(\lambda) \equiv -\frac{(1-\alpha\beta\lambda)Q(\beta\lambda)}{(1-\alpha)\beta\Xi\lambda}.$$

Similarly, (30) defines a function[16]

$$\tau(\lambda) \equiv \frac{\alpha\Xi\lambda}{1-\alpha\beta\lambda}. \tag{39}$$

Substituting these functions for $\psi$ and $\tau$ in (38), one obtains an equation in which $\lambda$ is the only unknown variable. Multiplying both sides of this equation by $(1-\alpha)\beta(1-\alpha\beta\lambda)\Xi\lambda$,[17] one obtains the equation

$$V(\lambda) = 0, \tag{40}$$

where $V(\lambda)$ is the quartic polynomial

$$\begin{aligned} V(\lambda) \equiv {} & [(1+\omega\theta)(1-\alpha\beta\lambda)^2 - \alpha^2\beta(\omega-\nu)\Xi\lambda]Q(\beta\lambda) \\ & + \beta(1-\alpha)(1-\alpha\beta)(\omega-\nu)\Xi\lambda. \end{aligned} \tag{41}$$

Finally, one can write the inequalities (32)–(34) as restrictions upon the value of $\lambda$ alone. One observes from the above discussion that the product $\tau(\lambda)\psi(\lambda)$ is well defined for all $\lambda$, and equal to $-(\alpha/1-\alpha)\beta^{-1}Q(\beta\lambda)$. Using this function of $\lambda$ to replace the terms $\tau\psi$ in the previous inequalities, one obtains an equivalent set of three inequalities,

$$\lambda < \alpha^{-1}, \tag{42}$$

[15]The function is not defined if $\lambda = 0$. However, since $Q(0) \neq 0$, it is clear from (29) that $\lambda \neq 0$, for any economy with some adjustment costs (so that $\Xi$ is finite).

[16]The function is not defined if $\lambda = (\alpha\beta)^{-1}$, but that value of $\lambda$ would be inconsistent with (33) and (34) holding jointly, as noted above.

[17]This expression is necessarily nonzero in the case of the kind of solution that we seek, for the reasons noted in the previous two footnotes.

$$\frac{\alpha}{1+\alpha}\beta^{-1}Q(\beta\lambda) - 1 < \lambda < \frac{\alpha}{1-\alpha}\beta^{-1}Q(\beta\lambda) + 1, \qquad (43)$$

that $\lambda$ must satisfy.

I can then summarize my characterization of a firm's optimal pricing and investment behavior as follows.

**Proposition 1.** Suppose that the firm's decision problem has a solution in which, for any small enough initial log relative capital stock and log relative price of the individual firm, and in the case that the exogenous disturbance $q_t$ and the aggregate variables $\hat{Y}_t, \hat{K}_t, \hat{\lambda}_t$, and $\pi_t$ forever satisfy tight enough bounds, both the conditional expectation $\mathrm{E}_t\hat{k}_{t+j}(i)$ and the conditional variance $\mathrm{var}_t\hat{k}_{t+j}(i)$ remain bounded for all $j$, with bounds that can be made as tight as one likes by choosing sufficiently tight bounds on the initial conditions and the evolution of the aggregate variables.[18] Then the firm's optimal decision rules can be approximated by log-linear rules of the form (25) for $\hat{p}_t^*(i)$ in periods when the firm reoptimizes its price and (28) for the investment decision $\tilde{k}_{t+1}(i)$ each period. The coefficient $\lambda$ in (28) is a root of the quartic equation (40), that satisfies the inequalities (42)–(43). The coefficient $\tau$ in (28) is furthermore equal to $\tau(\lambda)$, where the function $\tau(\cdot)$ is defined by (30), and the coefficient $\psi$ in (25) is equal to $\psi(\lambda)$, where the function $\psi(\cdot)$ is defined by (38). Finally, the intercept $\hat{p}_t^*$ in (25) is given by (37), in which expression the process $\{\hat{s}_t\}$ is defined by (19).

This result gives a straightforward algorithm that can be used to solve for the firm's decision rules, in the case that local methods suffice to give an approximate characterization of optimal behavior in the event of small enough disturbances and a small enough initial departure of the individual firm's situation from that of an average firm. The two decision rules (25) and (28), together with the law of motion

$$\tilde{p}_t(i) = \tilde{p}_{t-1}(i) - \pi_t$$

for any period $t$ in which $i$ does not reoptimize its price, then allow a complete solution for the evolution of the firm's relative capital stock and relative price, given its initial relative capital stock

[18] Note that this is the only condition under which local log-linearizations of the kind used above can suffice to approximately characterize the solution to the firm's problem.

and relative price and given the evolution of the aggregate variables $\{\hat{Y}_t, \hat{K}_t, \lambda_t, \pi_t, q_t\}$.

### *2.3 Existence of a Solution*

Proposition 1 does not guarantee the existence of a nonexplosive solution to the firm's decision problem. The following result, however, shows that at least in the case of large enough adjustment costs, there is a solution of the kind characterized in proposition 1.

**Proposition 2.** Let household preferences, the production function, the rate of depreciation of capital, and the frequency of price changes all be fixed, but consider alternative specifications of the investment adjustment-cost function $I(\cdot)$, all of which are twice differentiable, increasing, convex, and satisfy $I(1) = \delta$, $I'(1) = 1$. Then for any adjustment-cost function for which the value of $\epsilon_\psi \equiv I''(1) > 0$ is large enough, the polynomial (40) has a unique real root $\lambda$ satisfying (42)–(43). It follows that the firm decision problem has a solution of the kind described in proposition 1. Furthermore, in this solution $0 < \lambda < 1$, and $\tau$, $\phi$, and $\psi$ are all positive. In the limit as $\Xi \to 0$, $\lambda \to 1$, $\tau \to 0$, $\phi \to 1 + \omega\theta$, and

$$\phi \to \frac{\omega - \nu}{1 + \omega\theta} > 0.$$

This result can be established by considering the way in which the polynomial (40) depends on the value of $\Xi$, which in turn varies inversely with $\epsilon_\psi$. Note that the steady-state allocation associated with zero inflation (or flexible prices) is determined independently of the assumed degree of adjustment costs, and so the values of the parameters $\alpha$, $\beta$, $\delta$, $\nu$, $\omega$, $\theta$, $\rho_y$, and $\rho_k$ are all given, regardless of the variation considered in the value of $\epsilon_\psi$. The coefficient $\Xi$ is then equal to a positive constant divided by $\epsilon_\psi$, so that one may equivalently consider the consequences of varying the value of $\Xi$ while holding fixed the values of the parameters listed above. I am then interested in the roots of $V(\lambda)$ as the value of $\Xi$ approaches zero.

Since the definition (41) involves the polynomial $Q(z)$, it is first necessary to consider how this polynomial depends on the value of $\Xi$. One observes that

$$Q(z) = z^2 - (1 + \beta + c\Xi)z + \beta,$$

where

$$c \equiv \frac{\rho_k}{\rho_y \theta} > 0.$$

One can then write

$$V(\lambda;\Xi) = \bar{V}(\lambda) + V_\Xi(\lambda)\Xi + \frac{1}{2}V_{\Xi\Xi}(\lambda)\Xi^2,$$

where the polynomials

$$\bar{V}(\lambda) \equiv (1+\omega\theta)(1-\alpha\beta\lambda)^2\beta(1-\lambda)(1-\beta\lambda),$$

$$V_\Xi(\lambda) \equiv \beta(1-\alpha\beta\lambda)[1-\alpha(1+\beta)+\alpha\beta\lambda](\omega-\nu)\lambda-(1+\omega\theta)(1-\alpha\beta\lambda)^2 c\lambda,$$

and $V_{\Xi\Xi}(\lambda)$ are each independent of the value of $\Xi$.

When $\Xi = 0$, the roots of $V(\lambda)$ are simply the roots of $\bar{V}(\lambda)$, which are easily seen to be $\lambda_1 = 1$, $\lambda_2 = \beta^{-1}$, and $\lambda_3 = \lambda_4 = (\alpha\beta)^{-1}$. By continuity, any real roots in the case of a small enough positive value of $\Xi$ will also have to be close to one of the roots of $\bar{V}(\lambda)$.

It is easily seen that no such root can satisfy the inequalities (42)–(43), unless it is a root near 1. Because $Q(\beta\lambda_2;0)Q(1;0) = 0$, the right-most term in (43) is equal to 1, so that the second inequality is violated when $\lambda = \lambda_2, \Xi = 0$. By continuity, the second inequality of (43) will also necessarily be violated by any root near $\lambda_2$ in the case of any small enough value of $\Xi$. Similarly, because $Q(\beta\lambda_3;0) = Q(\alpha^{-1};0) = \alpha^{-1}(\alpha^{-1}-\beta)(1-\alpha)$, the right-most term is negative, and the second inequality is again violated, when $\lambda = \lambda_3 = \lambda_4$, $\Xi = 0$. Hence any roots near these will also violate the inequality in the case of any small enough value of $\Xi$. Thus there can be at most one root of (40) that satisfies the inequalities for small positive values of $\Xi$, and it must be near 1.

Because $\bar{V}'(1) < 0$, $V(\lambda)$ will continue to have a real root $\lambda_1(\Xi)$ near 1 for all small enough values of $\Xi$, and the implicit function theorem implies that

$$\frac{d\lambda_1}{d\Xi}(0) = -\frac{V_\Xi(1)}{\bar{V}'(1)}.$$

Since

$$\begin{aligned} V_\Xi(1) &= \beta(1-\alpha\beta)(1-\alpha)(\omega-\nu) - (1+\omega\theta)(1-\alpha\beta)^2 c \\ &< (1-\alpha\beta)^2[(\omega-\nu) - (1+\omega\theta)c] \\ &= (1-\alpha\beta)^2[(\omega\rho_y^{-1}-1)\nu - c] < 0, \end{aligned}$$

using the fact that $\rho_y > \omega$ in the final line and

$$\bar{V}'(1) = -(1+\omega\theta)\beta(1-\beta)(1-\alpha\beta)^2 < 0,$$

it follows that

$$\frac{d\lambda_1}{d\Xi}(0) < 0.$$

Thus there is a real root $0 < \lambda_1 < 1$ for all small enough positive values of $\Xi$. This root necessarily also satisfies (42).

Since $Q(\beta; 0) = 0$, the left-most term of (43) is near –1 for all small enough values of $\Xi$; hence the first inequality of (43) is satisfied by the root $\lambda_1$ as well. However, both sides of the second inequality are equal to 1 when $\Xi = 0$; thus in order to determine whether the inequality holds when $\Xi > 0$, one must determine the sign of the derivative

$$D \equiv \frac{d}{d\Xi}\left[\lambda_1(\Xi) - \frac{\alpha}{1-\alpha}\frac{Q(\beta\lambda_1(\Xi);\Xi)}{\beta}\right]$$

at $\Xi = 0$. Since

$$\frac{d}{d\Xi}Q(\beta\lambda_1(\Xi);\Xi) = -\beta(1-\beta)\frac{d\lambda_1}{d\Xi} - \beta c$$

at $\Xi = 0$, it follows that

$$\begin{aligned} D &= \frac{1-\alpha\beta}{1-\alpha}\frac{d\lambda_1}{d\Xi} + \frac{\alpha}{1-\alpha}c \\ &= \frac{(\omega-\nu) - (1+\omega\theta)c}{(1-\beta)(1+\omega\theta)} \\ &= \frac{[(\omega\rho_y^{-1} - 1)\nu - c]}{(1-\beta)(1+\omega\theta)} < 0. \end{aligned}$$

Thus for all small enough $\Xi > 0$, the second inequality of (43) holds as well, and $\lambda = \lambda_1(\Xi)$ is the solution asserted to exist in the proposition.

It then follows from (39) that associated with this solution is a positive value of $\tau$, and that $\tau \to 0$ as $\Xi \to 0$. It similarly follows from (36) that the associated value of $\phi$ is positive for all small enough values of $\Xi$, and that $\phi \to 1 + \omega\theta$ as $\Xi \to 0$. Finally, it follows from these results and (38) that the associated value of $\psi$ is positive,[19]

[19]Recall that our assumptions require that $\omega > \nu$.

and that it approaches the positive limit stated in the proposition as $\Xi \to 0$. Proposition 2 is thus established.

Proposition 2 guarantees that a solution to the firm's optimization problem that can be characterized using the local methods employed above will exist for at least some economies, namely, those in which adjustment costs are large enough. The proposition also implies that in the limit of large adjustment costs, the optimal price-setting rule approaches the one derived in Woodford (2003, chap. 3) under the assumption of an exogenously given capital stock for each firm. Thus the exogenous-capital model represents a useful approximation to the equilibrium dynamics in a model with endogenous capital accumulation, if adjustment costs are large enough.

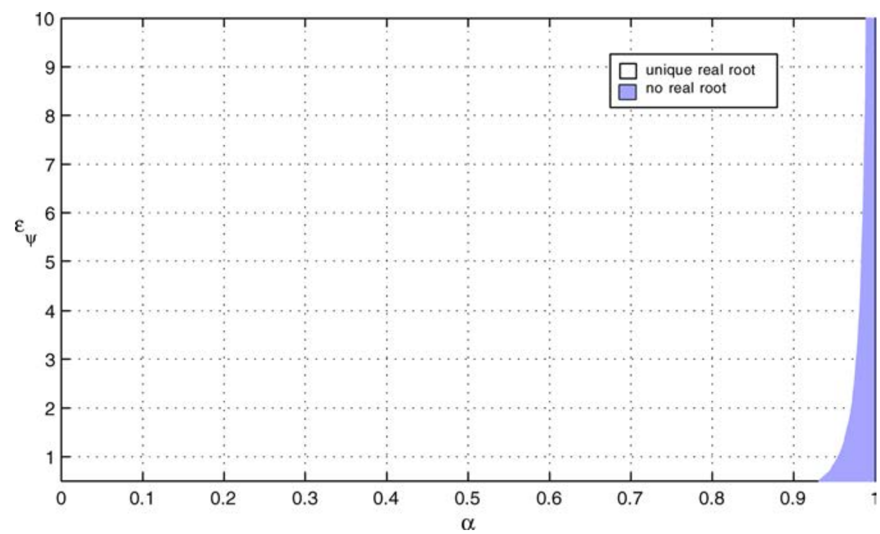
Numerical exploration of the properties of the polynomial (40) suggests that adjustment costs do not have to be large in order for the analysis given above to apply. In figure 1, model parameters are assigned the values given in table 1,[20] while the values of $\alpha$ and $\epsilon_\psi$ are allowed to vary. The figure indicates for which part of the $\alpha-\epsilon_\psi$ plane the polynomial (40) has a unique real root satisfying the bounds (42)–(43). Except in the case of very high values of $\alpha$ ($\alpha > 0.93$, corresponding to an average interval between price changes longer than three and one-half years), a unique real root of this kind exists

**Table 1. Numerical Parameter Values**

| | |
|---|---|
| $\beta$ | 0.99 |
| $\nu$ | 0.11 |
| $\phi_h^{-1}$ | 0.75 |
| $\omega_p$ | 0.33 |
| $(\theta - 1)^{-1}$ | 0.15 |
| $\delta$ | 0.12 |

[20] These are the same parameter values used in the numerical illustrations in Woodford (2004), which are in turn chosen for comparability with the numerical analyses of related models in Woodford (2003). (The justification for interest in these values is discussed in both of those sources.) Thus, for example, in figure 1, one sees that if $\alpha = 0.66$, a unique solution exists for all possible values of $\epsilon_\psi$; this explains why it is possible to present solutions for alternative values of $\epsilon_\psi$ in figure 1 of Woodford (2004). In this calibration of the model, periods are understood to correspond to quarters.

**Figure 1. Values of $\alpha$ and $\epsilon_\psi$ for which a Solution of the Kind Characterized in Proposition 1 Exists**




in the case of any $\epsilon_\psi > 0$. If we suppose that $\epsilon_\psi = 3$ (the calibration used in Woodford 2004), then a solution exists in the case of any $\alpha$ less than 0.978 (i.e., as long as prices are changed at least once every eleven years, on average). In the case of very high values of $\alpha$, a solution does *not* exist, except in the case of very high values of $\epsilon_\psi$,[21] and when it does not, the solution to the firm's problem cannot be characterized using the local methods employed above.[22] But such high values of $\alpha$ are clearly not empirically realistic, so we need not be concerned with this case.

[21]It may appear from the figure that no solution is possible when $\alpha$ exceeds 0.99, but this is because the vertical axis is truncated at $\epsilon_\psi = 10$. If $\alpha = 0.995$, a solution exists in the case of all $\epsilon_\psi > 22.2$; if $\alpha = 0.999$, a solution exists in the case of all $\epsilon_\psi > 88.2$. Thus a solution does always exist in the case of large enough adjustment costs, in accordance with proposition 2.

[22]This may, for example, be due to a failure of the firm's problem to be locally convex. I do not further investigate the problem here, as it does not appear to arise in cases of practical interest.

## 3. Inflation Dynamics

I now consider the implications of the analysis above for the evolution of the overall inflation rate. I show that the model of price setting presented above implies the existence of a New Keynesian Phillips curve of the form (1), and then consider the interpretation of empirical estimates of the slope coefficient $\xi$ in this relation.

### *3.1 A New Keynesian Phillips Curve*

Recall that the average log relative price set by firms that reoptimize at date $t$ is given by (37). This equation can be quasi-differenced (after dividing by $\phi$[23]) to yield

$$\hat{p}_t^* = (1 - \alpha\beta)\phi^{-1}\hat{s}_t + \alpha\beta E_t\pi_{t+1} + \alpha\beta E_t\hat{p}_{t+1}^*.$$

Then, using (26) to substitute for $\hat{p}_t^*$, one obtains a relation of the form (1), where

$$\xi \equiv \frac{(1-\alpha)(1-\alpha\beta)}{\alpha\phi}. \tag{44}$$

Equation (1) is the corrected form of equation (3.17) in Woodford (2003, chap. 5). Together with (19), it provides a complete characterization of the equilibrium dynamics of inflation, given the evolution of $\hat{Y}_t$, $\hat{K}_t$, and $\hat{\lambda}_t$. This pair of equations can be thought of as constituting the "aggregate supply block" of the model with endogenous capital. They generalize the aggregate-supply equation of the constant-capital model (expounded in Woodford 2003, chap. 3) to take account of the effects of changes in the capital stock on real marginal cost, and hence on the short-run trade-off between inflation and output.

In the constant-capital model, (19) (after using [12] to substitute for $\hat{\lambda}_t$) reduces to

$$\hat{s}_t = \omega(\hat{Y}_t - q_t) + \sigma^{-1}(\hat{Y}_t - g_t),$$

which can be equivalently written as

$$\hat{s}_t = (\omega + \sigma^{-1})\tilde{Y}_t, \tag{45}$$

[23]It follows from (38) that $\phi \neq 0$, given that (as already discussed) $\lambda \neq (\alpha\beta)^{-1}$.

where $\tilde{Y}_t$ is the "output gap," defined as the (log) difference between actual and flexible-price equilibrium output. Substituting this relation into (1), one obtains the familiar output-gap formulation of the New Keynesian Phillips curve,

$$\pi_t = \kappa \tilde{Y}_t + \beta E_t \pi_{t+1}, \tag{46}$$

where $\kappa \equiv (\omega + \sigma^{-1})\xi > 0$.

In the model with endogenous (and firm-specific) capital, instead, (45) takes the more general form

$$\hat{s}_t = (\omega + \sigma^{-1})\tilde{Y}_t - \sigma^{-1}\tilde{I}_t, \tag{47}$$

where $\tilde{I}_t$ indicates the gap between actual investment (specifically, the value of $\hat{I}_t$) and its flexible-price equilibrium level.[24] If one substitutes this relation instead into (1), one obtains a generalization of (46),

$$\pi_t = \kappa \tilde{Y}_t - \kappa_I \tilde{I}_t + \beta E_t \pi_{t+1},$$

where $\kappa$ is defined as before, but now $\kappa_I \equiv \sigma^{-1}\xi > 0$. Thus while (1) continues to apply, the relation between inflation and real activity is no longer as simple as (46). This is a further reason (in addition to the lack of simple empirical measures of the flexible-price equilibrium level of output) why it has been appropriate for the empirical literature to focus more on estimation of the inflation equation (1) than of the corresponding aggregate-supply relation.

As with equation (3.17) in Woodford (2003, chap. 5), equation (1) implies that one can solve for the inflation rate as a function of current and expected future real marginal cost, resulting in a relation of the form

$$\pi_t = \sum_{j=0}^{\infty} \Psi_j E_t \hat{s}_{t+j}. \tag{48}$$

The correct formula for these coefficients is given by

$$\Psi_j = \xi \beta^j,$$

just as in the model with constant capital discussed in Woodford (2003, chap. 3). Hence the coefficients do not decay as rapidly with

[24]See Woodford (2004) for further discussion of the definition of this and related "gap" variables in this model.

increasing $j$ as is shown in figure 5.6 of Woodford (2003), in the case of finite adjustment costs. Nor do the coefficients ever change sign with increasing $j$, as occurs in the figure. In the case that $\xi > 0$ (as implied by the calibrated parameter values proposed below), an increase in the expected future level of real marginal costs unambiguously requires that inflation increase, and the degree to which inflation determination is forward looking is even greater than is indicated by the figure in Woodford (2003).

### *3.2 The Case of a Rental Market for Capital*

I now briefly compare the results obtained above to those that would be obtained under the assumption of a competitive rental market for capital services.[25] In the literature, when models of staggered pricing have allowed for endogenous capital accumulation (as, for example, in Yun [1996] or Chari, Kehoe, and McGrattan [2000]), they have typically assumed that firms purchase capital services on a competitive rental market, rather than accumulating firm-specific capital as in the model above. This alternative assumption is of considerable convenience, since it allows price-setting decisions to be analyzed separately from the decision to accumulate capital.[26] However, while the assumption of an economy-wide rental market for capital is purely a convenience in the case of standard real business-cycle models (i.e., one-sector models with a competitive goods market), it is no longer innocuous in a model where firms are price setters, and so must consider the consequences for their profits of setting a price different from that of their competitors. As we shall see, alternative assumptions about the way in which capital services can be obtained (with a production technology that is otherwise the same) lead to different conclusions regarding aggregate dynamics. In particular, the predicted slope of the Phillips-curve trade-off can be affected to an extent that is quantitatively significant.

I shall consider two versions of a model with a competitive rental market for capital services. In each case, the production technology

[25] Sveen and Weinke (2004b) similarly compare the consequences of these two assumptions, but instead focus on the differences that result for the implied impulse responses to disturbances in a complete dynamic stochastic general equilibrium (DSGE) model.

[26] The same assumption was used, for example, in the DSGE model with oligopolistic pricing of Rotemberg and Woodford (1995).

and the technology of capital accumulation are as described in the introductory paragraphs of this paper, except that now capital goods are either accumulated by households and rented to the firms that produce the goods that are used for consumption and investment, or they are accumulated by a special set of firms that accumulate capital and then rent capital services to the goods-producing firms. (Our equilibrium relations will be the same, whether capital is accumulated by households or by a special set of firms.) There is assumed to be a competitive market for capital services each period, with rental rate $\rho_t$ in period $t$. (Note that this rental rate is no longer indexed by the firm that uses the capital.)

It follows that for each household or firm $i$ that accumulates capital, its holdings of capital $\{k_t(i)\}$ must evolve in accordance with the first-order condition (5), except that now the firm-specific shadow value $\rho_{t+1}(i)$ is replaced by the market rental rate $\rho_{t+1}$, with the same value for all $i$. Log-linearization of this condition again leads to a relation of the form (6) for each $i$, but with $\hat{\rho}_{t+1}(i)$ replaced simply by $\hat{\rho}_{t+1}$. Assuming that one starts from a symmetric distribution of capital $k_0(i) = K_0$ for all $i$, one will similarly have a common capital stock $k_t(i) = K_t$ in all subsequent periods, since each household or firm solves an identical optimization problem. The aggregate capital stock will then also evolve in accordance with (5) or, up to a log-linear approximation, in accordance with (6).

An optimal demand for capital services by a goods-producing firm $i$ (not to be confused with a firm $i$ that accumulates capital) again requires that the firm's output-capital ratio satisfy (7), though (7) is now a first-order condition for a firm that takes as given the cost of capital services $\rho_t$, rather than a definition of the shadow value of additional capital services, and $\rho_t(i)$ must now be replaced by the common rental rate $\rho_t$ for all $i$. There are two possible assumptions that may be made regarding labor inputs. In the literature, when a rental market for capital services is assumed, it is often also assumed that all sectors hire the same kind of labor, and that there is a single economy-wide labor market as well; this is the case of "homogeneous factor markets" treated in Woodford (2003, chap. 3).[27] In this case,

[27]It is the case assumed in the derivation of a New Keynesian Phillips curve in Galí and Gertler (1999) and in the baseline case considered in Sbordone (1998, 2002). Note that a single economy-wide labor market is also assumed in the analysis of the consequences of an exogenous firm-specific capital stock in Sbordone

every firm $i$ faces a common wage, so that $w_t(i) = w_t$. It then follows from (7) that each firm $i$ will choose a common output-capital ratio; firms with higher demand for their products (because of lower prices) will choose to use a proportionately higher quantity of capital services and a proportionately higher quantity of labor as well. It then follows from (14) that the marginal cost of output supply will be the same for all firms $i$ and independent of the quantity produced by any firm, so that $s_t(i) = s_t$ for all $i$, where the common real marginal cost $s_t$ is an increasing function of both $\rho_t$ and $w_t$. Equation (18) then reduces simply to

$$\hat{s}_t(i) = \hat{s}_t.$$

In this case, frequently assumed in previous derivations of the New Keynesian Phillips curve, (20) implies that the optimal relative price that should be chosen by a firm that resets its price at date $t$ is given by

$$\hat{p}_t^*(i) = (1-\alpha\beta)\sum_{k=0}^{\infty}(\alpha\beta)^k \hat{E}_t^i\left[\hat{s}_{t+k} + \sum_{j=1}^{k}\pi_{t+j}\right] \qquad (49)$$

instead of (23). In this case, the quantities inside the brackets are not firm specific, and there is no need to distinguish between the conditional expectations $\hat{E}_t^i[\cdot]$ and $E_t[\cdot]$. Nor is there any need to solve for the dynamics of a firm's relative capital stock in order to evaluate the right-hand side of (49). The right-hand side of (49) is the same for all $i$, and thus gives the value of $\hat{p}_t^*$. Equation (49) then leads directly to an inflation equation of the form (1), with

$$\xi_h \equiv \frac{(1-\alpha)(1-\alpha\beta)}{\alpha} > 0. \qquad (50)$$

Alternatively, we may assume the existence of a sector-specific labor market for each sector, as in the model developed in this paper for the case of firm-specific capital or the model of "specific

(1998, 2002) and in Galí, Gertler, and Lopez-Salido (2001). For this reason, the formula for $\xi(\alpha)$ presented by those authors for the case of firm-specific capital differs from the one derived in Woodford (2003, chap. 3) under the assumption of industry-specific labor markets. Eichenbaum and Fisher (2004) also assume an economy-wide labor market even in their model with firm-specific capital, though in their case each firm's capital stock is endogenous as in the model developed here.

factor markets" treated in Woodford (2003, chap. 3). In this case, the real wage for the type of labor hired by firm $i$ is given by a sector-specific labor supply equation (8). Substituting this into (7) and log-linearizing, we again obtain equilibrium relation (9) for each firm $i$, except that $\hat{\rho}_t(i)$ must now be replaced by the common rental rate $\hat{\rho}_t$. Because $\rho_t$ is now the same for all firms, this conditional for cost-minimizing production by firm $i$ implies that the firm's relative capital stock will be a monotonic function of its relative sales, so that

$$\rho_k(\hat{k}_t(i) - \hat{K}_t) = \rho_y(\hat{y}_t(i) - \hat{Y}_t) \tag{51}$$

for all $i$ at any date.

The marginal cost of production of each firm $i$ is again given by (17), but we can now use (51) to substitute for the firm's relative demand for capital as a function of its relative sales. Then, again using (4) to substitute for the relative sales of firm $i$, one obtains

$$\hat{s}_t(i) = \hat{s}_t - \chi\theta\tilde{p}_t(i) \tag{52}$$

instead of (18), where

$$\chi \equiv \frac{\omega\rho_k - (\omega - \nu)\rho_y}{\rho_k} = \frac{\nu\omega_p}{\rho_k(\phi_h - 1)} > 0.$$

Note that there is no longer any dependence on the firm's relative capital stock (which is no longer a state variable for the firm's optimization problem).

Once again substituting (52) for $s_{t+k}(i)$ and (21) for $\hat{E}_t^i\tilde{p}_{t+k}(i)$ in (20), one now obtains

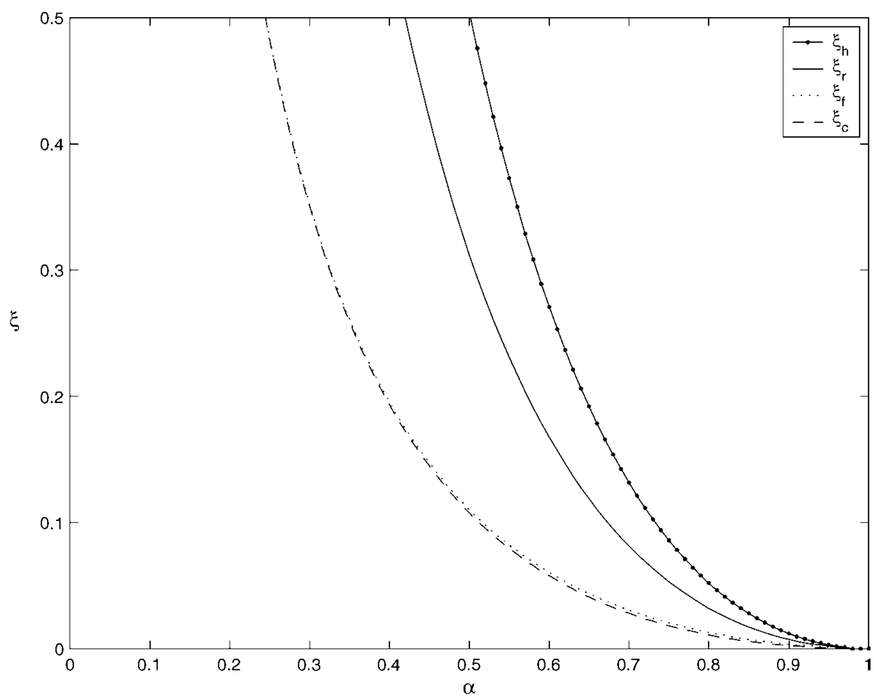
$$(1 + \chi\theta)\hat{p}_t^*(i) = (1 - \alpha\beta)\sum_{k=0}^{\infty}(\alpha\beta)^k\hat{E}_t^i\left[\hat{s}_{t+k} + (1 + \chi\theta)\sum_{j=1}^{k}\pi_{t+j}\right] \tag{53}$$

for the optimal relative price that should be chosen by a firm that resets its price at date $t$. One can again replace the conditional expectation $\hat{E}_t^i[\cdot]$ by $E_t[\cdot]$, and one again observes that $\hat{p}_t^*(i)$ is the same for all $i$, so that one can replace $\hat{p}_t^*(i)$ by $\hat{p}_t^*$. Relation (53) is then of the same form as relation (37) for the model above with endogenous but firm-specific capital, but with the coefficient $\phi$ in the earlier equation here replaced by $1 + \chi\theta$. One again obtains a pricing relation of the form (1), but with elasticity

$$\xi_r \equiv \frac{(1-\alpha)(1-\alpha\beta)}{\alpha(1+\chi\theta)} > 0. \qquad (54)$$

Thus each model leads to a Phillips-curve relation of the same form (1), except that in each case the elasticity $\xi > 0$ is a different function of underlying model parameters. The quantitative difference made by the alternative assumptions can be illustrated through a numerical example. Let us again assume the parameter values given in table 1, and furthermore now specify that $\epsilon_\psi = 3$, as assumed in Woodford (2004). Figure 2 then plots the value of $\xi$ corresponding to any given frequency of price change (indicated by

**Figure 2. The Relation between $\xi$ and $\alpha$ under Four Alternative Assumptions about Factor Markets**



the value of $\alpha$ on the horizontal axis) under each of four possible assumptions. The function $\xi_h(\alpha)$ defined in (50) indicates how the elasticity $\xi$ in (1) varies with $\alpha$ in the case of homogeneous factor markets. The function $\xi_r(\alpha)$ defined in (54) applies instead in the case of industry-specific labor but an economy-wide rental market for capital. The function $\xi_f(\alpha)$ defined in (44) applies instead in the case of industry-specific labor and firm-specific capital.[28] And finally, the function $\xi_c(\alpha)$ is the corresponding relation derived in Woodford (2003, chap. 3) for the case of the model with industry-specific labor and a constant quantity of firm-specific capital.[29] The function $\xi_c(\alpha)$ corresponds to the limit of $\xi_f(\alpha)$ as $\epsilon_\psi$ is made unboundedly large; it follows from proposition 1 that this is given by

$$\xi_c \equiv \frac{(1-\alpha)(1-\alpha\beta)}{\alpha(1+\omega\theta)} > 0.$$

We see from the figure that for any given value of $\alpha$ (in the range for which all four functions are defined), the model with homogeneous factor markets implies the highest value of $\xi$, as in this case the model possesses the fewest sources of "real rigidities" in the sense of Ball and Romer (1990). The fact that an increase in demand in one part of the economy bids up the price of factor inputs throughout the economy creates a source of "strategic substitutability" between the pricing decisions in different sectors of the economy (the fact that others keep their prices low increases your marginal cost of production, and so gives you a reason for *higher* prices, rather than lower ones); this speeds up the rate of adjustment of the aggregate price index to changes in demand conditions.[30] There are greater real rigidities, and hence a flatter Phillips curve, in the case of industry-specific labor markets, even if we continue to assume an economy-wide rental market for capital services; for in this case, an

[28]As shown in figure 1, the function $\xi_f$ is only defined for values of $\alpha$ lower than a critical value on the order of 0.978. The other functions are defined for all values of $\alpha$ between zero and one.

[29]This is called the model with "specific factor markets" in Woodford (2003, chap. 3).

[30]See Woodford (2003, chap. 3) for further discussion of why the Phillips curve is relatively steep in this case, building upon the seminal treatment by Kimball (1995). The discussion there, conducted under the assumption of an exogenously given capital stock, still gives the essential insight into why the specificity of factor markets matters.

increase in demand in one part of the economy still bids up the price of capital services throughout the economy, but does not similarly affect wages in other sectors. There are still greater real rigidities, and a still flatter Phillips curve, if we assume firm-specific investment, because in this case an increase in demand in one part of the economy that increases the shadow value of capital there has no immediate effect on the shadow cost of capital services in other parts of the economy.

Real rigidities are the greatest if we assume, as in the model with "specific factor markets" in Woodford (2003, chap. 3), that the capital stock of each firm is exogenously given, and hence *never* affected by differential shadow values of capital in different sectors. In the model with endogenous firm-specific capital developed here, a sustained higher shadow value of capital in part of the economy will eventually raise the shadow value of capital services everywhere, as a result of differential rates of investment in the sectors with differing shadow values of capital. Thus capital is still reallocated among sectors in response to rate-of-return differentials, albeit with a delay, as long as investment adjustment costs are not too large. However, the figure shows that in our calibrated example, an empirically realistic level of adjustment costs results in a value of $\xi$ that is quite close to what would be implied by the exogenous-capital model with firm-specific capital (though slightly larger), while it is considerably lower than would be implied by the assumption of instantaneous reallocation of capital across sectors so as to equalize the shadow value of capital services. Thus the implicit assumption of an exogenously evolving capital stock in derivations of the Phillips curve for models with firm-specific capital by authors such as Sbordone (1998) appears not to have been a source of any great inaccuracy.[31] The endogeneity of the capital stock is instead of greater significance for

[31] Coenen and Levin (2004) also discuss the role of firm-specific capital in increasing real rigidities, in the context of a model with Taylor-style fixed-period price commitments, which allows separate econometric identification of the length of time between price changes, on the one hand, and the elasticity of a firm's desired relative price with respect to aggregate output, on the other. They are concerned with whether the estimated value of the latter elasticity can be reconciled with the microfoundations of the firm's pricing decision, and argue that allowing for firm-specific capital is important in doing so. Like Sbordone (1998) and Galí, Gertler, and Lopez-Salido (2001), they assume that each firm's capital stock is fixed in analyzing this issue.

predictions about the equilibrium responses of inflation or output to aggregate disturbances, as shown in Woodford (2004), because of the effects of the endogenous rate of investment on the evolution of real marginal cost, as indicated by equation (47).

### *3.3 Additional Sources of Real Rigidities*

Even the model with firm-specific capital developed above still abstracts from a number of possible sources of real rigidities. Here I briefly consider the effects of two generalizations that are discussed in more detail (though in the context of a model with exogenous capital) in Woodford (2003, chap. 3, sec. 1.4).

First, I shall now suppose that each differentiated good is produced using not only labor and capital, but also intermediate inputs produced by other industries. As in Rotemberg and Woodford (1995), I assume a production function of the form

$$y_t(i) = \min\left[\frac{k_t(i) f(A_t h_t(i)/k_t(i))}{1 - s_m}, \ \frac{m_t(i)}{s_m}\right],$$

generalizing (3), where $f(\cdot)$ has the same properties as before, $m_t(i)$ denotes the quantity of materials inputs used by firm $i$ in period $t$, and $0 \leq s_m < 1$ is a parameter of the production technology that can be identified, for purposes of calibration, with the share of materials costs in the value of gross output. The materials inputs are measured in units of the composite good.

The shadow value to firm $i$ of an additional unit of capital is then given by

$$\rho_t(i) = \frac{w_t(i)}{A_t} f^{-1}((1 - s_m) y_t(i)/k_t(i))[\phi((1 - s_m) y_t(i)/k_t(i)) - 1],$$

generalizing (7), where $\phi(\cdot)$ is the same function as before. But because the log deviation of $(1 - s_m) y_t(i)/k_t(i)$ from its steady-state value is equal to the log deviation of $y_t(i)/k_t(i)$ from its steady-state value, the log-linear relation (9) continues to apply, regardless of the size of the materials share.

With intermediate inputs, the real marginal cost of production can be written as

$$s_t(i) = (1 - s_m) s_t^{VA}(i) + s_m, \qquad (55)$$

where $s_t^{VA}(i)$ is the real marginal cost of producing a unit of "real value added," by which I mean the homogeneous-degree-one aggregate of primary factors of production given by $k_t(i)f(A_t h_t(i)/k_t(i))$. Equation (15) furthermore takes the more general form

$$s_t^{VA}(i) = \frac{v_h(f^{-1}((1-s_m)y_t(i)/k_t(i))k_t(i)/A_t;\xi_t)}{\lambda_t A_t f'(f^{-1}((1-s_m)y_t(i)/k_t(i)))}.$$

Substituting this into (55) and log-linearizing, I obtain

$$\hat{s}_t(i) = (1-\mu s_m)[\omega(\hat{y}_t(i) - \hat{k}_t(i) - q_t) + \nu \hat{k}_t(i) - \hat{\lambda}_t], \qquad (56)$$

generalizing (16), where $\mu \equiv \theta/(\theta-1) > 1$ is the steady-state markup (ratio of price to marginal cost).[32] The reduced elasticity of real marginal cost with respect to the firm's level of production when $s_m$ is positive (but less than $\mu^{-1}$) indicates greater real rigidities.

Second, I shall suppose that substitution possibilities among the differentiated goods are no longer necessarily described by the familiar Dixit-Stiglitz aggregator that leads to the constant-elasticity demand function (4) for individual goods. If I instead assume only that the aggregator belongs to the more general family of homogeneous-degree-one functions considered by Kimball (1995), the elasticity of demand varies with the relative price of (and hence the relative demand for) individual good $i$. The relative demand for an individual good is again a decreasing function of the relative price,[33] but the function need not be a constant-elasticity function, as in (4).

As a result, the desired markup of the supplier's price over the marginal cost of supply will no longer be a constant $\mu > 1$, but rather a function $\mu(y_t(i)/Y_t)$ of the relative output of the good, where $Y_t$ is aggregate output, defined using the Kimball aggregator. To a log-linear approximation, the deviation of the log desired markup from its steady-state level (that I shall again call $\mu$) is equal to $\epsilon_\mu \tilde{y}_t(i)$,

[32] In the case of the model with generalized preferences introduced in the next paragraph, this relation still applies, but $\theta > 1$ indicates the steady-state elasticity of substitution among differentiated goods, rather than a coefficient of the Dixit-Stiglitz aggregator.

[33] As in the model developed above, it is assumed that both household preferences and the production technology of firms depend only on the quantity purchased of the composite good defined by this aggregator. Hence the purchases of each buyer, for whatever purpose, will be distributed across differentiated goods in the same proportions.

where $\epsilon_\mu$ is the elasticity of the function $\mu(\cdot)$, evaluated at the steady state (i.e., at a relative output of 1), and $\tilde{y}_t(i)$ is the log relative output. The demand function can again be log-linearized to yield

$$\tilde{y}_t(i) = -\theta \tilde{p}_t(i),$$

where $\theta > 1$ is now the steady-state elasticity of demand (and not necessarily also the elasticity near a relative price other than 1). One can then show that the first-order condition for optimal price setting under Calvo staggering of price changes takes the form

$$\sum_{k=0}^{\infty} (\alpha\beta)^k \hat{E}_t^i [\tilde{p}_{t+k}(i) - (1+\theta\epsilon_\mu)^{-1} \hat{s}_{t+k}(i)] = 0, \qquad (57)$$

generalizing (20), just as in Woodford (2003, chap. 3). The only difference here is that real marginal cost will depend on the firm's endogenous, firm-specific capital stock in the way treated above. One observes directly from (57) that a value $\epsilon_\mu > 0$ will increase the degree of real rigidities, by reducing the sensitivity of the desired relative price to variations in real marginal cost, and hence to variations in the firm's output.

Substituting (56) into (57), I now obtain

$$\Gamma_1 \hat{p}_t^*(i) = (1-\alpha\beta) \sum_{k=0}^{\infty} (\alpha\beta)^k \hat{E}_t^i \left[ \hat{s}_{t+k} + \Gamma_1 \sum_{j=1}^{k} \pi_{t+j} - \Gamma_2 \tilde{k}_{t+k}(i) \right], \qquad (58)$$

generalizing (23), where

$$\Gamma_1 \equiv 1 + \theta\epsilon_\mu + (1-\mu s_m)\omega\theta, \quad \Gamma_2 \equiv (1-\mu s_m)(\omega - \nu).$$

Here only the expressions for $\Gamma_1$ and $\Gamma_2$ have become more complex. One can then show, using the same reasoning as above, that the solution to the firm's optimization problem is characterized by proposition 1, *except* that (36) must be replaced by

$$\phi \equiv \Gamma_1 - \Gamma_2 \tau \frac{\alpha\beta}{1-\alpha\beta\lambda}, \qquad (59)$$

and (41) must be replaced by

$$V(\lambda) \equiv [\Gamma_1(1-\alpha\beta\lambda)^2 - \alpha^2\beta\Gamma_2\Xi\lambda] Q(\beta\lambda) + \beta(1-\alpha)(1-\alpha\beta)\Gamma_2\Xi\lambda. \qquad (60)$$

One can similarly obtain once again an aggregate-supply relation of the form (1), where the elasticity $\xi$ is defined by (44), but now using the generalized definition (59) of $\phi$. Alternatively, one can write the aggregate-supply relation as

$$\pi_t = \xi \hat{s}_t^{VA} + \beta E_t \pi_{t+1}, \tag{61}$$

in which case

$$\xi \equiv \frac{(1 - \mu s_m)(1 - \alpha)(1 - \alpha\beta)}{\alpha\phi}, \tag{62}$$

where $\phi$ is defined by (59), using the fact that

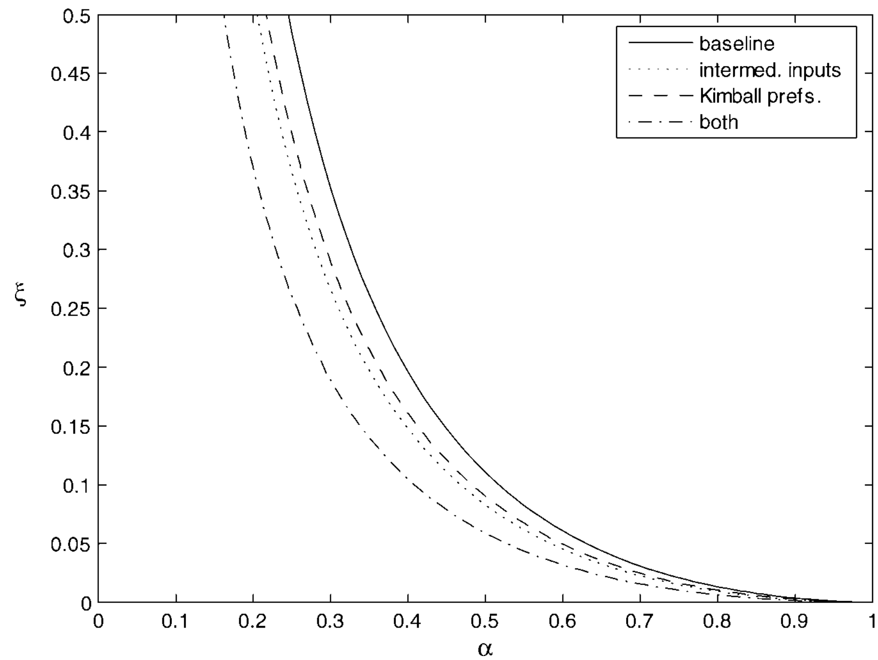
$$\hat{s}_t = (1 - \mu s_m)\hat{s}_t^{VA},$$

from a log-linearization of (55). The alternative form (61) is actually the one that is estimated in the literature, since (under the assumption of a Cobb-Douglas production function for "value added") it is $s_t^{VA}$ rather than $s_t$ that is proportional to real unit labor cost (the proxy for "marginal cost" that is used in empirical work).

The additional sources of real rigidities affect the value of $\xi$ associated with a given average frequency of price adjustment, as shown in figure 3. As in Woodford (2003, chap. 3, sec. 1.4), I shall consider the consequences of an intermediate input share such that $\mu s_m = 0.6$, and a nonconstant elasticity of substitution among differentiated goods such that $\theta\epsilon_\mu = 1$. The figure plots the functional relation $\xi(\alpha)$ defined by (44) for each of four possible combinations of parameter values: the baseline case, in which $s_m = 0, \epsilon_\mu = 0$; a case with intermediate inputs, in which $\mu s_m = 0.6$, though again $\epsilon_\mu = 0$; a case with Kimball preferences, in which $s_m = 0$ but $\theta\epsilon_\mu = 1$; and finally, a case with both additional sources of real rigidities, in which $\mu s_m = 0.6, \theta\epsilon_\mu = 1$. In all four cases, it is assumed that labor markets are industry specific, capital is endogenous and firm specific, and the numerical parameters other than those just listed are as in the case with firm-specific capital plotted in figure 2. (The function $\xi(\alpha)$ in the baseline case here corresponds to the function $\xi_f(\alpha)$ in figure 2.)

One observes that for each of the values of $\alpha$ considered, either intermediate inputs or Kimball preferences with $\epsilon_\mu > 0$ lower the implied value of $\xi$, and if both departures from the baseline model are considered simultaneously, the implied value of $\xi$ is still lower. Hence allowance for either of these empirically plausible additional

**Figure 3. The Relation between $\xi$ and $\alpha$ with Additional Sources of Real Rigidities**




sources of real rigidities further reduces the implied slope of the Phillips curve, without any change in the assumed frequency of price changes. The results obtained here are quite similar to those obtained in Woodford (2003, chap. 3) for the case of a model in which each firm's capital stock is given exogenously (see table 3.1 there).

### *3.4 Consequences for Estimates of the Frequency of Price Adjustment*

Because alternative assumptions about the specificity of factor markets affect the location of the curve $\xi(\alpha)$, as shown in figure 2, it follows that the consequences of an estimate of $\xi$ for the frequency of price adjustment are correspondingly different in the different cases. (One should note that estimation of the aggregate-supply relation [1] only allows an estimate of the elasticity $\xi$ and provides no direct evidence regarding the frequency of price adjustment, nor any way of testing which of the alternative possible assumptions about the specificity of factor markets is the correct one.) An assumption of specific factor markets—either that labor markets are industry specific, or that capital is firm specific—increases the degree of real rigidities, relative to an assumption of an economy-wide market for the services of that factor, and so lowers the value of $\xi$ corresponding to any given value of $\alpha$. Conversely, it follows that the value of $\alpha$ required to explain any given value of $\xi$—and hence the value of $\alpha$ implied by any given estimate of $\xi$—is *lower* the greater the degree of specificity of factors. Hence a given degree of sluggishness in the adjustment of the overall price index to changes in aggregate conditions can be reconciled with a greater degree of firm-level flexibility of prices in the case that one assumes more specific factors of production.

This is illustrated by the calculations reported in table 2. The numerical parameter values given in table 1 are again assumed, and in addition (in the case of the model with firm-specific capital) it is assumed that $\epsilon_\psi = 3$, as in the baseline case considered in Woodford

**Table 2. Interpretation of the Estimated Value of $\xi$ under Alternative Assumptions about Factor Markets**

| Implied Values of $\alpha$ | | | |
|---|---|---|---|
| $\xi$ | Homogeneous Factor | Rental Market | Firm Specific |
| 0.05 | .804 | .757 | .630 |
| 0.04 | .823 | .779 | .663 |
| 0.03 | .845 | .806 | .703 |
| 0.02 | .872 | .840 | .754 |

| Implied Values of $T$ | | | |
|---|---|---|---|
| $\xi$ | Homogeneous Factor | Rental Market | Firm Specific |
| 0.05 | 4.57 | 3.59 | 2.16 |
| 0.04 | 5.13 | 4.01 | 2.43 |
| 0.03 | 5.94 | 4.65 | 2.84 |
| 0.02 | 7.32 | 5.71 | 3.55 |

(2004). The first panel of the table then indicates the value of $\alpha$ that would be implied by a given estimate of the elasticity $\xi$, under each of three different possible assumptions about factor markets: homogeneous factor markets; industry-specific labor markets but a rental market for capital services; and industry-specific labor markets together with endogenous, firm-specific capital. The range of values considered for $\xi$ in the table corresponds to the range of values found in empirical estimates of the New Keynesian Phillips curve (1) for the United States.[34] The second panel of the table shows the implied values of

$$T \equiv \frac{-1}{\log \alpha},$$

the average time (in quarters) that a price remains fixed,[35] for each of the same possible estimates of $\xi$ under each of the same three possible assumptions about factor markets.

One observes that the assumption made regarding factor markets has a substantial effect on the implied frequency of price adjustment, given any estimate of the slope of the Phillips curve $\xi$. If, for example, one estimates a slope $\xi = 0.02$—and some estimates using U.S.

[34]For example, Galí and Gertler (1999) report an estimate of 0.023 when they estimate the "reduced form" equation (1) using U.S. data. When they use an alternative generalized method of moments (GMM) estimation approach that yields estimates of $\alpha$ (under the assumption of homogeneous factor markets) rather than of $\xi$, the values of $\xi$ implied by the reported estimates (that vary depending on the sample and moment conditions used) are mostly in the range of 0.02 to 0.04. (It should be noted that when Galí and Gertler report estimates of $\alpha$, they are really only estimating a nonlinear transformation of the elasticity $\xi$ that would correspond to $\alpha$ under the assumption of homogeneous factor markets, and do not attempt any test of the homogeneous-factor assumption.) Galí, Gertler, and Lopez-Salido (2001) similarly report estimates of $\alpha$ using U.S. data that imply values of $\xi$ equal to 0.03 or 0.04. Sbordone (2002) obtains an estimate of 0.055 for U.S. data using a different estimation technique, while Sbordone (2004) obtains an estimate of 0.025 for U.S. data using yet another approach.

[35]In the literature, estimates of $\alpha$ are often converted into estimates of the average time between price changes using the alternative formula $T = 1/(1-\alpha)$. This latter formula is correct if one takes the discrete-time model (in which all prices change, if they change at all, at a single time each quarter) literally; but it has the unappealing feature that no matter how flexible prices may be (and how steep the estimated Phillips curve may be as a result), $T$ must always equal at least three months. The formula here assumes instead that there is a constant hazard rate $\rho$ in continuous time for price changes, and that an estimate of $\alpha$ is an estimate of $e^{-\rho}$. This means that if one estimates a steep enough Phillips curve, and hence infers a value of $\alpha$ close enough to 1, the inferred value of $T$ may be arbitrarily small.

data are this low, though most reported estimates have been at least somewhat larger—then under the assumption of homogeneous factor markets (and the other parametric assumptions in table 1), one would conclude that the estimate implied an average time between price changes of over seven quarters. This is implausibly long, given microeconomic evidence on the frequency of price changes, so that one might well conclude that the model cannot account for the observed facts about price adjustment, no matter how well it might fit the joint evolution of overall inflation and average marginal cost. Assuming instead that labor markets are industry specific, however, would reduce the implied average time between price changes to less than six quarters, even if one continues to assume a rental market for capital services. And allowing for firm-specific capital would further reduce the implied average time between price changes, to only three and one-half quarters. This is no longer so implausible, given the evidence in surveys such as that of Blinder et al. (1998) that many prices in the United States are changed only once a year or less.

My finding that an assumption that capital is firm specific reduces the average time between price changes implied by estimates of the aggregate-supply relation (1) confirms the previous results of Sbordone (1998) and Galí, Gertler, and Lopez-Salido (2001), obtained under an implicit assumption that each firm's capital stock is constant, or evolves exogenously. Figure 2 shows that the assumption of a constant (or exogenous) capital stock would imply even slightly greater real rigidities than exist in the case of an endogenous but firm-specific capital stock; but the numerical error resulting from that simplifying assumption is not great, at least if investment adjustment costs are of the size assumed here.[36] For example, in the case that $\xi = 0.02$, under the assumptions of exogenous capital and industry-specific labor markets, the value of $\alpha$ would be 0.740 and the value of $T$ would be 3.28, rather than the values shown in the third column of table 2. Allowance for a realistic degree of endogenous adjustment of each firm's capital stock does not dramatically change those conclusions.

[36] As shown in Woodford (2004), adjustment costs of roughly this size are needed to explain the observed size of output response to an identified monetary policy shock, in the context of a simple New Keynesian model that incorporates the model of investment and price-setting decisions developed here.

The implied average times between price adjustments shown in table 2 may still seem a bit too long to square with microeconomic evidence, even in the case of firm-specific capital, especially if $\xi$ is estimated to take a value between 0.2 and 0.3. (Blinder et al. [1998] report a median time between price changes of three quarters, but Bils and Klenow [2004] instead report a median of less than two quarters.) However, a value of $\xi$ of this magnitude can be reconciled with even greater frequencies of adjustment of individual prices, if additional empirically plausible sources of real rigidities are taken into account.

Table 3 shows the values of $\alpha$ and $T$ implied by alternative estimates of $\xi$, under alternative assumptions about the importance of intermediate inputs and the degree to which the aggregator that defines the composite good differs from the Dixit-Stiglitz form. All numerical parameters except $s_m$ and $\epsilon_\mu$ take the same values as in table 2, and in each case it is now assumed that labor markets are

**Table 3. Interpretation of the Estimated Value of $\xi$ under Alternative Assumptions about Input/Output Structure and Substitutability of Differentiated Goods**

| Implied Values of $\alpha$ | | | | |
|---|---|---|---|---|
| $\xi$ | Baseline | Intermediate Inputs | Kimball | Both |
| 0.05 | .630 | .584 | .598 | .528 |
| 0.04 | .663 | .619 | .633 | .564 |
| 0.03 | .703 | .662 | .674 | .609 |
| 0.02 | .754 | .716 | .728 | .669 |

| Implied Values of $T$ | | | | |
|---|---|---|---|---|
| $\xi$ | Baseline | Intermediate Inputs | Kimball | Both |
| 0.05 | 2.16 | 1.86 | 1.95 | 1.56 |
| 0.04 | 2.43 | 2.09 | 2.18 | 1.75 |
| 0.03 | 2.84 | 2.42 | 2.54 | 2.02 |
| 0.02 | 3.55 | 3.00 | 3.15 | 2.48 |

industry specific and capital is firm specific. (Thus the “baseline” case in table 3 corresponds to the “firm-specific” column of table 2.) The values assumed for $s_m$ and $\epsilon_\mu$ in the alternative cases are the same as in figure 3.

One observes the average time between price changes implied by any given estimate of $\xi$ falls in the case that one assumes either of the additional sources of real rigidities, and falls by even more if one assumes both. Making corrections of both type that remain within the range of empirically plausible parameter values, one finds that a Phillips-curve slope of only 0.02 can be consistent with an average period between price changes that is less than two and one-half quarters. A Phillips-curve slope of 0.04 can instead be consistent with an average period between price changes that is well below two quarters. Since point estimates of this magnitude for $\xi$ are obtained in a number of studies (and it is within the 95 percent confidence interval in an even larger number of cases), one cannot say that estimates of $\xi$ are too small to be consistent with microeconomic evidence regarding the frequency with which prices change.

I conclude that there is no necessary conflict between the parameter values that are required to explain the co-movement between overall inflation and aggregate output, as indicated by Phillips curves estimated using aggregate time series, on the one hand, and the parameter values required for consistency with microeconomic observations, on the other. The appearance of a “micro/macro conflict” results from simplifying assumptions in familiar derivations of the New Keynesian Phillips curve that are not actually necessary in order to obtain a relation between aggregate time series of that form, and that are not realistic, either. When one adopts more realistic (or at the very least, no less realistic) assumptions—industry-specific labor markets, firm-specific capital, intermediate inputs required for production, and a nonconstant elasticity of substitution among differentiated goods for both consumption and investment purposes—the discrepancy between the frequency of price adjustment that is required to explain the aggregate co-movements and the one that is indicated by microeconomic data disappears.

A similar conclusion is reached by Eichenbaum and Fisher (2004), Altig et al. (2005), and Matheron (2005) in the context of econometric models that allow for endogenous, firm-specific capital, following the analysis presented above. While the first two papers place particular stress on the role of firm-specific capital in reconciling the microeconomic and macroeconomic evidence, the assumption of an aggregator of the Kimball form that departs substantially from the Dixit-Stiglitz case is also important for the quantitative results of Eichenbaum and Fisher. Matheron stresses the importance of allowing for industry-specific labor as well as firm-specific capital. In the case of his analysis with euro-area data, a specification with firm-specific capital but homogeneous labor, as assumed by the other authors, reduces the estimated time between price revisions relative to the specification with economy-wide markets for both factors, but not by nearly enough to reconcile the model with microeconomic evidence on the frequency of price changes; allowing for both firm-specific capital and industry-specific labor, as proposed here, results in a substantial further reduction in the estimated time between price revisions.

I have given particular attention to the importance of allowing for firm-specific capital because, in the case that one allows for endogenous capital accumulation, the assumption that capital is firm specific results in a nontrivial complication in the analysis. It turns out, however, that the same form of equilibrium relation between inflation dynamics and the evolution of average real marginal cost can be derived under this assumption. Moreover, the relation between the slope of the Phillips curve and the frequency of price adjustment that can be derived under the simpler assumption of an exogenously given capital stock for each firm turns out to be fairly accurate as an approximation to the correct relation in the case of an empirically realistic size of adjustment costs for investment. Hence the conclusions of the earlier literature (beginning with Sbordone 1998) that drew inferences about the frequency of price adjustment from estimated Phillips curves under the implicit assumption of an exogenous capital stock are found to have been essentially correct, even if a more precise inference can be made using the analysis given here.

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# Liquidity, Risk Taking, and the Lender of Last Resort*

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This paper studies the strategic interaction between a bank whose deposits are randomly withdrawn and a lender of last resort (LLR) that bases its decision on supervisory information on the quality of the bank's assets. The bank is subject to a capital requirement and chooses the liquidity buffer that it wants to hold and the risk of its loan portfolio. The equilibrium choice of risk is shown to be decreasing in the capital requirement and increasing in the interest rate charged by the LLR. Moreover, when the LLR does not charge penalty rates, the bank chooses the same level of risk and a smaller liquidity buffer than in the absence of an LLR. Thus, in contrast with the general view, the existence of an LLR does not increase the incentives to take risk, while penalty rates do.

JEL Codes: E58, G21, G28.

From their inception, central banks have assumed as one of their key responsibilities the provision of liquidity to banks unable to find it elsewhere. The classical doctrine on the lender of last resort (LLR) was put forward by Bagehot (1873, 96–7): "Nothing, therefore, can be more certain than that the Bank of England . . . must in time of panic do what all other similar banks must do. . . . And for this purpose there are two rules: First. That these loans should only be made at a very high rate of interest. . . . Secondly. That at this rate these advances should be made on all good banking securities, and as largely as the public ask for them." The contemporary literature on the LLR has disagreed on whether the aim of "staying

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the panic" may be achieved by open market operations (see, for example, Goodfriend and King [1988] or Kaufman [1991]) or whether it requires lending to individual banks (see, for example, Flannery [1996] or Goodhart [1999]).[1] However, both sides seem to agree on the proposition that such lending creates a moral hazard problem. As argued by Solow (1982, 242): "The existence of a credible LLR must reduce the private cost of risk taking. It can hardly be doubted that, in consequence, more risk will be taken."

The purpose of this paper is to show that this proposition is not generally true. Specifically, we model the strategic interaction between a bank and an LLR. The bank is funded with insured deposits and equity capital, is subject to a minimum capital requirement, and can invest in two assets: a safe and perfectly liquid asset, and a risky and illiquid asset, whose risk is privately chosen by the bank. Deposits are randomly withdrawn. If the withdrawal is larger than the funds invested in the safe asset (the liquidity buffer), the bank will be forced into liquidation unless it can secure emergency lending from the LLR. In this setting, we show that when the LLR does not charge penalty rates, the bank chooses the same level of risk and a smaller liquidity buffer than in the absence of an LLR. Moreover, the equilibrium choice of risk is increasing in the penalty rate.

To explain the basic intuition for these results, consider a setup in which a risk-neutral bank raises a unit of insured deposits at an interest rate that is normalized to zero, and invests all these funds in an illiquid asset that yields a gross return $R_1 = R(p)$ with probability $p$, and $R_0 = 0$ otherwise. Moreover, suppose that $p$ is chosen by the bank at the time of investment, and that the success return $R(p)$ is decreasing in $p$, so safer investments yield a lower success return.

Without deposit withdrawals, and hence without the need for an LLR, with probability $p$ the bank gets the return $R(p)$ of its investment in the risky asset minus the amount due to the depositors, that is, $R(p) - 1$, and with probability $1 - p$ the bank fails. Under limited liability, the bank then maximizes $p[R(p) - 1]$, which gives $p^* = \arg\max p[R(p) - 1]$.

Suppose now that a certain fraction of the deposits are withdrawn, and that there is an LLR that only provides the required

[1] All these references (and more) are usefully collected in Goodhart and Illing (2002).

funding if its supervisory information on the quality of the bank's asset is good. Specifically, let $s_1$ denote the good supervisory signal, and let $q = \Pr(s_1 \mid R_1)$ denote the quality of the supervisory information. If the LLR only charges the zero deposit rate, the bank will get $R(p) - 1$ with probability $\Pr(s_1, R_1) = \Pr(s_1 \mid R_1) \Pr(R_1) = qp$. Since the constant $q$ factors out of the maximization problem, we get the same $p^* = \arg\max p[R(p) - 1]$. Hence we conclude that the introduction of deposit withdrawals and an LLR does not affect the bank's incentives to take risk.

As for the result on penalty rates, the intuition is that they increase the expected interest payments in the high-return state and, consequently, push the bank toward choosing higher risk and higher return strategies (i.e., a lower $p$). This positive relationship between the bank's (expected) funding costs and its portfolio risk is not new, since it is a simple implication of the analysis in the classical paper on credit rationing of Stiglitz and Weiss (1981). In particular, they show (p. 393) how "higher interest rates induce firms to undertake projects with lower probabilities of success but higher payoffs when successful." Applying the same argument to banks instead of firms gives the key result.

To endogenize the decision of the LLR, we adopt a political economy perspective according to which government agencies have objectives that need not correspond with the maximization of social welfare. In particular, following Repullo (2000) and Kahn and Santos (2001), we assume that the LLR cares about (1) the revenues and costs associated with its lending activity and (2) whether the bank fails. This may be justified by relating the payoff of the officials in charge of LLR decisions with the surpluses or deficits of the agency, as well as with the possible reputation costs associated with a bank failure.

Specifying an objective function for the LLR would not be needed if the supervisory information were verifiable, because then the intervention rule could be specified ex-ante, possibly in order to implement a socially optimal decision. However, the information coming from bank examinations is likely to contain many subjective elements that are difficult to describe ex-ante, so it seems reasonable to assume that it is nonverifiable. In this case, the decision will have to be delegated to the LLR, which will simply compare its conditional expected payoff of supporting and not supporting the bank.

To facilitate the presentation, the analysis starts with a basic model in which the bank is fully funded with deposits and can only invest in the risky asset. Then the model is extended to the case where the bank can invest in a safe and perfectly liquid asset and can raise equity capital. In the general model, we also assume that the bank is subject to a minimum capital requirement, and that (in line with Basel bank capital regulation) investment in the safe asset does not carry a capital charge.

We characterize the Nash equilibrium of the game between the bank and the LLR, where the former chooses the level of risk (and, in the general model, its capital and liquidity buffer) and the latter its contingent lending policy. The LLR's equilibrium strategy is straightforward: it will support the bank if and only if the liquidity shortfall is smaller than or equal to a critical value that is decreasing in the ex-post (i.e., conditional on the supervisory signal) probability of bank failure. The bank's equilibrium strategy is, however, more difficult to characterize. The reason is that its objective function is likely to be convex in the capital decision, which leads to a corner solution where the bank's capital is equal to the minimum required by regulation. In this case, the equilibrium level of risk only depends on the capital requirement, with higher capital increasing the bank shareholders' losses in case of default and reducing their incentives for risk taking. We complete the analysis by deriving numerically, for a simple parameterization of the model, the bank's equilibrium liquidity. We show that in equilibrium, the bank chooses the same level of risk of its illiquid portfolio and a lower liquidity buffer than in the absence of an LLR.

Four extensions are then discussed. First, we derive the result that penalty rates increase the equilibrium choice of risk. Next, we examine the second of Bagehot's rules—namely, that last-resort lending be collateralized—and show that this protection translates into a lower liquidity buffer and therefore a higher probability that the bank will require emergency liquidity assistance, but without any effect on risk taking. Third, we consider the effects of introducing a higher discount rate for the LLR, which yields a forbearance result: in equilibrium, the bank is more likely to receive support from the LLR, and hence it will hold a lower liquidity buffer. Finally, we look at the case where the LLR shares a fraction of the deposit insurance payouts (which includes, in the limit, the case where the LLR is the deposit insurer) and show that in this case the LLR's decision becomes more sensitive to the supervisory information.

It is important to stress that the key result on the zero effect on risk taking of having an LLR crucially depends on the specification of the order of moves, in particular the fact that the bank cannot modify the level of risk after receiving the support of the LLR (or cannot borrow from the LLR to undertake new investments). But in such a situation, the LLR is likely to carefully monitor the bank, preventing it from engaging in any significant risk shifting, so this seems a reasonable assumption.

Although the literature on the LLR is extensive (see Freixas et al. [2000] for a recent survey), somewhat surprisingly there has been little formal modeling of the issues discussed in this paper. Most of the relevant papers invoke general results on the link between any form of insurance and moral hazard. Moreover, liquidity support is not always distinguished from capital support, which clearly has bad incentive effects whenever it translates into rescuing the shareholders of a distressed bank. This paper restricts attention to liquidity support based on supervisory information on the quality of the bank's assets, and shows that under fairly general conditions this support does not encourage risk taking.

The paper is organized as follows. Section 1 presents the basic model of the game between the bank and the LLR. Section 2 introduces equity capital and a minimum capital requirement, and allows the bank to invest in a safe asset, characterizing the equilibrium with and without an LLR and discussing its comparative statics properties. Section 3 analyzes the effects of Bagehot's rules of charging penalty rates and requiring collateral, as well as changing the objective function of the LLR to allow for higher discounting of future payouts and sharing deposit insurance payouts. Section 4 offers some concluding remarks.

## 1. The Basic Model

Consider an economy with three dates ($t = 0, 1, 2$) and two risk-neutral agents: a *bank* and a *lender of last resort* (LLR). At date 0 the bank raises one unit of deposits at an interest rate that is normalized to zero, and invests these funds in an asset that yields a random *return* $R$ at date 2. The probability distribution of $R$ is described by

$$R = \begin{cases} R_0, & \text{with probability } 1-p, \\ \\ R_1, & \text{with probability } p, \end{cases} \tag{1}$$

where $p \in [0,1]$ is a parameter chosen by the bank at date 0. We assume that $R_0 < 1 < R_1$, so $1-p$ measures the riskiness of the bank's portfolio. The risky asset is illiquid in that there is no secondary market where it can be traded at date 1. However, the asset can be fully liquidated at this date, which yields a *liquidation value* $L \in (0,1)$.[2] Deposits are fully insured and can be withdrawn at either date 1 or date 2. To simplify the presentation, deposit insurance premia are set equal to zero.

At date 1 a fraction $v \in [0,1]$ of the deposits are withdrawn. Since the bank's asset is illiquid, if $v > 0$ the bank faces a liquidity problem that can only be solved by borrowing from the LLR. If such funding is not provided, the bank is liquidated at date 1. Otherwise, the bank stays open until the final date 2. The liquidity shock $v$ is observable, and we initially suppose that the LLR only charges the deposit rate, which has been normalized to zero.

In order to decide whether to provide this emergency funding, the LLR supervises the bank, which yields a *signal* $s \in \{s_0, s_1\}$ on the return of the bank's risky asset. Signal $s$ is assumed to be nonverifiable, so the LLR's decision rule cannot be designed ex-ante, but will be chosen ex-post by the LLR in order to maximize an objective function that will be specified below.

We introduce the following assumptions.

**Assumption 1.** $R_0 = 0$ and $R_1 = R(p)$, where $R(p)$ is decreasing and concave, with $R(1) \geq 1$ and $R(1) + R'(1) < 0$.

**Assumption 2.** $\Pr(s_0 \mid R_0) = \Pr(s_1 \mid R_1) = q \in [\frac{1}{2}, 1]$.

Assumption 1, together with (1), implies that the expected final return of the risky asset, $E(R) = pR(p)$, reaches a maximum at $\widehat{p} \in (0,1)$, which is characterized by the first-order condition

$$R(\widehat{p}) + \widehat{p}R'(\widehat{p}) = 0. \tag{2}$$

[2]The liquidation value $L$ could be correlated with the final return $R$, but this would not change the results.

To see this, notice that the first derivative of $pR(p)$ with respect to $p$ equals $R(0) > 0$ for $p = 0$ and $R(1) + R'(1) < 0$ for $p = 1$, and the second derivative satisfies $2R'(p) + pR''(p) < 0$. Thus, increases in $p$ below (above) $\widehat{p}$ increase (decrease) the expected final return of the risky asset. Moreover, we have $\widehat{p}R(\widehat{p}) > R(1) \geq 1$. Assumption 1 is borrowed from Allen and Gale (2000, chap. 8) and allows us to analyze in a continuous manner the risk-shifting effects of different institutional settings.[3]

Assumption 2 introduces a parameter $q$ that describes the quality of the supervisory information.[4] This information is only about whether the final return of the risky asset will be low ($R_0$) or high ($R_1$), and not about the particular value $R(p)$ taken by the high return. By Bayes' law, it is immediate to show that

$$\Pr(R_1 \mid s_0) = \frac{p(1-q)}{p(1-q) + (1-p)q}, \tag{3}$$

and

$$\Pr(R_1 \mid s_1) = \frac{pq}{pq + (1-p)(1-q)}. \tag{4}$$

Hence when $q = \frac{1}{2}$ we have $\Pr(R_1 \mid s_0) = \Pr(R_1 \mid s_1) = p$, so the supervisory signal is uninformative, while when $q = 1$ we have $\Pr(R_1 \mid s_0) = 0$ and $\Pr(R_1 \mid s_1) = 1$, so the signal completely reveals whether the final return will be low or high. Since $\Pr(R_1 \mid s_0) < p < \Pr(R_1 \mid s_1)$ for $p < 1$ and $q > \frac{1}{2}$, $s_0$ and $s_1$ will be called the bad and the good signal, respectively.

From the point of view of the initial date 0, the deposit withdrawal $v$ is a continuous random variable with support $[0, 1]$ and cumulative distribution function $F(v)$.[5] Since deposits are fully insured, it is natural to assume that the withdrawal $v$ is independent of the final return $R$. Also, $v$ is assumed to be independent of the supervisory signal $s$.

The bank and the LLR play a sequential game in which the bank chooses at date 0 the riskiness of its portfolio $p$, and if $v > 0$, the LLR

[3]This assumption has also been used by Cordella and Levy-Yeyati (2003) and Repullo (2005).

[4]More generally, we could have $\Pr(s_0 \mid R_0) \neq \Pr(s_1 \mid R_1)$, but this would not change the results.

[5]The distribution function $F(v)$ could have a mass point at $v = 0$, in which case $F(0) > 0$ would be the probability that the bank does not suffer a liquidity shock at date 1.

decides at date 1 whether to support the bank based on two pieces of information: the size of the liquidity shock $v$, and the supervisory signal $s$. Importantly, the LLR does not observe the bank's choice of $p$, so we have a game of complete but imperfect information.

In this game, the LLR is assumed to care about the expected value of its final wealth net of a share $\alpha$ of the *social cost* $c$ incurred in the event of a bank failure. Such cost comprises the administrative costs of closing the bank and paying back depositors and the negative externalities associated with the failure (contagion to other banks, breakup of lending relationships, distortions in the monetary transmission mechanism, etc.). As noted above, the LLR's objective function may be justified by relating the payoff of the officials in charge of its decisions with the income generated or lost through its lending activity and the social cost associated with a bank failure. To simplify the presentation, we assume that $\alpha = 1$, so the LLR fully internalizes the social cost of bank failure.[6]

Consider a situation in which $v > 0$, and let $s$ be the signal observed by the LLR. The payoff of the LLR if it supports the bank is computed as follows. With probability $\Pr(R_1 \mid s)$ the bank will be solvent at date 2 and the LLR will recover its loan $v$, while with probability $\Pr(R_0 \mid s)$ the bank will fail and the LLR will lose $v$ and incur the cost $c$, so the LLR's expected payoff is $-(v+c)\Pr(R_0 \mid s)$. On the other hand, if the LLR does not provide the liquidity support, the bank will be liquidated at date 1, and the LLR's payoff will be $-c$. Hence the LLR will support the bank if

$$-(v+c)\Pr(R_0 \mid s) \geq -c.$$

Using the fact that $\Pr(R_1 \mid s) = 1 - \Pr(R_0 \mid s)$, this condition simplifies to

$$v \leq \frac{c\Pr(R_1 \mid s)}{\Pr(R_0 \mid s)}.$$

Substituting (3) and (4) into this expression, it follows that when the LLR observes the bad signal $s_0$, it will support the bank if

$$v \leq v_0 \equiv \frac{cp(1-q)}{(1-p)q}, \tag{5}$$

[6]Clearly, this assumption does not affect the characterization of the equilibrium of the game, since it is equivalent to a change in the cost $c$. Interestingly, Repullo (2000) assumes $\alpha < 1$, while Kahn and Santos (2001) assume $\alpha > 1$.

and when the LLR observes the good signal $s_1$, it will support the bank if

$$v \leq v_1 \equiv \frac{cpq}{(1-p)(1-q)}. \tag{6}$$

The critical values $v_0$ and $v_1$ defined in (5) and (6) satisfy

$$v_1 = \left(\frac{q}{1-q}\right)^2 v_0,$$

which implies $v_1 > v_0$ whenever $q > \frac{1}{2}$. Hence if the signal is informative, the LLR is more likely to provide support to the bank when it observes the good signal $s_1$ than when it observes the bad signal $s_0$. Moreover, the critical value $v_0$ is decreasing in the quality $q$ of the supervisory information, with $\lim_{q\to 1} v_0 = 0$, and the critical value $v_1$ is increasing in $q$, with $\lim_{q\to 1} v_1 = \infty$.[7] Thus, when the signal is perfectly informative, the bank will never be supported if the signal is bad, and will always be supported if it is good.

The critical values $v_0$ and $v_1$ are increasing in the social cost of bank failure $c$, because when this cost is high, the LLR has a stronger incentive to lend to the bank in order to save $c$ when the high return $R_1$ obtains. They are also increasing in $p = \Pr(R_1)$, because when this probability is high, the LLR is more likely to recover its loan $v$ and save the cost $c$.

By limited liability, the bank gets a zero payoff if it is liquidated at date 1 or fails at date 2, and gets $R(p) - 1$ if it succeeds at date 2. This event happens when the high return $R_1$ obtains and either the LLR observes the bad signal $s_0$ and the liquidity shock satisfies $v \leq v_0$, or it observes the good signal $s_1$ and the liquidity shock satisfies $v \leq v_1$. By assumption 2 and the independence of $v$ we have

$$\begin{aligned}\Pr(R_1, s_0, \text{ and } v \leq v_0) &= \Pr(R_1)\Pr(s_0 \mid R_1)\Pr(v \leq v_0)\\ &= p(1-q)F(v_0),\\ \Pr(R_1, s_1, \text{ and } v \leq v_1) &= \Pr(R_1)\Pr(s_1 \mid R_1)\Pr(v \leq v_1)\\ &= pqF(v_1).\end{aligned}$$

Hence, the bank's objective function is

$$U_B = p[(1-q)F(v_0) + qF(v_1)][R(p) - 1]. \tag{7}$$

[7]The fact that $v_1$ may be greater than one is not a problem, because since the support of $v$ is $[0, 1]$, we have $\Pr(v \leq v_1) = F(v_1) = 1$, so in this case the bank would be supported with probability one.

A Nash equilibrium of the game between the bank and the LLR is a choice of risk $p^*$ by the bank, and a choice of maximum liquidity support by the LLR contingent on the bad and the good signal, $v_0^*$ and $v_1^*$, such that $p^*$ maximizes

$$p[(1-q)F(v_0^*) + qF(v_1^*)][R(p) - 1],$$

and

$$v_0^* = \frac{cp^*(1-q)}{(1-p^*)q} \quad \text{and} \quad v_1^* = \frac{cp^*q}{(1-p^*)(1-q)}.$$

In this definition, it is important to realize that since the LLR does not observe the bank's choice of risk, the critical values $v_0^*$ and $v_1^*$ only depend on the equilibrium $p^*$. This in turn implies that the term $[(1-q)F(v_0^*) + qF(v_1^*)]$ factors out in the bank's objective function, so its problem reduces to maximize $p[R(p) - 1]$.[8]

The first-order condition that characterizes the equilibrium choice of risk $p^*$ is

$$R(p^*) + p^*R'(p^*) = 1. \tag{8}$$

Since, by assumption 1, $R(p) + pR'(p)$ is decreasing in $p$, conditions (2) and (8) imply that $p^*$ is strictly below the first-best $\widehat{p}$, so the bank will be choosing too much risk. This is just the standard risk-shifting effect that follows from debt financing under limited liability.

It should be noted that the bank's choice of $p$ changes the probability distribution of the signals, increasing $\Pr(s_1) = pq+(1-p)(1-q)$ and decreasing $\Pr(s_0) = 1 - \Pr(s_1)$ (as long as $q > \frac{1}{2}$). However, by assumption 2, $p$ does not affect the distribution of the signals conditional on the realization of the high return $R_1$, which implies that the bank's probability of getting $R(p) - 1$ is linear in $p = \Pr(R_1)$.

To sum up, we have set up a model of a bank and an LLR in which the former chooses the riskiness of its portfolio and the latter chooses whether to lend to the bank to cover random deposit withdrawals, a decision that depends on a signal on the ex-post quality of the portfolio. We have shown that the bank's equilibrium choice of risk is independent of the distribution of the liquidity shocks and the other parameters that determine the LLR's decision, such as the quality of the supervisory information or the social cost of bank failure.

[8]In a sequential game of complete information, the characterization of equilibrium would be more complicated, since the critical values $v_0^*$ and $v_1^*$ would depend on the bank's choice of $p$.

## 2. The General Model

We now introduce in our basic model two features of banking in the real world that are relevant to the problem under discussion. First, on the asset side of the bank's balance sheet, we suppose that, apart from the risky asset, the bank can invest in a safe and perfectly liquid asset that can be used as a buffer against liquidity shocks. Second, on the liability side, we suppose that the bank can be funded with both deposits and equity capital, and that the bank is subject to a minimum capital requirement. The LLR observes both the bank's equity capital and its investment in the liquid asset. We characterize the equilibrium of the new game between the bank and the LLR, compare it with that of a model without an LLR, and examine its comparative statics properties.

Specifically, suppose that at date 0 the bank raises $k$ equity capital and $1-k$ deposits, and invests $\lambda$ in the safe asset and $1-\lambda$ in the risky asset, so the size of its balance sheet is normalized to one.[9] Bank capital has to satisfy the constraint $k \geq \kappa(1-\lambda)$, where $\kappa \in (0,1)$. Thus, the capital requirement depends on the (observable) bank's investment in the risky asset, but not on the (unobservable) bank's choice of risk.

We assume that the return of the safe asset is equal to the deposit rate, which has been normalized to zero, and that bank capital is provided by a special class of agents, called bankers, who require an expected rate of return $\delta \geq 0$ on their investment. A strictly positive value of $\delta$ captures either the scarcity of bankers' wealth or, perhaps more realistically, the existence of a premium for the agency and asymmetric information problems faced by the bank shareholders.[10]

### *2.1 Characterization of Equilibrium*

At date 1 a fraction $v \in [0,1]$ of the deposits are withdrawn. Since the bank has $1-k$ deposits, then $v(1-k)$ deposits are withdrawn

[9]This assumption is made without loss of generality. The same results would obtain if, for example, the bank raised one unit of deposits and $k$ units of capital, and invested $\lambda$ in the safe asset and $1+k-\lambda$ in the risky asset.

[10]See Holmström and Tirole (1997) and Diamond and Rajan (2000) for explicit models of why $\delta$ might be positive. The same assumption is made by Bolton and Freixas (2000), Hellmann, Murdock, and Stiglitz (2000), and Repullo and Suarez (2004), among others.

at this date. There are two cases to consider. First, if $v(1-k) \leq \lambda$, the bank can repay the depositors by selling the required amount of the safe asset, so it keeps $\lambda - v(1-k)$ invested in the safe asset. In this case the bank's payoff in the high-return state equals the return of its investment in the safe asset, $\lambda - v(1-k)$, plus the return of its investment in the risky asset, $(1-\lambda)R(p)$, minus the amount paid to the remaining depositors, $(1-v)(1-k)$, that is,

$$\lambda - v(1-k) + (1-\lambda)R(p) - (1-v)(1-k) = (1-\lambda)[R(p)-1] + k.$$

Second, if $v(1-k) > \lambda$, the bank needs to borrow $v(1-k) - \lambda$ from the LLR in order to avoid liquidation. If such funding is obtained, the bank's payoff in the high-return state equals the return of its investment in the risky asset, $(1-\lambda)R(p)$, minus the amount paid to the remaining depositors, $(1-v)(1-k)$, minus the amount paid to the LLR, $v(1-k) - \lambda$, that is,

$$(1-\lambda)R(p) - (1-v)(1-k) - [v(1-k) - \lambda] = (1-\lambda)[R(p)-1] + k.$$

In both cases, if the low-return state obtains, the bank's net worth is $\lambda - (1-k)$, which will be negative as long as the bank's investment in the liquid asset, $\lambda$, does not exceed its deposits, $1-k$, which will generally obtain in equilibrium.[11] Hence, by limited liability, the bank's payoff in the low-return state will be zero. Obviously, its payoff will also be zero when $v(1-k) > \lambda$ and the LLR does not support the bank.

The decision of the LLR in the case in which the bank requires emergency lending, $v(1-k) > \lambda$, is characterized as follows. If the LLR observes signal $s$ and decides to support the bank, with probability $\Pr(R_1 \mid s)$ the bank will be solvent at date 2 and the LLR will recover its loan $v(1-k) - \lambda$, while with probability $\Pr(R_0 \mid s)$ the bank will fail and the LLR will lose $v(1-k) - \lambda$ and incur the cost $c$. If, on the other hand, the LLR does not provide the liquidity support, the bank will be liquidated at date 1, and the LLR's payoff will be $-c$. Hence the LLR will support the bank if

$$-[v(1-k) - \lambda + c] \Pr(R_0 \mid s) \geq -c.$$

[11] In particular, we show below that under plausible conditions the capital requirement will be binding, so $k = \kappa(1-\lambda)$, which implies $\lambda - (1-k) = -(1-\kappa)(1-\lambda) < 0$.

As before, substituting (3) and (4) into this expression, it follows that when the LLR observes the bad signal $s_0$, it will support the bank if the liquidity shortfall, $v(1-k)-\lambda$, is smaller than or equal to the critical value $v_0$ given by (5), that is, if

$$v \leq \frac{v_0+\lambda}{1-k}, \tag{9}$$

and when it observes the good signal $s_1$, it will support the bank if the liquidity shortfall, $v(1-k)-\lambda$, is smaller than or equal to the critical value $v_1$ given by (6), that is, if

$$v \leq \frac{v_1+\lambda}{1-k}. \tag{10}$$

Thus, the probability that the bank will reach the final date 2 is increasing in its investment in the safe asset $\lambda$ and its equity capital $k$. This is explained by the role of the safe asset as a buffer against liquidity shocks, and by the fact that the higher the bank capital, the lower its deposits and hence the size of the liquidity shocks.

The bank's objective function is to maximize the expected value of the shareholders' payoff net of the opportunity cost of their initial infusion of capital. The latter is simply $(1+\delta)k$. To compute the former, notice that bank shareholders get a zero payoff if the bank is liquidated at date 1 or fails at date 2, and they get $(1-\lambda)(R(p)-1)+k$ if it succeeds at date 2. This event happens when the high return $R_1$ obtains and either the LLR observes the bad signal $s_0$ and the liquidity shock $v$ satisfies (9), or it observes the good signal $s_1$ and the liquidity shock $v$ satisfies (10). As before, we have

$$\Pr\left(R_1, s_0, \text{ and } v \leq \frac{v_0+\lambda}{1-k}\right) = p(1-q)F\left(\frac{v_0+\lambda}{1-k}\right),$$
$$\Pr\left(R_1, s_1, \text{ and } v \leq \frac{v_1+\lambda}{1-k}\right) = pqF\left(\frac{v_1+\lambda}{1-k}\right).$$

Hence, the bank's objective function in the general model is

$$U_B = p\left[(1-q)F\left(\frac{v_0+\lambda}{1-k}\right) + qF\left(\frac{v_1+\lambda}{1-k}\right)\right] \times [(1-\lambda)(R(p)-1)+k] - (1+\delta)k. \tag{11}$$

Obviously, $U_B$ coincides with the objective function (7) in the previous section when $\lambda = 0$ and $k = 0$.

A Nash equilibrium of the game between the bank and the LLR is a choice of liquidity $\lambda^*$, capital $k^*$, and risk $p^*$ by the bank, and a choice of maximum liquidity support by the LLR contingent on the bad and the good signal, $v_0^*$ and $v_1^*$, such that $(\lambda^*, k^*, p^*)$ maximizes

$$p\left[(1-q)F\left(\frac{v_0^*+\lambda}{1-k}\right)+qF\left(\frac{v_1^*+\lambda}{1-k}\right)\right] \\ \times\left[(1-\lambda)(R(p)-1)+k\right]-(1+\delta)k, \qquad (12)$$

subject to the capital requirement $k \geq \kappa(1-\lambda)$, and

$$v_0^* = \frac{cp^*(1-q)}{(1-p^*)q} \quad \text{and} \quad v_1^* = \frac{cp^*q}{(1-p^*)(1-q)}. \qquad (13)$$

As in the basic model, it is important to note that since the LLR does not observe the bank's choice of risk, the critical values $v_0^*$ and $v_1^*$ only depend on the equilibrium $p^*$. This in turn implies that the bank's problem reduces to maximize $p\left[(1-\lambda)(R(p)-1)+k\right]$. Thus, the bank's choice of risk is characterized by the first-order condition

$$(1-\lambda^*)[R(p^*)-1]+k^*+p^*(1-\lambda^*)R'(p^*)=0,$$

which simplifies to

$$R(p^*)+p^*R'(p^*)=1-\frac{k^*}{1-\lambda^*}. \qquad (14)$$

Comparing this expression with (2) and (8), and taking into account that, by assumption 1, $R(p)+pR'(p)$ is decreasing in $p$, it follows that the bank's equilibrium choice of risk $p^*$ will be closer to the first-best $\widehat{p}$ than in the model without the capital requirement. This is just the standard *capital-at-risk effect*: higher capital implies higher losses for the bank's shareholders in case of default and hence lower incentives for risk taking.[12]

If the bank's equilibrium choice of capital $k^*$ is at the corner $\kappa(1-\lambda^*)$, then the first-order condition (14) further simplifies to

$$R(p^*)+p^*R'(p^*)=1-\kappa. \qquad (15)$$

[12] See Hellmann, Murdock, and Stiglitz (2000) and Repullo (2004) for a recent discussion of this effect.

In this case, the equilibrium $p^*$ only depends on the capital requirement $\kappa$. Moreover, assumption 1 implies that $R(p) + pR'(p)$ is decreasing in $p$, which gives $dp^*/d\kappa > 0$. Hence, the higher the capital requirement, the lower the risk chosen by the bank.[13]

In general, it is difficult to prove that this corner solution will obtain, since the properties of the bank's objective function (12) depend on the shape of the distribution function of the liquidity shock $F(v)$.[14] For this reason, in what follows we work with a specific parameterization of $F(v)$, namely $F(v) = v^\eta$, where $\eta \in (0, 1)$.[15] In this case it can be checked that the bank's objective function (12) is convex in $k$, so we can only have either $k^* = \kappa(1 - \lambda^*)$ or $k^* = 1$. But for large $k$ we have

$$F\left(\frac{v_0^* + \lambda}{1 - k}\right) = F\left(\frac{v_1^* + \lambda}{1 - k}\right) = 1,$$

so the derivative of (12) with respect to $k$ is $p - (1 + \delta) < 0$. Hence $k = 1$ cannot be a solution, which gives $k^* = \kappa(1 - \lambda^*)$.

This result implies that the equilibrium of the game between the bank and the LLR is easy to characterize. The risk $p^*$ chosen by the bank is the unique solution of the first-order condition (15). This in turn determines the critical values $v_0^*$ and $v_1^*$ that characterize the behavior of the LLR. Substituting $p = p^*$ and $F(v) = v^\eta$ into the bank's objective function (12), we then find the value of $\lambda^*$ by maximizing

$$\begin{aligned} p^* &\left[(1 - q)\left(\frac{v_0^* + \lambda}{1 - k}\right)^\eta + q\left(\frac{v_1^* + \lambda}{1 - k}\right)^\eta\right] \\ &\times [(1 - \lambda)(R(p^*) - 1) + k] - (1 + \delta)k, \end{aligned} \tag{16}$$

subject to $k = \kappa(1 - \lambda)$. Finally, we get $k^* = \kappa(1 - \lambda^*)$.

[13] Notice that for $\kappa = 1$, that is, a 100% capital requirement, (2) and (15) imply $p^* = \widehat{p}$.

[14] However, finding that $k$ is at the minimum required by regulation is standard in both static and dynamic models of banking; see, for example, Repullo and Suarez (2004) and Repullo (2004).

[15] Notice that this is a simple special case of a beta distribution for which the density function $F'(v) = \eta v^{\eta - 1}$ is decreasing in $v$, so small liquidity shocks are more likely than large shocks.

Going analytically beyond this point is, however, complicated, because although the bank's objective function (16) is concave in $\lambda$, this is in general no longer the case once we substitute the constraint $k = \kappa(1-\lambda)$ into (16), since this function is convex in $k$. For this reason, our results on equilibrium liquidity and capital will be derived from numerical solutions.

### *2.2 Equilibrium without an LLR*

We now compare the equilibrium behavior of the bank when there is an LLR with its behavior when there is no LLR. The objective function of the bank in such a model is a special case of (11) when we set $v_0 = v_1 = 0$ (i.e., no last-resort lending), which gives

$$U_B = pF\left(\frac{\lambda}{1-k}\right)[(1-\lambda)(R(p)-1)+k] - (1+\delta)k.$$

Thus, the bank gets $(1-\lambda)(R(p)-1)+k$ in the high-return state, which obtains with probability $p$, but only if it has sufficient liquidity to cover the deposit withdrawals at date 1—that is, if $v(1-k) \leq \lambda$, an event that happens with probability $F(\lambda/(1-k))$.

From here we can follow our previous steps to conclude that when the bank's capital $k$ is at the corner $\kappa(1-\lambda^*)$, its choice of risk is characterized by the first-order condition (15), so we get exactly the same $p^*$ as in the model with the LLR. In other words, contrary to what has been taken for granted in the banking literature, our model predicts that *the existence of an LLR does not have any effect on the bank's incentives to take risk.*

Computation of the effects of having an LLR on the liquidity decision of the bank requires us to specify the functional forms of the high return of the risky asset, $R(p)$, and the cumulative distribution function of the liquidity shock, $F(v)$, as well as the parameter values of the capital requirement $\kappa$, the cost of capital $\delta$, the informativeness of the supervisory signal $q$, and the social cost of bank failure $c$. Since our focus is on qualitative results, we will not calibrate the model to obtain plausible numerical results, but instead choose simple functional forms and parameter values. Specifically, the functional forms $R(p) = 3 - 2p^2$ and $F(v) = v^{\eta}$, with $\eta = 0.25$,

**Table 1. Equilibria with and without an LLR**

| | With LLR | Without LLR |
|---|---|---|
| $\lambda^*$ | 0.11 | 0.23 |
| $k^*$ | 0.09 | 0.08 |
| $p^*$ | 0.59 | 0.59 |
| $v_0^*$ | 0.10 | – |
| $v_1^*$ | 0.22 | – |

will be maintained in all our simulations,[16] and our baseline parameter values are $L = 0.50$, $\kappa = \delta = c = 0.10$, and $q = 0.60$.

The corresponding equilibria with and without an LLR are shown in table 1. As noted above, the level of risk $p^*$ chosen by the bank is the same in both models, and may be obtained by substituting $R(p) = 3 - 2p^2$ into (15), which gives $p^* = \sqrt{(2+\kappa)/6} = 0.59$. Not surprisingly, the results in table 1 show that the liquidity buffer $\lambda^*$ is much larger in the absence of an LLR.[17] Given that $k^* = \kappa(1 - \lambda^*)$, this in turn implies a lower level of capital.

When the deposit withdrawal $v(1-k^*)$ is below the bank's liquidity $\lambda^*$ (an event that happens with probability 0.58 in the model with the LLR, and with probability 0.71 in the model without it), the bank will be able to repay the depositors by selling the required amount of the safe asset. Moreover, in the first model, when $v(1 - k^*) > \lambda^*$, the LLR will provide liquidity up to $v_0^* = 0.10$ when it observes the bad signal $s_0$, and up to $v_1^* = 0.22$ when it observes the good signal $s_1$.

The probability that the bank gets a positive payoff in the model with an LLR is

$$p^* \left[ (1-q) F\left( \frac{v_0^* + \lambda^*}{1 - k^*} \right) + qF\left( \frac{v_1^* + \lambda^*}{1 - k^*} \right) \right] = 0.44,$$

[16] Observe that $R(p) = 3 - 2p^2$ is decreasing and concave, with $R(1) = 1$ and $R(1) + R'(1) = -3 < 0$, so assumption 1 is satisfied. Also, the median liquidity shock is $F^{-1}(0.5) = 0.0625$.

[17] Gonzalez-Eiras (2003) provides some interesting empirical evidence on this result. He shows that the contingent credit line agreement signed by the Central Bank of Argentina with a group of international banks in December 1996 enhanced that central bank's ability to act as an LLR and led to a significant decrease in the liquidity holdings of domestic Argentinian banks.

while the corresponding probability in the model without an LLR is

$$p^* F\left(\frac{\lambda^*}{1-k^*}\right) = 0.42.$$

Since in the first model the bank is investing a higher proportion of its portfolio in the risky asset, its equilibrium expected payoff is significantly higher with an LLR (0.45 against 0.37).

### *2.3 Comparative Statics*

We next analyze the effect on the equilibrium of the game between the bank and the LLR of changes in the capital requirement $\kappa$, the cost of capital $\delta$, the informativeness of the supervisory signal $q$, and the social cost of bank failure $c$. The results summarized in table 2 are derived by computing the equilibrium corresponding to deviations in $\kappa$, $\delta$, $q$, and $c$ from the baseline case.

As noted above, an increase in the capital requirement $\kappa$ leads to an increase in $p^*$, which by (13) increases the maximum support provided by the LLR contingent on the bad and the good signal, $v_0^*$ and $v_1^*$. The effect on $\lambda^*$ is also positive. Two reasons explain this result. First, the higher capital requirement makes investment in the risky asset, which does not carry a capital charge, relatively less attractive for the bank than investing in the safe asset. Second, the higher $p^*$ reduces the success payoff of the risky asset, $R(p^*)$, and also makes it relatively less attractive for the bank than the safe

**Table 2. Equilibrium Effects of Changes in the Capital Requirement $\kappa$, the Cost of Capital $\delta$, the Informativeness of the Supervisory Signal $q$, and the Social Cost of Bank Failure $c$**

| $x$ | $\frac{dp^*}{dx}$ | $\frac{d\lambda^*}{dx}$ | $\frac{dk^*}{dx}$ | $\frac{dv_0^*}{dx}$ | $\frac{dv_1^*}{dx}$ |
|---|---|---|---|---|---|
| $\kappa$ | + | + | + | + | + |
| $\delta$ | 0 | + | − | 0 | 0 |
| $q$ | 0 | − | + | − | + |
| $c$ | 0 | − | + | + | + |

asset. On the other hand, the higher liquidity support offered by the LLR reduces the bank's incentives to invest in the safe asset, but the numerical results show that this effect is dominated by the other two.

With regard to the other comparative statics results, note first that, as shown analytically, the value of $p^*$ chosen by the bank only depends on the capital requirement $\kappa$, so the effect of the other three parameters is zero.

Since the cost of capital $\delta$ does not affect $p^*$, it does not have any effect either on the maximum liquidity support provided by the LLR contingent on the bad and the good signal, $v_0^*$ and $v_1^*$. The cost of capital $\delta$ has a positive effect on equilibrium liquidity $\lambda^*$, because when capital is more expensive, investing in the safe asset (which does not carry a capital charge) is relatively more attractive than investing in the risky asset. Since $k^* = \kappa(1 - \lambda^*)$, this also explains why an increase in $\delta$ has a negative effect on $k^*$.

As noted in section 1, the critical value $v_0^*$ is decreasing in the quality $q$ of the supervisory information, while the critical value $v_1^*$ is increasing in $q$, so with better information the bank is less (more) likely to be supported by the LLR when the signal is bad (good). The sign of the derivative of $\lambda^*$ with respect to $q$ is negative, which means that the positive effect of having less support when the signal is bad is dominated by the negative effect of having more support when the signal is good. Since $k^* = \kappa(1 - \lambda^*)$, this in turn explains why an increase in $q$ has a positive effect on $k^*$. However, when $q$ is sufficiently large, we may get to the corner $\lambda^* = 0$ and $k^* = \kappa$, where these derivatives become zero.

As also noted in section 1, the critical values $v_0^*$ and $v_1^*$ are increasing in the social cost of bank failure $c$, because when this cost is high, the LLR has a stronger incentive to lend to the bank in order to save $c$ when the higher return state obtains. This explains why the bank wants to hold a lower liquidity buffer $\lambda^*$, so $k^* = \kappa(1 - \lambda^*)$ will be higher. However, as in the case of parameter $q$, when $c$ is sufficiently large, we may get to the corner $\lambda^* = 0$ and $k^* = \kappa$, where these derivatives become zero.

If we consider that the social cost of failure increases more than proportionately with the size of the bank's balance sheet (which we have normalized to one), $c$ will be higher for large banks, which implies a "too big to fail" result: large banks are more likely to be supported by the LLR, and consequently they will hold smaller liquidity buffers.

## 3. Extensions

### *3.1 Penalty Rates*

The classical doctrine on the LLR put forward by Bagehot (1873) required "that these loans should only be made at a very high rate of interest." We now examine how the results in section 2 are modified when the LLR charges a penalty rate $r > 0$. Importantly, we assume that $r$ is exogenously given ex-ante, and not chosen by the LLR ex-post.

To characterize the equilibrium of the new game between the bank and the LLR, suppose that $v(1-k)$ deposits are withdrawn at date 1. If $v(1-k) \leq \lambda$, the bank can repay the depositors by selling the required amount of the safe asset, so there is no change with respect to our previous analysis. If, on the other hand, $v(1-k) > \lambda$, the bank needs to borrow $v(1-k)-\lambda$ from the LLR. If such funding is obtained, the bank's payoff in the high-return state equals the return of its investment in the risky asset, $(1-\lambda)R(p)$, minus the amount paid to the remaining depositors, $(1-v)(1-k)$, minus the amount paid to the LLR, $(1+r)[v(1-k)-\lambda]$, that is,

$$\begin{aligned}(1-\lambda)R(p) - (1-v)(1-k) - (1+r)[v(1-k)-\lambda] \\ = (1-\lambda)[R(p)-1] + k - r[v(1-k)-\lambda].\end{aligned}$$

The last term in this expression accounts for the interest payments to the LLR.

The decision of the LLR in the case when $v(1-k) > \lambda$ is now characterized as follows. If the LLR observes signal $s$ and decides to support the bank, with probability $\Pr(R_1 \mid s)$ the bank will be solvent at date 2 and the LLR will recover its loan $v(1-k)-\lambda$ and net $r[v(1-k)-\lambda]$ in interest payments, while with probability $\Pr(R_0 \mid s)$ the bank will fail and the LLR will lose $v(1-k)-\lambda$ and incur the cost $c$, so the LLR's expected payoff is

$$r[v(1-k)-\lambda]\Pr(R_1 \mid s) - [v(1-k)-\lambda+c]\Pr(R_0 \mid s).$$

On the other hand, if the LLR does not provide the liquidity support, the bank will be liquidated at date 1, and the LLR's payoff will be $-c$. Hence the LLR will support the bank if

$$r[v(1-k)-\lambda]\Pr(R_1 \mid s) - [v(1-k)-\lambda+c]\Pr(R_0 \mid s) \geq -c.$$

Substituting (3) and (4) into this expression, it follows that when the LLR observes the bad signal $s_0$, it will support the bank if the liquidity shortfall, $v(1-k)-\lambda$, is smaller than or equal to the critical value

$$v_0 \equiv \frac{c\Pr(R_1 \mid s_0)}{\Pr(R_0 \mid s_0) - r\Pr(R_1 \mid s_0)} = \frac{cp(1-q)}{(1-p)q - rp(1-q)},$$

and when the LLR observes the good signal $s_1$, it will support the bank if the liquidity shortfall, $v(1-k)-\lambda$, is smaller than or equal to the critical value

$$v_1 \equiv \frac{c\Pr(R_1 \mid s_1)}{\Pr(R_0 \mid s_1) - r\Pr(R_1 \mid s_1)} = \frac{cpq}{(1-p)(1-q) - rpq}.$$

As before, it is easy to check that $v_1 > v_0$ whenever $q > \frac{1}{2}$. Also, notice that both $v_0$ and $v_1$ are increasing in $r$, so with penalty rates the LLR will be softer with the bank, providing emergency funding for a larger range of liquidity shocks.

To compute the bank's new objective function, we have to subtract from $U_B$ in (11) the expected interest payments to the LLR. If the LLR observes the bad signal $s_0$, the bank borrows from the LLR when $0 < v(1-k)-\lambda \leq v_0$, that is, when

$$\frac{\lambda}{1-k} < v \leq \frac{v_0+\lambda}{1-k},$$

so the conditional expected cost of this borrowing is

$$r\left[\int_{\frac{\lambda}{1-k}}^{\frac{v_0+\lambda}{1-k}} [v(1-k)-\lambda]\; dF(v)\right]\Pr(s_0 \mid R_1).$$

Similarly, if the LLR observes the good signal $s_1$, the conditional expected cost of the bank's borrowing is

$$r\left[\int_{\frac{\lambda}{1-k}}^{\frac{v_1+\lambda}{1-k}} [v(1-k)-\lambda]\; dF(v)\right]\Pr(s_1 \mid R_1).$$

Hence the bank's new objective function is

$$\begin{aligned} U_B = p &\left[(1-q)F\left(\frac{v_0+\lambda}{1-k}\right) + qF\left(\frac{v_1+\lambda}{1-k}\right)\right] \\ &\times [(1-\lambda)(R(p)-1)+k] - (1+\delta)k \\ &- rp\left[(1-q)\int_{\frac{\lambda}{1-k}}^{\frac{v_0+\lambda}{1-k}} [v(1-k)-\lambda]\, dF(v)\right. \\ &\left. + q\int_{\frac{\lambda}{1-k}}^{\frac{v_1+\lambda}{1-k}} [v(1-k)-\lambda]\, dF(v)\right]. \end{aligned}$$

Assuming that $F(v) = v^\eta$, the integrals in this expression can be easily solved, and we can compute for the baseline parameters the equilibrium effects of charging a penalty rate $r$.[18] The results are presented in table 3.

Thus, an increase in the penalty rate $r$ leads to a reduction in $p^*$, so the bank's portfolio becomes riskier. The reason for this result is that penalty rates increase the expected interest payments in the high-return state and, consequently, the bank tries to compensate this effect by choosing a higher risk and higher return portfolio (recall that by assumption 1, a decrease in $p$ increases $R(p)$). The reduction in $p^*$ would ceteris paribus lead to a decrease in both $v_0^*$ and $v_1^*$, but this is more than compensated by the positive effect of the interest payments on the LLR's willingness to lend. The increase in $v_0^*$ and $v_1^*$ in turn explains why the bank chooses a lower liquidity buffer $\lambda^*$, so $k^* = \kappa(1-\lambda^*)$ will be higher.

**Table 3. Equilibrium Effects of Changes in the Penalty Rate $r$**

| $x$ | $\frac{dp^*}{dx}$ | $\frac{d\lambda^*}{dx}$ | $\frac{dk^*}{dx}$ | $\frac{dv_0^*}{dx}$ | $\frac{dv_1^*}{dx}$ |
|---|---|---|---|---|---|
| $r$ | − | − | + | + | + |

[18] It should be noted that this computation is complicated because now $p^*$ depends on $r$, and cannot be directly solved from the first-order condition (15). The equilibrium is obtained by numerical iteration of the best response functions of the two players.

### *3.2 Collateralized Lending*

The classical doctrine on the LLR put forward by Bagehot (1873) not only required charging "a very high rate of interest," but also "that at this rate these advances should be made on all good banking securities." We now examine how the results in section 2 are modified when last-resort lending is collateralized, so the LLR becomes a senior claimant when it provides the liquidity support and the bank subsequently fails.[19] Obviously, for this to make any difference the failure return should be positive, so in this subsection we assume that $R_0 = l \in (0, L)$.[20]

To analyze the effect of this change, consider a situation in which $v(1-k) > \lambda$, and let $s$ be the signal observed by the LLR. There are two cases to consider. First, if the liquidity shortfall, $v(1-k)-\lambda$, is smaller than or equal to the collateral $l$, the LLR is fully covered, so its expected payoff if it supports the bank, $-c\Pr(R_0 \mid s)$, is greater than the payoff if it does not, $-c$, so the bank will always be supported. Second, if the liquidity shortfall, $v(1-k)-\lambda$, is greater than the collateral $l$, the expected payoff of the LLR if it supports the bank is

$$-\left[v(1-k)-\lambda-l+c\right]\Pr(R_0 \mid s),$$

since with probability $\Pr(R_0 \mid s)$ the bank will fail at date 2 and the LLR will lose $v(1-k)-\lambda-l$ and incur the cost $c$. On the other hand, if the LLR does not provide the liquidity support, the bank will be liquidated at date 1, and the LLR's payoff will be $-c$. Hence the LLR will support the bank if

$$-\left[v(1-k)-\lambda-l+c\right]\Pr(R_0 \mid s) \geq -c.$$

[19]The referee criticized this interpretation, noting that "if we follow Bagehot's second rule and only provide liquidity backed by 'good banking securities,' it is not clear why the central bank will end up underwriting risky investments." In the referee's view, in the present model, "a fairer interpretation of Bagehot might imply no lending by the central bank at all." However, as noted by Goodhart (1999, 343), "Bagehot's proposal related simply to the collateral that the applicant could offer, and the effect of this rule in practice was to distinguish, in part, between those loans on which the central bank might expect with some considerable probability to make a loss and those on which little, or no, loss should eventuate."

[20]The assumption that the failure return $l$ at date 2 is smaller than the liquidation value $L$ at date 1 is not required for our analysis, but makes a lot of sense in the context of the model.

**Table 4. Equilibrium Effects of Collateralization**

| $x$ | $\frac{dp^*}{dx}$ | $\frac{d\lambda^*}{dx}$ | $\frac{dk^*}{dx}$ | $\frac{d(v_0^*+l)}{dx}$ | $\frac{d(v_1^*+l)}{dx}$ |
|---|---|---|---|---|---|
| $l$ | 0 | − | + | 1 | 1 |

Substituting (3) and (4) into this expression, it follows that when the LLR observes the bad signal $s_0$, it will support the bank if the liquidity shortfall, $v(1-k)-\lambda$, is smaller than or equal to the critical value $v_0+l$, where $v_0$ is given by (5); when the LLR observes the good signal $s_1$, it will support the bank if the liquidity shortfall, $v(1-k)-\lambda$, is smaller than or equal to the critical value $v_1+l$, where $v_1$ is given by (6).

Hence the bank's objective function becomes

$$U_B = p\left[(1-q)F\left(\frac{v_0+l+\lambda}{1-k}\right)+qF\left(\frac{v_1+l+\lambda}{1-k}\right)\right] \times [(1-\lambda)(R(p)-1)+k]-(1+\delta)k.$$

Using the same arguments as in section 2, it follows that the bank's choice of risk $p^*$ will also be characterized by the first-order condition (15), so it only depends on the capital requirement $\kappa$.

As for the effect of collateralization on the bank's liquidity and capital decisions, note that the case in which the LLR's loan is not collateralized (and it is junior to the claim of the deposit insurer) is equivalent to the case $l=0$ analyzed in section 2, so the signs of derivatives with respect to $l$ in table 4 indicate the effect of collateralization on $\lambda^*$ and $k^*$.

Thus, collateralization of last-resort lending does not have any effect on the bank's incentives to take risk, but increases the maximum support that the LLR is willing to provide contingent on the bad and the good signal, $v_0^*+l$ and $v_1^*+l$. This explains why the bank wants to hold a lower liquidity buffer $\lambda^*$, so $k^*=\kappa(1-\lambda^*)$ will be higher. In other words, the protection for the LLR advocated by Bagehot translates into a lower liquidity buffer and hence a higher probability that the bank will require emergency liquidity assistance, but without any effect on risk taking.

### 3.3 Discounting of Future Payoffs

The LLR is a public institution that is run by officials that may have fixed terms of office. If these terms are short or the officials are close to finishing their terms, the officials may have an incentive to avoid current costs possibly at the expense of some larger future costs that would be assumed by their successors. Formally, we can incorporate this possibility into our model by introducing a discount factor $\beta < 1$ for the LLR.[21]

To analyze the effect of such discounting, consider a situation in which $v(1-k) > \lambda$, and let $s$ be the signal observed by the LLR. The expected discounted payoff of the LLR if it supports the bank is now

$$-\beta\,[v(1-k)-\lambda+c]\,\Pr(R_0 \mid s),$$

since with probability $\Pr(R_0 \mid s)$ the bank will fail at date 2 and the LLR will lose $v(1-k)-\lambda$ and incur the cost $c$. On the other hand, if the LLR does not provide the liquidity support, the bank will be liquidated at date 1, and the LLR's payoff will be $-c$. Hence the LLR will support the bank if

$$-\beta\,[v(1-k)-\lambda+c]\,\Pr(R_0 \mid s) \geq -c.$$

Substituting (3) and (4) into this expression, it follows that when the LLR observes the bad signal $s_0$, it will support the bank if the liquidity shortfall, $v(1-k)-\lambda$, is smaller than or equal to the critical value

$$v_0 \equiv \frac{c[1-\beta\Pr(R_0 \mid s_0)]}{\beta\Pr(R_0 \mid s_0)} = \frac{c[p(1-q)+(1-\beta)(1-p)q]}{\beta(1-p)q},$$

and when the LLR observes the good signal $s_1$, it will support the bank if the liquidity shortfall, $v(1-k)-\lambda$, is smaller than or equal to the critical value

$$v_1 \equiv \frac{c[1-\beta\Pr(R_0 \mid s_1)]}{\beta\Pr(R_0 \mid s_1)} = \frac{c[pq+(1-\beta)(1-p)(1-q)]}{\beta(1-p)(1-q)}.$$

[21] This assumption is justified by Kaufman (1991) in the following terms: "The discount rate used by policy makers, who are under considerable political pressure to optimize economic performance in the short-term and whose terms of office are relatively short and not guaranteed to last until the next crisis, is likely to be overestimated."

**Table 5. Equilibrium Effects of Changes in the LLR's Discount Factor $\beta$**

| $x$ | $\frac{dp^*}{dx}$ | $\frac{d\lambda^*}{dx}$ | $\frac{dk^*}{dx}$ | $\frac{dv_0^*}{dx}$ | $\frac{dv_1^*}{dx}$ |
|---|---|---|---|---|---|
| $\beta$ | 0 | + | − | − | − |

As before, it is easy to check that $v_1 > v_0$ whenever $q > \frac{1}{2}$. Also, notice that both $v_0$ and $v_1$ are decreasing in the discount factor $\beta$. This means that an LLR with $\beta < 1$ will be softer with the bank, providing funding for a larger range of liquidity shocks.

We can now compute for the baseline parameters the equilibrium effects of introducing a discount factor $\beta < 1$ for the LLR. As in the model in section 2, the bank's choice of risk $p^*$ is again characterized by the first-order condition (15), so the discount factor $\beta$ does not have any effect on the bank's incentives to take risk. The full comparative statics results are presented in table 5.

As expected, an increase in the discount factor $\beta$ (that is, a decrease in the corresponding discount rate) makes the LLR more willing to incur the current costs of not supporting the bank in order to save some larger future costs, so the derivative of $v_0^*$ and $v_1^*$ with respect to $\beta$ is negative. The reduction in $v_0^*$ and $v_1^*$ in turn explains why the bank chooses a higher liquidity buffer $\lambda^*$, so $k^* = \kappa(1 - \lambda^*)$ will be lower.

Thus we conclude that a high LLR discount rate leads to forbearance, but in line with our previous results, this only translates into a lower liquidity buffer, without any effect on risk taking.

### *3.4 Internalizing Deposit Insurance Payouts*

We have assumed so far that the LLR is institutionally separated from the deposit insurer, so the former does not take into account deposit insurance payouts in deciding whether to support the bank. We now consider a situation in which the LLR either internalizes or assumes a fraction $\gamma \in [0, 1]$ of these payouts. When $\gamma = 0$, the LLR is completely independent from the deposit insurer (e.g., a central bank with no deposit insurance role), whereas when $\gamma = 1$, the LLR also acts as deposit insurer.[22]

To analyze the effect of such possible connection between the LLR and the deposit insurer, consider a situation in which $v(1-k) > \lambda$, and let $s$ be the signal observed by the LLR. The expected payoff of the LLR if it supports the bank is now

$$-[v(1-k) - \lambda + c + \gamma(1-v)(1-k)] \Pr(R_0 \mid s),$$

since with probability $\Pr(R_0 \mid s)$ the bank will fail at date 2 and the LLR will lose $v(1-k) - \lambda$, incur the cost $c$, and assume a fraction $\gamma$ of the deposit insurance payouts that are given by $(1-v)(1-k)$. On the other hand, if the LLR does not provide the liquidity support, the bank will be liquidated at date 1, and the LLR will incur the cost $c$ and assume a fraction $\gamma$ of the deposit insurance payouts that are given by $(1-k) - \lambda - (1-\lambda)L$, where $(1-\lambda)L$ is the liquidation value of the bank's risky asset.[23] Hence the LLR will support the bank if

$$\begin{aligned} &-[v(1-k) - \lambda + c + \gamma(1-v)(1-k)] \Pr(R_0 \mid s) \\ &\quad \geq -[c + \gamma[(1-k) - \lambda - (1-\lambda)L]]. \end{aligned}$$

Substituting (3) and (4) into this expression, it follows that when the LLR observes the bad signal $s_0$, it will support the bank if the liquidity shortfall, $v(1-k)-\lambda$, is smaller than or equal to the critical value

$$\begin{aligned} v_0 &\equiv \frac{[c + \gamma(1-k-\lambda)] \Pr(R_1 \mid s_0) - \gamma(1-\lambda)L}{(1-\gamma) \Pr(R_0 \mid s_0)} \\ &= \frac{[c + \gamma(1-k-\lambda)]\, p(1-q) - \gamma(1-\lambda)L[p(1-q) + (1-p)q]}{(1-\gamma)(1-p)q}, \end{aligned}$$

[22]Intermediate cases are also relevant. For example, until 1998 the Bank of Spain matched the contribution of the Spanish banks to the deposit insurance fund, so $\gamma$ was $\frac{1}{2}$.

[23]We are implicitly assuming that the amount of deposits is greater than or equal to the liquidation value of the bank at date 1, that is, $1-k \geq \lambda + (1-\lambda)L$. Notice that if $k$ is at the corner $\kappa(1-\lambda)$, this condition reduces to $(1-\lambda)(1-\kappa-L) \geq 0$. In our numerical analysis we take $\kappa = 0.10$ and $L = 0.50$, so it holds.

and when the LLR observes the good signal $s_1$, it will support the bank if the liquidity shortfall, $v(1-k)-\lambda$, is smaller than or equal to the critical value

$$v_1 \equiv \frac{[c+\gamma(1-k-\lambda)]\Pr(R_1 \mid s_1) - \gamma(1-\lambda)L}{(1-\gamma)\Pr(R_0 \mid s_1)}$$
$$= \frac{[c+\gamma(1-k-\lambda)]\,pq - \gamma(1-\lambda)L[pq+(1-p)(1-q)]}{(1-\gamma)(1-p)(1-q)}.$$

As before, one can check that $v_1 > v_0$ whenever $q > \frac{1}{2}$.

We can now compute for the baseline parameters the equilibrium effects of internalizing a fraction $\gamma$ of the deposit insurance payouts. As before, the bank's choice of risk $p^*$ is again characterized by the first-order condition (15), so the share $\gamma$ does not have any effect on the bank's incentives to take risk. The full comparative statics results are presented in table 6.

An increase in the share $\gamma$ makes the LLR tougher when it observes the bad signal $s_0$ (since the critical value $v_0^*$ is decreasing in $\gamma$), and makes it softer when it observes the good signal $s_1$ (since the critical value $v_1^*$ is increasing in $\gamma$). The sign of the derivative of $\lambda^*$ with respect to $\gamma$ is negative for small values of $\gamma$, for which the positive effect of having less support when the signal is bad is dominated by the negative effect of having more support when the signal is good. However, for sufficiently high values of $\gamma$, the critical value $v_1^*$ reaches the value of 1, which means that the bank will always be supported when the signal is good, and so the only remaining effect will be the positive one associated with further reductions in $v_0^*$. Since $k^* = \kappa(1-\lambda^*)$, this explains the two possible signs of the effect of $\gamma$ on $k^*$.

**Table 6. Equilibrium Effects of Changes in the LLR's Share of Deposit Insurance Payouts $\gamma$**

| $x$ | $\frac{dp^*}{dx}$ | $\frac{d\lambda^*}{dx}$ | $\frac{dk^*}{dx}$ | $\frac{dv_0^*}{dx}$ | $\frac{dv_1^*}{dx}$ |
|---|---|---|---|---|---|
| $\gamma$ | 0 | $-/+$ | $+/-$ | $-$ | $+$ |

## 4. Concluding Remarks

Goodhart (1999, 339–40) has argued that "there are few issues so subject to myth, sometimes unhelpful myths that tend to obscure rather than illuminate real issues, as is the subject of whether a central bank . . . should act as a lender of last resort." The third myth in his list is that "moral hazard is everywhere and at all times a major consideration."[24] This paper provides a rationale for the claim that this is indeed a myth. Specifically, it shows that the existence of a lender of last resort does not have any effect on the risk of the banks' illiquid portfolios, but simply reduces their incentives to hold liquid assets.

Although our model is special in a number of respects, we believe that the results are fairly robust. In particular, neither full deposit insurance nor the assumption that deposit withdrawals are purely random is essential. To see this, suppose that in the context of our basic model (without liquidity $\lambda$ and capital $k$) there is an exogenous fraction $u \in (0, L]$ of junior uninsured deposits that require an expected return equal to zero,[25] and let $d$ denote the corresponding interest rate. We assume that uninsured depositors observe the same signal $s \in \{s_0, s_1\}$ as the LLR,[26] and run on the bank at date 1 if and only if they observe the bad signal $s_0$.

In this situation, the LLR will support the bank if the withdrawal $u$ is smaller than the critical value $v_0$ given by (5), so the bank's objective function (7) becomes

$$U_B = p[(1-q)1(u \leq v_0) + q][R(p) - 1 - ud],$$

where $1(u \leq v_0)$ is an indicator function that takes the value 1 if $u \leq v_0$, and 0 otherwise. From here it follows that the first-order condition that characterizes the equilibrium choice of risk $p^*$ is

$$R(p^*) + p^* R'(p^*) = 1 + ud^*, \tag{17}$$

[24] The other myths are that it is generally possible to distinguish between illiquidity and insolvency, that national central bank LLR capabilities are unrestricted whereas international bodies cannot function as LLRs, and that it is possible to dispense with an LLR altogether.

[25] The assumption that $u \leq L$ is made for simplicity, in order to ensure that the junior uninsured depositors get zero when the bank is liquidated at date 1.

[26] This assumption is also made for simplicity. See Repullo (2005) for a model where the depositors' signal is different (but coarser) than that of the LLR.

where $d^*$ is the equilibrium deposit rate. Assuming that uninsured depositors can only claim at date 1 the principal (and not the interest), they receive $1(u \leq v_0^*)u$ at date 1 with probability $\Pr(s_0) = q + (1-2q)p^*$, and $u(1+d^*)$ at date 2 with probability $\Pr(R_1, s_1) = qp^*$. Hence their participation constraint is

$$[q + (1-2q)p^*]1(u \leq v_0^*) + qp^*(1+d^*) = 1. \tag{18}$$

Solving equations (17) and (18) gives the equilibrium values of $p^*$ and $d^*$.[27]

On the other hand, in the absence of an LLR, the bank's objective function becomes $U_B = pq[R(p) - 1 - ud]$, so the first-order condition (17) does not change, while the participation constraint (18) simplifies to $qp^*(1+d^*) = 1$. Hence we conclude that when $u > v_0^*$ the existence of an LLR does not have any effect on the bank's incentives to take risk. Moreover, when $u \leq v_0^*$, the existence of an LLR reduces the deposit rate that satisfies the participation constraint (18), which in turn, by the Stiglitz and Weiss (1981) argument noted in the introductory paragraphs of this paper, increases the equilibrium value of $p^*$. Hence having an LLR may actually reduce the bank's incentives to take risk.

The stark contrast between our results and the extant literature deserves further discussion. It is true that in general any form of insurance (e.g., against liquidity shocks) has the potential to create a moral hazard problem. In the context of our model, this clearly shows in the effect on the holding of liquid assets. But to have an effect on risk taking, something else is needed. One such case would be the following. Suppose that instead of observing a signal $s$ on the return of the bank's risky asset, the uninsured depositors observe the bank's choice of $p$. Furthermore, suppose that, in the absence of an LLR, they can make the deposit rate $d$ contingent on the choice of $p$ (for example, by threatening to withdraw their funds). In this case, the bank would maximize $U_B = p[R(p) - 1 - ud(p)]$ subject to the uninsured depositors' participation constraint $p[1 + d(p)] = 1$. Substituting the constraint into the bank's objective function and maximizing the resulting expression with respect to $p$ gives the first-order condition

[27] Since the relationship between $p^*$ and $d^*$ in both the first-order condition (17) and the participation constraint (18) is decreasing, we may have multiple equilibria, in which case we focus on the one that is closest to the first-best $\widehat{p}$—that is, the one with the highest $p^*$.

$$R(\widetilde{p}) + \widetilde{p}R'(\widetilde{p}) = 1 - u. \tag{19}$$

Since $R(p) + pR'(p)$ is decreasing, the bank's choice of risk $\widetilde{p}$ is increasing in the proportion $u$ of uninsured deposits, and converges to the first-best $\widehat{p}$ when $u$ tends to 1. So we conclude that the existence of uninsured depositors that observe the bank's choice of risk and use this information to renegotiate the terms of their contract ameliorates the bank's risk-shifting incentives. Moreover, the introduction of an LLR that facilitates the withdrawal of the funds at date 1 may upset this disciplining mechanism, bringing us back to the $p^* < \widetilde{p}$ characterized above.

Two objections can be made to this argument. The standard one is that small depositors do not have the ability or the incentives to monitor banks.[28] The nonstandard one that we are putting forward here is that one should distinguish between the *monitoring of actions* and the *monitoring of the consequences of those actions*.[29] In the absence of an LLR, the former ameliorates the moral hazard problem, but the latter does not, because it simply changes the bank's objective function from $p[R(p) - 1 - ud]$ to $pq[R(p) - 1 - ud]$. Clearly, multiplying the function by a constant does not have any effect on the first-order condition that characterizes the bank's choice of risk. And the same result obtains when there is an LLR. Since arguably the second is the most plausible type of monitoring,[30] we conclude that there should be no presumption that the existence of an LLR worsens the bank's risk-shifting incentives—except, as shown in section 3.1, when it charges penalty rates.

Finally, it is worth noting that our model also provides a rationale for a standard feature of LLR policy, namely the principle of "constructive ambiguity." This is taken to mean that LLRs do not typically spell out beforehand the procedural and practical details of their policy. One possible rationalization of this principle is based on the idea of the LLR committing to a mixed strategy; see

[28] As forcefully argued by Corrigan (1991, 49–50), "I think it is sheer fantasy to assume that individual investors and depositors—and perhaps even large and relatively sophisticated investors and depositors—can make truly informed credit judgements about highly complex financial instruments and institutions."

[29] See Prat (2003) for a detailed discussion of the related distinction between signals on actions and signals on the consequences of actions.

[30] This is, for example, the assumption made in the recent work of Rochet and Vives (2004) on the LLR.

Freixas (1999). Our model supports a different story, suggested by Goodfriend and Lacker (1999), according to which the policy is not random from the perspective of the LLR, but it is perceived as such by outsiders that cannot observe the supervisory information on the basis of which decisions are made. Thus, the randomness lies in the supervisory information, not in the policy rule.

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# Learning about Monetary Policy Rules when Long-Horizon Expectations Matter*

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This paper considers the implications of an important source of model misspecification for the design of monetary policy rules: the assumed manner of expectations formation. In the model considered here, private agents seek to maximize their objectives subject to standard constraints and the restriction of using an econometric model to make inferences about future uncertainty. Because agents solve a multiperiod decision problem, their actions depend on forecasts of macroeconomic conditions many periods into the future, unlike the analysis of Bullard and Mitra (2002) and Evans and Honkapohja (2002). A Taylor rule ensures convergence to the rational expectations equilibrium associated with this policy if the so-called Taylor principle is satisfied. This suggests the Taylor rule to be desirable from the point of view of eliminating instability due to self-fulfilling expectations.

JEL Codes: E52, D83, D84.

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Over the past decade, monetary policy theory and central banking practice have underscored various desiderata for judicious policy. It is often argued that social welfare can be improved by arranging for the central bank to conduct monetary policy according to a suitably chosen instrument rule, dictating how interest rates should be adjusted in response to particular disturbances to the economy. The discussions of Clarida, Galí, and Gertler (1999) and Woodford (2003) present a coherent theory of monetary policy and make the case for such rules. In practice, however, monetary policy contends with many difficulties. Among these, the absence of a correctly specified model of the economy with which to formulate policy is paramount.

This paper considers a potentially important source of model misspecification in the design of instrument rules: the assumed manner in which expectations are formed. The motivation is two-fold. First, even if rational expectations provide a reasonably accurate description of economic agents' behavior, a prudent policy should be robust to small deviations from rationality. Given two policies that both implement a particular desired equilibrium, the policy that results in this equilibrium under more general assumptions on expectations formation is presumably preferred.

Second, some have argued that policies that appear to be desirable, because they are consistent with a desirable equilibrium, will almost surely have disastrous consequences in practice, by allowing self-fulfilling expectations to propagate. For example, Friedman (1968) argued that a monetary policy aimed at pegging the nominal interest rate would inevitably lead to economic instability via a Wicksellian cumulative process. Moreover, he argued that due to small implementation errors, this would occur even if the nominal interest rate target was optimally chosen. The argument proceeds as follows: Suppose the monetary authority pegs the nominal interest rate below the natural rate of interest. This policy would give rise to expectations of future inflation, with the resulting lower real rate of interest tending to stimulate output and prices. Such price rises would engender expectations of further price inflation, in turn further lowering the expected real rate and so on—generating self-fulfilling expectations of ever higher inflation. This paper seeks to build on the research of Howitt (1992) by providing a formal analysis of such self-fulfilling expectations in the context of a model with optimizing behavior. Unlike the analysis of Howitt, this paper postulates a framework where agents optimally make forecasts of macroeconomic conditions many periods into the future when making current decisions.

The rational expectations paradigm comprises two stipulations: (1) agents optimize given their beliefs about the joint probability distribution for various state variables that are independent of their actions and that matter for their payoffs and (2) the probabilities that they assign coincide with the predictions of the model. Following a considerable literature on learning (see Sargent [1993] and Evans and Honkapohja [2001] for reviews), this paper retains the first stipulation while replacing the second with the assumption that the joint probabilities are formed using an econometric model. The predictions of this econometric model need not coincide with the predictions of the theoretical model. The central question posed by the analysis is whether, given sufficient data, the predictions of the econometric model eventually converge to those of the economic model.

Having departed from the rational expectations paradigm, some care must be taken in specifying an individual agent's knowledge. Agents are assumed to know what they need to know to behave according to the first stipulation above: they know their own preferences and constraints, and, more generally, they correctly understand the mapping from their actions to their expected payoff, given a probability distribution for the variables that are outside their control. However, they are not assumed to know anything of the true economic model of how those variables outside of their control are determined. For instance, they do not know that other agents have preferences just like their own and that agents form expectations the way that they do, even if these things are true within the model. It therefore is not appropriate to assume agents use knowledge that other agents' consumption decisions satisfy a subjective Euler equation (for example) in deciding what to do themselves. This has the crucial implication that agents have to make long-horizon forecasts in the framework proposed by this paper.

Recent work by Bullard and Mitra (2002) and Evans and Honkapohja (2003) is similarly motivated. These authors, however, assume a log-linear model of the monetary transmission mechanism in which agents need only forecast inflation and aggregate income one period in advance. In contrast, this paper assumes that agents face a multiperiod decision problem, as in the microfoundations used in recent analysis of the implications of monetary policy rules under rational expectations (see Bernanke and Woodford 1997; Clarida, Galí, and Gertler 1999; and Woodford 1999). This paper demonstrates that the aggregation of rationally modeled decisions, when these decisions are based on subjective expectations, does not predict the aggregate dynamics that depend only on forecasts a single period in the future, even though the aggregate dynamics under rational expectations can be described in that way. In fact, in making current decisions about spending and pricing of their output, agents must make forecasts of macroeconomic conditions many periods into the future. This prediction is a direct result of agents not being able to base their decisions on knowledge of the actions of other agents in the economy. The central methodological contribution of this paper is demonstrating that long-horizon forecasts matter in the determination of current economic conditions in a simple model of output gap and inflation determination with subjective expectations. As such, it builds on the work of Marcet and Sargent (1989), which shows that the optimal decision rule in a partial equilibrium model of investment determination necessarily depends on long-horizon forecasts.

In the model proposed here, learning occurs in the following manner. Agents conjecture the form of the equilibrium dynamics of state variables and estimate an econometric model of this form. This econometric model describes the agents' perceived law of motion. The estimated model is then used to evaluate forecasts of the future paths of state variables that are exogenous to private agents' decision problems. These forecasts, in conjunction with agents' optimal decision rules, can be solved to provide a solution for the actual path of aggregate variables as a function of the current state. This is the actual law of motion. Each period, this process is repeated as additional data become available. A principal focus of the analysis is the manner in which agents update their decision rules, and whether additional data lead them to adopt perceived laws of motion that are closer to the actual laws of motion of the economy. In particular, do agents learn the rational expectations dynamics over time?

The criterion by which this paper judges convergence of learning dynamics to rational expectations dynamics is the notion of expectational stability, or E-stability, proposed by Evans and Honkapohja (2001). Given the requirements of E-stability and the aggregate economic dynamics implied by the model's microfoundations, the analysis considers the implications of learning for several standard prescriptions for monetary policy—specifically, whether certain policy rules are able to ensure least-squares convergence to the associated rational expectations dynamics.

In this paper, monetary policy is specified as a commitment to one of two classes of state-contingent instrument rules: (1) nominal interest-rate rules that depend only on the history of exogenous disturbances and (2) Taylor rules that specify a path for the nominal interest rate that depends on the model's endogenous variables. The former class of rule is of considerable interest, as it has been argued to be a natural way to implement optimal monetary policy, by specifying the optimal action in each possible state of the world. However, such rules, which include nominal interest-rate pegs as a special case, are subject to the critique of Friedman (1968) and also Sargent and Wallace (1975), who showed that commitment to exogenously determined interest-rate paths can lead to multiple rational expectations equilibria. The latter feedback rules, introduced by Taylor (1993), have been used in monetary policy both as a prescriptive and descriptive tool. As initially demonstrated by McCallum (1983), interest-rate rules that possess sufficient feedback from endogenous variables can often deliver a determinate equilibrium. In the present model under rational expectations, Woodford (2003, chap. 4) shows that a Taylor rule leads to a determinate equilibrium if the so-called Taylor principle is satisfied.

Two main results emerge from the analysis of learning dynamics. First, interest-rate rules that are specified as depending only on the history of exogenous disturbances are not expectationally stable under learning dynamics. Such rules are therefore subject to self-fulfilling expectations, consistent with the concerns of Friedman (1968). This, combined with the indeterminacy of rational expectations equilibrium of this class of policy rule, suggests such rules to be ineffective in eliminating economic instability due to self-fulfilling expectations, and therefore undesirable as a means to implement optimal monetary policy. Second, for the Taylor rule, expectational stability hinges critically on satisfaction of the so-called Taylor principle (which stipulates that feedback from endogenous variables to nominal interest rates be sufficiently strong to ensure that increases in inflation be associated with increases in the real interest rate). These findings are invariant to the nature of learning dynamics considered and suggest the Taylor principle to be a remarkably robust feature of the policy environment in the context of this model.

The analysis of this paper is most closely related to the work of Bullard and Mitra (2002) and Evans and Honkapohja (2003), who analyze a log-linear model of the monetary transmission mechanism, where agents forecast inflation and aggregate spending one period in advance.[1] The latter show that instrument rules that require the nominal interest rate to respond only to the history of exogenous disturbances are not stable under learning dynamics. Moreover, they show how to implement the optimal rational expectations equilibrium when the monetary authority is constrained to be a discretionary optimizer and that this equilibrium is also E-stable. Bullard and Mitra (2002) show in the same model that for a monetary authority that is assumed to be able to commit to a number of Taylor-type interest-rate rules, the associated rational expectations equilibrium is E-stable under learning dynamics so long as the Taylor principle is satisfied. That these findings concur with the results of this paper is not necessarily to be expected. The presence of long-horizon forecasts in the present paper gives rise to dynamics that are distinct from those predicted by these analyses.

The paper proceeds as follows. Section 1 sketches the microfoundations of a simple dynamic stochastic general equilibrium model under a general assumption on expectations, provides commentary on the irreducibility of long-horizon forecasts, and highlights some attractive features of the framework proposed to model learning dynamics. Section 2 develops the expectations formation mechanism adopted in this paper. Section 3 discusses the notion of expectational stability and provides a simple abstract example of learning analysis. Section 4 considers the robustness of some common prescriptions for monetary policy to the presence of learning dynamics. The final section concludes.

## 1. The Framework

To develop a framework suitable for the analysis of monetary policy under alternative assumptions on expectations formation, we

[1] See also Bullard and Mitra (2000), Evans and Honkapohja (2002), Honkapohja and Mitra (2004), and Honkapohja and Mitra (2005) for further analyses of issues in monetary policy under learning dynamics in the same framework.

make use of a simple dynamic stochastic general equilibrium model with microfoundations found in Clarida, Galí, and Gertler (1999) and Woodford (2003). To simplify the exposition, the analysis abstracts from monetary frictions that would allow money to be held despite being dominated in rate of return, as in the "cashless" baseline model developed in Woodford (2003, chap. 2). The model is developed in several steps. The household's intertemporal allocation problem is considered, followed by the firm's optimal pricing problem. The implications of the assumed expectations formation mechanism for monetary policy are then explored.

### *1.1 Household's Intertemporal Problem*

The economy is populated by a continuum of households that seek to maximize future expected discounted utility

$$\hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[ U(C_T^i; \xi_T) - \int_0^1 v(h_T^i(j); \xi_T) dj \right], \tag{1}$$

where utility depends on a consumption index, $C_t^i$, of the economy's available goods (to be specified); a vector of aggregate preference shocks, $\xi_t$; and the amount of labor supplied for the production of each good $j$, $h^i(j)$. The second term in the brackets therefore captures the total disutility of labor supply. The consumption index, $C_t^i$, is the Dixit-Stiglitz constant-elasticity-of-substitution aggregator of the economy's available goods and has an associated price index written, respectively, as

$$C_t^i \equiv \left[ \int_0^1 c_t^i(j)^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} \quad \text{and} \quad P_t \equiv \left[ \int_0^1 p_t(j)^{1-\theta} dj \right]^{\frac{1}{1-\theta}},$$

where $\theta > 1$ is the elasticity of substitution between any two goods, and $c_t^i(j)$ and $p_t(j)$ denote household $i$'s consumption and the price of good $j$. The absence of real money balances from the period utility function (1) reflects the assumption that there are no transaction frictions that can be mitigated by holding money balances. However, agents may nonetheless choose to hold money if it provides comparable returns to other available financial assets.

$\hat{E}_t^i$ denotes the subjective beliefs of household $i$ about the probability distribution of the model's state variables—that is, variables that are beyond agents' control, though relevant to their decision problems. The presence of a hat, "ˆ", denotes nonrational expectations, and the special case of rational expectations will be denoted by the usual notation, $E_t$. Beliefs are assumed to be homogenous across households for the purposes of this paper (though this is not understood to be the case by agents) and to satisfy standard probability laws so that $\hat{E}_t^i \hat{E}_{t+1}^i = \hat{E}_t^i$. In forming beliefs about future events, agents do not take into account that they will update their own beliefs in subsequent periods, and this is the source of nonrational behavior in this model. However, when households solve their decision problem at time $t$, beliefs held at that time satisfy standard probability laws, so that standard solution methods apply. The specific details of beliefs and the manner in which agents update beliefs are developed in section 2. The discount factor is assumed to satisfy $0 < \beta < 1$. The function $U(C_t; \xi_t)$ is concave in $C_t$ for a given value of $\xi_t$, and $v(h_t(i); \xi_t)$ is convex in $h_t(i)$ for a given value of $\xi_t$.

Asset markets are assumed to be *incomplete*: there is a single one-period riskless nonmonetary asset available to transfer wealth intertemporally. Under this assumption, the household's flow budget constraint can be written as

$$M_t^i + B_t^i \leq \left(1 + i_{t-1}^m\right) M_{t-1}^i + \left(1 + i_{t-1}\right) B_{t-1}^i + P_t Y_t^i - T_t - P_t C_t^i, \quad (2)$$

where $M_t^i$ denotes the household's end-of-period holdings of money, $B_t^i$ denotes the household's end-of-period nominal holdings of riskless bonds, $i_t^m$ and $i_t$ are the nominal interest rates paid on money balances and bonds held at the end of period $t$, $Y_t^i$ is the period income (real) of households, and $T_t$ denotes lump sum taxes and transfers. The household receives income in the form of wages paid, $w(j)$, for labor supplied in the production of each good, $j$. Furthermore, all households $i$ are assumed to own an equal part of each firm and therefore receive a common share of profits $\Pi_t(j)$ from the sale of each firm's good $j$ (though agents do not know this to be true). Period nominal income is therefore determined as

$$P_t Y_t^i = \int_0^1 [w_t(j) h_t^i(j) + \Pi_t(j)] dj$$

for each household $i$. The flow budget constraint indicates that financial assets at the end of period $t$ can be no more than the value of assets brought into this period, plus nonfinancial income after taxes and consumption spending. This constraint must hold in all future dates and states of uncertainty. Fiscal policy is assumed to be Ricardian so that goods prices, asset prices, and output are determined independently of fiscal variables.[2] It will be assumed that the fiscal authority pursues a zero-debt policy, so that bonds are in zero net supply.

To summarize, the household's problem in each period $t$ is to choose $\{c_t^i(j), h_t^i(j), M_t^i, B_t^i\}$ for all $j \in [0,1]$ so as to maximize (1) subject to the constraint (2), taking as parametric the variables $\{p_T(j),\ w_T(j),\ \Pi_T,\ i_{T-1},\ i_{T-1}^m,\ \xi_T\}$ for $T \geq t$. The first-order conditions characterizing the solution to this optimization problem are detailed in appendix 1.

### *1.1.1 A Consumption Rule Derived*

In order to derive a linear decision rule describing the household's optimal intertemporal allocation of consumption, a log-linear approximation to the household's first-order conditions is employed. Appendix 1 shows that a log-linear approximation to the household's Euler equation and the intertemporal budget constraint imply the relations

$$\hat{C}_t^i = \hat{E}_t^i \hat{C}_{t+1}^i - \sigma(\hat{\imath}_t - \hat{E}_t^i \hat{\pi}_{t+1}) + g_t - \hat{E}_t^i g_{t+1} \tag{3}$$

and

$$\hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \hat{C}_T^i = \varpi_t^i + \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \hat{Y}_T^i, \tag{4}$$

where $\sigma \equiv -U_c/(U_{cc}\bar{C})$ is the intertemporal elasticity of substitution, $g_t \equiv \sigma U_{c\xi}\xi_t/U_c$, and where for any variable $z_t$, $\hat{z}_t \equiv \ln(z_t/\bar{z})$ denotes the log deviation of the variable from its steady-state value, $\bar{z}$, defined in appendix 1. $\varpi_t^i \equiv W_t^i/(P_t\bar{Y})$ is the share of the household's real wealth as a fraction of steady-state income, where $W_t^i \equiv (1+i_{t-1})\,B_{t-1}^i$. Solving (3) backwards recursively from date $T$ to date $t$ and taking expectations at that time gives

[2] Preston (2002) considers the fiscal theory of the price level under learning dynamics.

$$\hat{E}_t^i \hat{C}_T^i = \hat{C}_t^i - g_t + \hat{E}_t^i \left[ g_T + \sigma \sum_{T=t}^{T-1} (\hat{\imath}_t - \hat{\pi}_{t+1}) \right],$$

which on substitution into the intertemporal budget constraint yields

$$\hat{C}_t^i = (1-\beta)\,\varpi_t^i + \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \times \left[ (1-\beta)\hat{Y}_T^i - \beta\sigma(\hat{\imath}_T - \hat{\pi}_{T+1}) + \beta(g_T - g_{T+1}) \right] \quad (5)$$

as the desired decision rule: it describes optimal behavior given arbitrary beliefs (so long as such beliefs satisfy standard probability laws). It follows that households necessarily make long-horizon forecasts of macroeconomic conditions to determine their optimal current consumption choice. Consumption varies across households according to differences in wealth and income. Section 5 discusses why optimizing agents necessarily make decisions according to (5), rather than just making use of the Euler equation (3) as has been assumed in the recent literature.

It is useful to contrast this derived decision rule to the predicted consumption allocation under the permanent income hypothesis. Indeed, the first two terms capture precisely the basic insight of the permanent income hypothesis that agents should consume a constant fraction of the expected future discounted wealth, given a constant real interest rate equal to $\beta^{-1} - 1$. The third term arises from the assumption of a time-varying real interest rate, and represents deviations from this constant real rate due to either variation in the nominal interest rate or inflation. The final term results from allowing stochastic disturbances to the economy.

To determine aggregate behavior, integrate over $i$ to give

$$\hat{C}_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[ (1-\beta)\hat{Y}_T - \beta\sigma(\hat{\imath}_T - \hat{\pi}_{T+1}) + \beta(g_T - g_{T+1}) \right],$$

using the fact that $\int_i \varpi_t^i di = 0$ from market clearing (bonds are in zero net supply) and introducing the notation $\int_i z_t^i di = z_t$ for any variable $z$ and, specifically, $\int_i \hat{E}_t^i di = \hat{E}_t$ to define the average expectations operator. (In aggregating, we have made use of the equilibrium property that all agents will receive the same wage for each type of labor supplied. Since all agents hold the same diversified portfolio of firm profits, it is necessarily true that $\hat{Y}_t^i = \hat{Y}_t^j$ for all $i$, $j$ and we call this common income stream $\hat{Y}_t$.) It is important to note that the expectations operator, $\hat{E}_t$, possesses no behavioral content, and simply defines the average expectations of a distribution of agents in the economy. That this is true follows immediately from the assumed knowledge of agents: they do not know the tastes and beliefs of other agents in the economy and therefore do not have a complete economic model with which to infer the true aggregate probability laws and how state variables beyond their control are determined.

Since equilibrium requires $\hat{C}_t = \hat{Y}_t$, the aggregation of household decision rules can be written in terms of the output gap, $x_t \equiv \hat{Y}_t - \hat{Y}_t^n$, to give

$$x_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)x_{T+1} - \sigma(\hat{\imath}_T - \hat{\pi}_{T+1}) + r_T^n\right], \qquad (6)$$

where $\hat{Y}_t^n$ is the natural rate of output (to be defined) and $r_t^n \equiv (\hat{Y}_{t+1}^n - g_{t+1}) - (\hat{Y}_t^n - g_t)$ is a composite of exogenous disturbances. The current output gap is therefore determined by the current nominal interest rate and exogenous disturbance and the average of households' long-horizon forecasts of both these variables and also output and inflation into the indefinite future.

### 1.2 Optimal Price Setting

Now consider the firm's problem, again relegating details to the appendix. Calvo price setting is assumed so that a fraction $0 < \alpha < 1$ of goods prices are held fixed in any given period, while a fraction $1 - \alpha$ of goods prices are adjusted. Given homogeneity of beliefs, all firms having the opportunity to change their price in period $t$ face the same decision problem and therefore set a common price $p_t^*$. The Dixit-Stiglitz aggregate price index must therefore evolve according to the relation

$$P_t = \left[\alpha P_{t-1}^{1-\theta} + (1-\alpha)\, p_t^{*1-\theta}\right]^{\frac{1}{1-\theta}}. \qquad (7)$$

Firms setting prices in period $t$ face a demand curve $y_t(i) = Y_t(p_t(i)/P_t)^{-\theta}$ for their good and take aggregate output $Y_t$ and aggregate prices $P_t$ as parametric. Good $i$ is produced using a single labor input $h(i)$ according to the relation $y_t(i) = A_t f(h_t(i))$, where $A_t$ is an exogenous technology shock and the function $f(\cdot)$ satisfies the standard Inada conditions.

When setting prices in period $t$, firms are assumed to value future streams of income at the marginal value of aggregate income in terms of the marginal value of an additional unit of aggregate income today. That is, a unit of income in each state and date $T$ is valued by the stochastic discount factor

$$Q_{t,T} = \beta^{T-t} \cdot \frac{P_t}{P_T} \cdot \frac{U_c(Y_T, \xi_T)}{U_c(Y_t; \xi_t)}.$$

This simplifying assumption is appealing in the context of the symmetric equilibrium that is examined in this model. Since all agents are assumed to have common beliefs and tastes, and because all households are assumed to own an equal share of firm profits, it follows that in equilibrium each receives a common income stream that is necessarily equal to aggregate income. Having firms value future profits at the marginal value of aggregate income therefore corresponds to each shareholder's valuation.[3]

The firm's price-setting problem in period $t$ is therefore to maximize the expected present discounted value of profits

$$\hat{E}_t^i \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left[\Pi_T^i \left(p_t(i)\right)\right], \tag{8}$$

where

$$\Pi_T^i (p) = Y_t P_t^{\theta} p^{1-\theta} - w_t(i) f^{-1}(Y_t P_t^{\theta} p^{-\theta}/A_t), \tag{9}$$

with the notation $f^{-1}(\cdot)$ denoting the inverse function of $f(\cdot)$. The factor $\alpha^{T-t}$ in the firm's objective function is the probability that

[3]This assumption is not particularly important. In the employed log-linear approximation, firms only use knowledge of the long-run average value of $Q_{t,T}$, which equals the discount factor $\beta$. So long as firms know $\beta$, any number of assumptions on the price-setting behavior of firms would be consistent with the presented analysis. Firms could hold different beliefs about fluctuations in $Q_{t,T}$ so long as they all know the long-run average to be equal to $\beta$.

the firm will not be able to adjust its price for the next $(T-t)$ periods.

To summarize, the firm's problem is to choose $\{p_t(i)\}$ to maximize (8), taking as given the variables $\{Y_T,\ P_T,\ w_T(j),\ A_T,\ Q_{t,T}\}$ for $T \geq t$ and $j \in [0,1]$. The first-order conditions characterizing optimality are contained in appendix 2.

#### *1.2.1 Price Decision Rule Derived*

As for the household problem, we seek a log-linear approximation to firms' price-setting behavior. Appendix 2 demonstrates that the first-order condition of the firm's optimal pricing problem satisfies the approximate log-linear relation

$$\widehat{p}_t^*(i) = \hat{E}_t^i \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[ \frac{1-\alpha\beta}{1+\omega\theta} \cdot (\omega + \sigma^{-1}) x_T + \alpha\beta\hat{\pi}_{T+1} \right], \quad (10)$$

where $\omega > 0$ is the elasticity of firm $i$'s real marginal cost function (defined in the appendix) with respect to its own output, $y_t(i)$. Thus firm $i$'s optimal price is determined as a linear function of the future expected paths of the output gap and inflation. Analogously to the household's problem, firms optimally make long-horizon forecasts of general macroeconomic conditions in deciding their current price, $p_t^*(i)$.

To infer the aggregate implications of the maintained theory of pricing, integrate (10) over $i$ to give

$$\widehat{p}_t^* = \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[ \frac{1-\alpha\beta}{1+\omega\theta} \cdot (\omega + \sigma^{-1}) x_T + \alpha\beta\hat{\pi}_{T+1} \right].$$

Noting that a log-linear approximation to the price index (7) gives $\hat{\pi}_t = \widehat{p}_t^* \cdot (1-\alpha)/\alpha$, the above expression can be written as

$$\hat{\pi}_t = \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[ \kappa\alpha\beta \cdot x_{T+1} + (1-\alpha)\beta\hat{\pi}_{T+1} \right], \quad (11)$$

where

$$\kappa \equiv \frac{(1-\alpha)}{\alpha} \frac{1-\alpha\beta}{1+\omega\theta} (\omega + \sigma^{-1}) > 0.$$

Equation (11) indicates that current inflation is determined by today's output gap and the average of firms' expectations of the future time path of both the output gap and the inflation rate. As for the households' problem, since private agents do not know the tastes and beliefs of others and therefore are unable to infer the true aggregate probability laws, this relation cannot be quasi-differenced to deliver a relationship between current inflation and expectations of next period's inflation rate. To simplify notation, for the remainder of the paper the "ˆ" is omitted, with the understanding that all variables are defined as log deviations from steady-state values.

It is worth noting that the foregoing methodology is not specific to the model at hand. Different theories of price setting or consumer behavior could be adopted. For instance, perfect competition could be assumed to give fully flexible prices. Alternatively, it could be assumed that some fraction $\gamma$ of firms have flexible prices, while a fraction $1 - \gamma$ set prices a period in advance. This would give the Lucas supply curve, as shown by Woodford (2003, chap. 3). Long-horizon forecasts do not matter under these theories of pricing because firms do not face a multiperiod decision problem—they are static and two-period problems, respectively. Assuming Calvo pricing is a tractable way to develop a minimally realistic model for the analysis of monetary policy and facilitates comparison to the recent literature on monetary policy and learning. It is an open question whether other, possibly more realistic theories of pricing have important implications for monetary policy under learning dynamics.

### *1.3 The Irreducibility of Long-Horizon Forecasts*

A number of previous papers have proposed analyses of learning dynamics in the context of models where agents solve multiperiod (indeed, infinite horizon) decision problems, but without requiring that agents make forecasts regarding outcomes more than one period in the future. In these papers, agents' decisions depend only on forecasts of future variables that appear in the Euler equations that can be used to characterize rational expectations equilibrium. For example, Bullard and Mitra (2002) propose an analysis of learning dynamics in a model that is intended to have the same underlying microfoundations as the model presented above—that is, intended to consider the consequences of least-squares learning in the context of the standard New Keynesian model of inflation and output gap determination. However, section 1 demonstrates that under learning dynamics, private-sector optimization implies the aggregate structural relations (6) and (11) so that long-horizon expectations of general macroeconomic conditions matter for the evolution of aggregate output and inflation.

Since these relations hold for arbitrary beliefs satisfying standard probability laws, they must also hold under rational expectations. Under this assumption, (6) and (11) can be simplified by application of the law of iterated expectations, as agents—having complete knowledge of the tastes and beliefs of other agents—are able to compute the equilibrium probabilities and associated laws, ensuring that individual beliefs coincide with the aggregate probability laws implied by the economic model. Leading the aggregate demand relation (6) one period and taking rational expectations at date $t$ gives

$$\begin{aligned} E_t x_{t+1} &= E_t E_{t+1} \sum_{T=t+1}^{\infty} \beta^{T-t-1} \left[ (1-\beta) x_{T+1} - \sigma(i_T - \pi_{T+1}) + r_T^n \right] \\ &= E_t \sum_{T=t+1}^{\infty} \beta^{T-t-1} \left[ (1-\beta) x_{T+1} - \sigma(i_T - \pi_{T+1}) + r_T^n \right], \end{aligned} \tag{12}$$

where the second equality follows from the law of iterated expectations. It follows that

$$\begin{aligned} x_t &= E_t[(1-\beta)x_{t+1} - \sigma(i_t - \pi_{t+1}) + r_t^n] + \\ &\quad E_t \sum_{T=t+1}^{\infty} \beta^{T-t} \left[ (1-\beta) x_{T+1} - \sigma(\beta \cdot i_T - \pi_{T+1}) + r_T^n \right] \\ &= E_t[(1-\beta)x_{t+1} - \sigma(i_t - \pi_{t+1}) + r_t^n] + \beta E_t x_{t+1} \\ &= E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1}) + r_t^n, \end{aligned}$$

where the second equality makes use of (12). Similar manipulations for the Phillips curve relation give the rational expectations model of the monetary transmission mechanism

$$x_t = E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1}) + r_t^n$$
$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1}.$$

This simple model of the economy has been used in recent studies of monetary policy rules by Bernanke and Woodford (1997), Clarida, Galí, and Gertler (1999), and Woodford (1999). A rational expectations equilibrium analysis therefore predicts that only one-period-ahead forecasts of inflation and the output gap matter for the evolution of the economy. The approach of Bullard and Mitra (2002) is to take these relations and replace the rational expectations assumption with the learning assumption outlined in section 2. This gives the system

$$x_t = \hat{E}_t x_{t+1} - \sigma(i_t - \hat{E}_t \pi_{t+1}) + r_t^n \tag{13}$$
$$\pi_t = \kappa x_t + \beta \hat{E}_t \pi_{t+1}, \tag{14}$$

obtained by substituting the rational expectations operator, $E_t$, with the learning dynamics operator, $\hat{E}_t$. But the system (13)–(14) is not equivalent to the model consisting of equations (6) and (11) under most possible specifications of subjective expectations.

The proposed learning procedure has the advantage that if the econometric model used by agents to produce forecasts is correctly specified, then the resulting behavior is *asymptotically* optimal. That is, behavior under the learning algorithm differs from what would be optimal behavior under the true probability laws by an amount that is eventually arbitrarily small. For the examined monetary policies, a correctly specified econometric model posits inflation, output, and the nominal interest rate to be linear functions of the lagged natural rate disturbance, with a residual term orthogonal to the natural rate. The consistency of the ordinary least squares estimator implies that the coefficients that agents use in forming their beliefs are eventually close to the true coefficients. Since the optimal decision rule is a continuous function of the coefficients of the agents' forecasting rule, beliefs that are arbitrarily close to the correct ones imply behavior that is arbitrarily close to being optimal.

In general, this is not a property of the Euler equation approach. To make this clear, recall that the optimal decision rule is given by

$$\tilde{C}_t^i = (1-\beta)\,\varpi_t^i + \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)x_T - \beta\sigma(i_T - \pi_{T+1}) + \beta r_T^n\right],$$

where we have defined $\tilde{C}_t^i \equiv C_t^i - Y_t^n$. Agents having a positive initial wealth endowment, $\varpi_t^i > 0$, will have higher than average consumption (given that the income process is the same for all agents in the equilibrium described in section 1), while those having a negative initial wealth endowment, $\varpi_t^i < 0$, will have lower than average consumption.

It is immediate, then, that using the Euler equation alone cannot lead individual households to make the optimal consumption allocation each period, since it does not lead them to take account of their wealth in any way whatsoever. Suppose we interpret the Euler equation (13) as saying that household $i$ forecasts the aggregate output gap, $x_{t+1}$, then bases its consumption decision $\tilde{C}_t^i$ on this, so that

$$\tilde{C}_t^i = \hat{E}_t^i x_{t+1} - \sigma(i_t - \hat{E}_t^i \pi_{t+1}) + r_t^n \tag{15}$$

describes household behavior. Such a procedure will lead to systematic underconsumption by households with $\varpi_t^i > 0$ and overconsumption of those households with $\varpi_t^i < 0$. If the Euler equation approach is instead interpreted as saying that the household forecasts its own future consumption $\tilde{C}_{t+1}^i$ (based on the past time series of own consumption spending) and then bases current consumption on this, we have

$$\tilde{C}_t^i = \hat{E}_t^i \tilde{C}_{t+1}^i - \sigma(i_t - \hat{E}_t^i \pi_{t+1}) + r_t^n, \tag{16}$$

and similar conclusions present themselves.[4]

Such suboptimal behavior is a manifestation of the following more general point: forecasting $\hat{E}_t^i \tilde{C}_{t+1}^i$ as $\hat{E}_t^i x_{t+1}$ is internally inconsistent with household optimization. It represents a forecast of the agent's own future decision that differs from what it expects to be optimal given its current forecasts of future income, inflation, and interest rates and given the agent's understanding of its own decision rule. Moreover, if agents have internally consistent beliefs, such forecasts would differ from what the agent should now be forecasting

[4]Note that under this proposed learning mechanism, the interpretation of $\hat{E}_t^i$ is distinct from the behavior postulated in this paper: agents, rather than forecasting future state variables that are beyond their control though pertinent to their decision problem, adopt a pure statistical model of their own future consumption choice—expectations are not taken with respect to the probability distribution induced by the optimal decision rule and beliefs about exogenous state variables.

about their own period $t+1$ forecasts. Forecasts of this kind therefore represent a less sophisticated approach to forecasting, because they fail to make use of information that the agent necessarily possesses. The model of learning proposed in this paper induces a more sophisticated approach to forecasting that ensures consistency among the various things that the agent is assumed to simultaneously believe.

As an example, consider the model in the case of a zero initial wealth endowment. The optimal decision rule is

$$\tilde{C}_t^i = \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[ (1-\beta) x_T - \beta\sigma(i_T - \pi_{T+1}) + \beta r_T^n \right].$$

Since in the optimal program this rule governs consumption decisions in all future periods, it follows that households expect next period's optimal consumption choice to be

$$\hat{E}_t^i \tilde{C}_{t+1}^i = \hat{E}_t^i \sum_{T=t+1}^{\infty} \beta^{T-t} \left[ (1-\beta) x_T - \beta\sigma(i_T - \pi_{T+1}) + \beta r_T^n \right], \quad (17)$$

obtained by forwarding the optimal decision rule one period and taking expectations at time $t$. It follows immediately that for the Euler equation to provide the optimal consumption allocation, under the interpretations in the preceding paragraph given to (15) or (16) above, $\hat{E}_t^i x_{t+1}$ and $\hat{E}_t^i \tilde{C}_{t+1}$, respectively, must coincide with this optimal forecast given by (17). But in general, there is no reason for forecasts of $x_{t+1}$ and $\tilde{C}_{t+1}^i$ constructed from regression of past observations of these variables on observed aggregate disturbances to coincide with (17). The optimal forecast is a particular linear combination of forecasts of the state variables relevant to the household's decision problem. Importantly, such forecasts will only coincide in a rational expectations equilibrium; that is, when agents know the true probability laws—the very laws agents are attempting to learn. Thus consumption decisions made according to either Euler equation (15) or (16) lead to suboptimal behavior.

Honkapohja, Mitra, and Evans (2002) argue that such Euler equations can be derived from the framework of this paper (in the case of zero initial wealth endowments) with the additional assumption that agents understand that market clearing requires $\tilde{C}_t^i = x_t$

(or $C_t^i = Y_t$) in all periods. While this assumption might appear appealing in the context of a model with homogenous agents that are constrained in equilibrium to consume identical incomes, more generally it lacks appeal on two grounds. First, market-clearing conditions are part of the set of rational expectations equilibrium restrictions that agents are attempting to learn—why are they any more likely to be endowed with knowledge of one restriction over another? This will be particularly important in more general models when agents receive differing income streams and have incentives to trade assets in equilibrium. Second, even if it is assumed that agents are aware of this market-clearing condition, so that the Euler equation of the form (15) can be derived, such a decision rule does not describe optimal behavior: households would never choose to adopt such a learning rule given the maintained assumption that agents optimize conditional on their beliefs.

## 2. Expectations Formation

The previous section derives the aggregate implications of household and firm behavior. Equation (6) specifies the evolution of aggregate demand, while equation (11) is analogous to a forward-looking Phillips curve determining current inflation as a function of expected future inflation and the output gap. To close this stylized model of the macroeconomy, assumptions on the expectations formation mechanism and the nature of monetary policy—which determines the evolution of the nominal interest rate $\{i_t\}$—are required. Given expectations, so long as monetary policy is specified as being determined by the model's exogenous variable and/or permitted to depend only on the endogenous variables, inflation, and the output gap (including future expected and past values), then this equation together with (6) and (11) is sufficient to determine $\{\pi_t,\ x_t,\ i_t\}$. The monetary policies considered in this paper satisfy this requirement. It remains to specify the expectations formation mechanism.

### *2.1 Recursive Learning*

To be precise about the learning dynamics of this model, adjoin an equation for the interest rate to equations (6) and (11) to give the system

$$x_t = -\sigma i_t + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)x_{T+1} - \sigma(\beta \cdot i_{T+1} - \pi_{T+1}) + r_T^n\right] \tag{18}$$

$$\pi_t = \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\kappa\alpha\beta \cdot x_{T+1} + (1-\alpha)\beta \cdot \pi_{T+1}\right] \tag{19}$$

$$i_t = i(x, \pi, r^n). \tag{20}$$

The final equation defines a general specification for monetary policy that satisfies the requirements discussed above. Conditional on expectations, there are three equations that determine the three unknown endogenous variables $\{\pi_t,\ x_t,\ i_t\}$. It is clear from equations (18) and (19) that agents require forecasts of the entire future path of each endogenous variable. Agents therefore estimate a linear model in inflation, the output gap, and the nominal interest rate, using as regressors variables that appear in the minimum-state-variable solution to the model under rational expectations. This conjectured model represents agents' beliefs of the equilibrium dynamics of the model's state variables.[5]

Under the classes of monetary policies considered in section 4, the minimum-state-variable solution is always linear in the disturbance term, $r_t^n$. Suppose the natural rate of interest, $r_t^n$, is determined by the stochastic process

$$r_t^n = f' s_t,$$

where

$$s_t = C s_{t-1} + \varepsilon_{s,t} \tag{21}$$

[5]One might query the assumption that agents construct forecasts using just variables that appear in the minimum-state-variable solution. After all, it is clear from the optimizing model developed that agents also observe (simultaneously) aggregate output and prices when making their own decisions about consumption and price setting. It follows that these aggregate variables might be thought useful in constructing forecasts about the future evolution of the economy. This informational assumption leads to the same substantive conclusions on E-stability as the case in which agents do not use this additional information, and results are available from the author. To keep the ideas at the fore and the algebra at bay, the main analysis works under the assumption that forecasts are constructed using only variables that appear in the minimum-state-variable solution.

and $s_t$ is an $(n \times 1)$ vector, $f$ is an $(n \times 1)$ coefficient vector, $\varepsilon_{s,t}$ is an i.i.d disturbance vector, and $C$ is a matrix with all eigenvalues being real and inside the unit circle. Thus the natural rate shock is specified as a fairly arbitrary linear combination of exogenous disturbances. Defining $z_t \equiv (\pi_t, x_t, i_t)'$, the estimated linear model is assumed to be

$$z_t = a_t + b_t \cdot s_t + \epsilon_t,$$

where $\epsilon_t$ is the usual error term and $(a_t, b_t)$ are coefficient vectors of dimension $(3 \times 1)$ and $(3 \times n)$, respectively. The estimation procedure makes use of the entire history of available data in period $t$, $\{z_t, 1, s_t\}_0^{t-1}$. As additional data become available in subsequent periods, agents update their estimates of the coefficients $(a_t, b_t)$. This is neatly represented as the recursive least-squares formulation

$$\phi_t = \phi_{t-1} + t^{-1} R_t^{-1} w_{t-1}(z_{t-1} - \phi'_{t-1} w_{t-1}) \tag{22}$$

$$R_t = R_{t-1} + t^{-1}(w_{t-1} w'_{t-1} - R_{t-1}), \tag{23}$$

where the first equation describes how the forecast coefficients, $\phi_t = (a'_t, vec(b_t)')'$, are updated with each new data point and the second equation describes the evolution of the matrix of second moments of the appropriately stacked regressors $w_t \equiv \{1, s_t\}_0^t$. The forecasts can then be constructed as

$$\hat{E}_t z_T = a_{t-1} + b_{t-1} \cdot C^{T-t} \cdot s_t \tag{24}$$

for $T \geq t$. The matrix $C$ is assumed to be known to agents for algebraic convenience. This is not important to the conclusions of this paper—all results hold when agents have to learn the nature of the autoregressive process describing $s_t$, and the results are available from the author. That agents form beliefs using (24) makes clear their irrationality—at time $t$, agents make use of an econometric model to assign probabilities to the evolution of state variables that does not account for their own subsequent updating of beliefs at $t+1$ by use of (22) and (23). This completes the description of the model.

To summarize, the model of the macroeconomy comprises an aggregate demand equation, (6); a Phillips curve, (11); a monetary policy rule; and the forecasting system (22), (23), and (24).

## 3. Analyzing Learning Dynamics

Subsequent analysis answers two related questions for a given assumption on monetary policy: Under what conditions does a unique rational expectations equilibrium obtain? And, given the existence of such an equilibrium, what conditions guarantee convergence to this equilibrium when agents' expectations are formed using a recursive least-squares algorithm rather than using rational expectations? While analysis of determinacy is now commonplace in the monetary policy literature, the conditions for convergence under least-squares learning dynamics are less familiar.[6] The criterion adopted in this paper to judge convergence under recursive learning is the notion of expectational stability of rational expectations equilibrium, called E-stability by Evans and Honkapohja (2001). Evans and Honkapohja show that local real-time convergence of a broad class of dynamic models under recursive learning is governed by E-stability. The following section draws on Evans and Honkapohja (2001) to develop the ideas of E-stability.

### *3.1 Expectational Stability*

Agents use their econometric model to construct forecasts of the future path of endogenous variables. For expositional purposes, this subsection assumes the evolution of $r_t^n$ is a standard AR(1) process, with coefficient $|\rho| < 1$. If monetary policy is conducted so that the minimum-state-variable solution is linear in $r_t^n$, then forecasts can be constructed using

$$\hat{E}_t z_T = a_{t-1} + b_{t-1} \cdot \rho^{T-t} \cdot r_t^n$$

for $T \geq t$. To obtain the actual evolution of the economy, substitute (24) into the system of equations (18), (19), and (20). Collecting like terms gives a general expression of the form

$$z_t = (Q + Aa_{t-1}) + (Bb_{t-1} + D)\, r_t^n,$$

where the matrices $A$ and $B$ collect coefficients on the estimated parameter vectors $(a_t', b_t')$, $Q$ collects constant terms, and $D$ collects

[6] See Blanchard and Kahn (1980) for a detailed discussion of the conditions for uniqueness of rational expectations equilibrium.

remaining coefficients on the state variable, $r_t^n$. Leading this expression one period and taking expectations (rational) provides

$$E_t z_{t+1} = (Q + Aa_{t-1}) + \rho\,(Bb_{t-1} + D)\,r_t^n,$$

which describes the optimal rational forecast conditional on private-sector behavior. Comparison with (24) makes clear that agents are estimating a misspecified model of the economy—agents assume a stationary model when in fact the true model has time-varying coefficients. Taken together with (24) at $T = t + 1$, it defines a mapping that determines the optimal forecast coefficients given the current private-sector forecast parameters $(a'_{t-1}, b'_{t-1})$, written as

$$T(a_{t-1}, b_{t-1}) = (Q + Aa_{t-1}, Bb_{t-1} + D). \tag{25}$$

A rational expectations equilibrium (REE) is a fixed point of this mapping. For such REE, we are then interested in asking, under what conditions does an economy with learning dynamics converge to this equilibrium? Using stochastic approximation methods, Evans and Honkapohja (2001) show that the conditions for convergence of the learning algorithm (22) and (23) are neatly characterized by the local stability properties of the associated ordinary differential equation

$$\frac{d}{d\tau}(a, b) = T(a, b) - (a, b), \tag{26}$$

where $\tau$ denotes "notional" time. The REE is said to be expectationally stable, or E-stable, if this differential equation is locally stable in the neighborhood of the REE. From standard results for ordinary differential equations, a fixed point is locally asymptotically stable if all eigenvalues of the Jacobian matrix $D\,[T(a, b) - (a, b)]$ have negative real parts (where $D$ denotes the differentiation operator and the Jacobian understood to be evaluated at the rational expectations equilibrium of interest). See Evans and Honkapohja (2001) for further details on expectational stability.

In the context of the above model, the Jacobian matrices are $(A - I)$ and $(B - I)$ and have dimension $(3 \times 3)$ (corresponding to the number of state variables that agents are forecasting). For such matrices to have roots all having negative real parts, the coefficients of the associated characteristic equation must satisfy three restrictions. It follows that E-stability imposes six restrictions on model parameters. Details of these conditions are provided in appendix 3. The remainder of the paper concerns itself with the relationship between the conditions for expectational stability and the requirements for determinacy when monetary policy is specified as a commitment to a variety of interest-rate rules.

## 4. Monetary Policy and Learning

The first part of this paper develops a framework in which agents face multiperiod decision problems and have subjective expectations. It shows that the aggregation of rationally formed decisions of individual agents with such subjective expectations implies that current output and inflation are determined by long-horizon forecasts of general macroeconomic conditions. The remainder of the paper is devoted to the question of whether certain policy rules in such an economy lead the learning dynamics to converge to the dynamics predicted by rational expectations equilibrium analysis—that is, in the language of the previous section, whether given sufficient data agents adopt perceived laws of motion that converge to the actual laws of motion of the economy.

Since Taylor (1993) there has been a revived interest in monetary policy rules, both as a prescriptive and descriptive tool. Taylor proposed a simple rule of the form

$$i_t = \bar{\imath}_t + \psi_\pi \pi_t + \psi_x x_t \tag{27}$$

prescribing the nominal interest rate to be adjusted in response to variations in inflation and in the output gap and $\bar{\imath}_t$ is a stochastic constant.[7] This work and Taylor (1999) provides evidence that a rule of this form gives a remarkably good characterization of U.S. monetary policy from the mid-1980s onward. More generally, some have argued that interest-rate rules should be an integral part of a framework for monetary policy, as such rules provide a possible solution to the pitfalls of discretionary behavior by the central bank: by providing a systematic response to economic shocks, the central bank might be better able to stabilize inflation and output and

[7] The actual rule proposed by Taylor (1993) was $i_t = \pi_t + 0.5(\pi_t - 2) + 0.5x_t$, interpreting $\pi_t$ as the four-quarter-ended inflation rate.

therefore improve social welfare. Furthermore, by specifying the optimal choice of the nominal interest rate in each state of the world, interest-rate rules can, in principle, be designed to implement optimal monetary policy. Clarida, Galí, and Gertler (1999) and Woodford (2003) develop these ideas in considerable detail and present a coherent theory of monetary policy that makes the case for such rules.

However, much of the literature on monetary policy rules regarding the desirability of one rule versus another rests on the assumption of rational expectations. And while rational expectations has obvious appeal as a modeling device, there is good reason to be cautious about policy recommendations derived under its assumption. The model of this paper provides a natural framework to evaluate the desirability of monetary policy rules given alternative assumptions on the expectations formation mechanism. Indeed, the adaptive learning framework has considerable appeal, as it includes the rational expectations paradigm as a special limiting case. It follows that such an expectations formation mechanism presents a minimal deviation from rational expectations and, therefore, that any rules that are found to induce economic instability under its assumption are likely to be undesirable as a recommended policy. Indeed, Howitt (1992), Evans and Honkapohja (2003), and Bullard and Mitra (2002) argue convergence of least-squares learning to the predictions of rational expectations equilibrium analysis to be a minimal requirement of any proposed policy.

The use of interest-rate rules as a means to conduct monetary policy has also been criticized on the grounds that even though a policy is consistent with a desirable equilibrium, it will almost surely have disastrous consequences in practice by allowing for self-fulfilling expectations to propagate. For example, Friedman (1968) argued that any attempt by the monetary authority to peg the nominal interest rate, even at an optimally chosen value, would inevitably lead to economic instability via a cumulative Wicksellian process. Following Howitt (1992), the basic logic of this criticism can be neatly formulated in a model where agents form expectations of the future path of the economy by extrapolating from historical relationships in observed data. The remaining analysis examines the possibility of self-fulfilling expectations when agents must form long-horizon forecasts in order to make current decisions. Thus, given a candidate monetary policy, the central question of interest is whether, given sufficient data, agents with subjective expectations will be able to learn the predictions of rational expectations equilibrium analysis.

### *4.1 Monetary Policy Rules*

Consider two classes of instrument rules: (1) nominal interest-rate rules that depend only on the history of exogenous disturbances and (2) Taylor-type feedback rules that specify a path for the nominal interest rate that depends on the path of endogenous variables. The former class of rule is of considerable interest since specifying the optimal action of the monetary authority in each state of the world is a natural way to implement optimal monetary policy. However, such rules are an example of the type of rule critiqued by Friedman (1968) and have also been criticized by Sargent and Wallace (1975), who showed that commitment to exogenously determined interest-rate paths can lead to indeterminacy of rational expectations equilibria. The possibility of indeterminacy of rational expectations equilibria raises an important challenge for the design of optimal monetary policy as underscored by the work of Svensson and Woodford (2002), Woodford (1999), Giannoni and Woodford (2002a), and Giannoni and Woodford (2002b): even though an optimal interest-rate rule, expressed as a function of the history of exogenous disturbances, can be designed to be consistent with the optimal equilibrium, such rules are also equally consistent with many other undesirable equilibria. In the context of the monetary policy literature under learning, Evans and Honkapohja (2003) have demonstrated an analogous result: such policy rules are in fact subject to self-fulfilling expectations as argued by Friedman.

Importantly, indeterminacy of rational expectations equilibrium is not a general property of interest-rate rules. McCallum (1983) showed that rules that allow appropriate feedback from endogenous variables can deliver a unique equilibrium. As mentioned, a prominent recent example due to Taylor (1993) is given by (27). Woodford (2003, chap. 4) shows that this rule leads to determinacy of rational expectations equilibrium if the so-called Taylor principle is satisfied. This rule will be the central focus of our study of learning dynamics in this economy.

There are clearly many other possible rules for the conduct of monetary policy. Clarida, Galí, and Gertler (1998) and Clarida, Galí, and Gertler (2000) have found that estimated central bank reaction functions often find an important role for expectations of future inflation in the setting of the current interest rate. This suggests rules of the form

$$i_t = \bar{\imath}_t + \psi_x E_t x_{t+1} + \psi_\pi E_t \pi_{t+1} \tag{28}$$

to be of practical interest.[8]

Alternatively, as argued by McCallum (1999), the informational assumptions implicit in the Taylor rule are tenuous in practice. Monetary authorities typically do not have available current-dated observations on the output gap and inflation rate when setting the current interest rate. Many researchers have responded to this criticism by modifying the information set available to the monetary authority when determining its instrument setting. Hence, the nominal interest rate could be argued to be better modeled as being determined by lagged expectations of current-dated output and inflation to give an instrument rule of the form

$$i_t = \bar{\imath}_t + \psi_x E_{t-1} x_t + \psi_\pi E_{t-1} \pi_t. \tag{29}$$

Finally, for a monetary authority concerned with stabilizing variation in output and inflation, the optimal commitment equilibrium in the present model under rational expectations can be shown to be implemented by a rule of the form

$$\begin{aligned} i_t = \frac{1}{\sigma} \Bigg[ & E_t x_{t+1} - \frac{\lambda}{\lambda + \kappa^2} x_{t-1} + \left( \frac{\beta\kappa}{\lambda + \kappa^2} + \sigma \right) E_t \pi_{t+1} \\ & + \frac{\kappa}{\lambda + \kappa^2} u_t + r_t^n \Bigg], \end{aligned}$$

where $\lambda$ gives the weight placed on stabilizing output variation and $u_t$ a cost-push shock.[9] All three classes of monetary policy rules certainly warrant careful analysis when private agents have subjective

[8] This is not the form of rule that these authors find best characterizes the central bank's policy reaction function. It is presented as being illustrative of a type of rule that might be of practical interest.

[9] As done in many recent analyses of this model under rational expections, a cost-push shock can be introduced into the aggregate supply curve to ensure a nontrivial optimal monetary policy problem.

expectations and these rules are interpreted as responding to observed private forecasts. Indeed, Evans and Honkapohja (2002) and Bullard and Mitra (2002) have examined rules of these types under learning dynamics in the context of a model where expectations of inflation and output one period in advance matter. In the context of the model of this paper, analysis of these rules leads to quite different conclusions about their desirability as a guideline for the conduct of monetary policy. For this reason, discussion of these classes of rules is contained in companion papers, Preston (forthcoming) and Preston (2004), that seek to understand the desirability of central bank decision procedures that attempt to implement monetary policy using forecast-based instrument rules.

## *4.2 Learning Dynamics*

### *4.2.1 Exogenous Interest-Rate Processes*

To begin analysis of the model under learning, consider a monetary policy specified as a commitment to an instrument rule of the form

$$i_t = \bar{\imath}_t + \psi_c + \psi_r r_t^n \tag{30}$$

that posits the nominal interest rate to be set in response to the disturbance in the natural rate of interest. In the following propositions, we assume the exogenous variables $(\bar{\imath}_t, r_t^n)$ are determined as

$$\begin{bmatrix} \bar{\imath}_t \\ r_t^n \end{bmatrix} = f' s_t,$$

redefining $f$ as an $(n \times 2)$ matrix and with $s_t$ determined as in (21). This postulates both the natural rate disturbance and the stochastic constant of the Taylor rule to be a particular linear combination of the elements of the disturbance vector $s_t$.

**Proposition 1.** Under the interest-rate rule (30), the associated REE of the economy given by (18) and (19) is linear in the state variables $s_t$ and is not E-stable under least-squares learning dynamics.

**Proof.** It is easy to verify the existence of an REE that is linear in the state variable, $s_t$. Therefore, assume that agents have forecast functions of the form (24). Substituting the assumed instrument rule and forecast functions (24) into the system (18), (19) gives

$$z_t = Aa_{t-1} + \{\text{terms independent of } a_{t-1}\}, \tag{31}$$

where

$$A = \begin{bmatrix} \frac{(1-\alpha)\beta}{1-\alpha\beta} + \frac{\kappa\sigma}{1-\beta} & \frac{\kappa\alpha\beta}{1-\alpha\beta} + \kappa & -\frac{\kappa\sigma\beta}{1-\beta} \\ \frac{\sigma}{1-\beta} & 1 & -\frac{\sigma\beta}{1-\beta} \\ 0 & 0 & 0 \end{bmatrix}.$$

Leading (31) and taking expectations delivers the optimal forecast of the evolution of the endogenous variables given the current forecast parameters of private agents. The required mapping between the perceived and optimal laws of motion follows immediately. E-stability requires $\det(A - I) < 0$, but

$$\det(A - I) = \frac{\kappa\sigma}{1-\beta} + \frac{\alpha\beta\kappa\sigma}{(1-\beta)(1-\alpha\beta)} > 0.$$

The desired result follows.

Not only do exogenous interest-rate rules suffer from an indeterminacy of equilibrium, but also any such equilibrium fails to be expectationally stable, giving credence to Friedman's critique of nominal interest-rate pegs (a form of interest-rate rule).[10] This result is related to that of Evans and Honkapohja (2003), who find an analogous result in the context of a model with a more restrictive class of learning dynamics. The present paper assumes agents know less about the economy and, as one might expect, this does not make agents better able to learn the rational expectations equilibrium. It is also worth noting that one of the motivations of the bounded rationality literature in macroeconomics was the possibility that learning mechanisms would provide an equilibrium selection criterion in the case of multiple rational expectations equilibria.[11] In the context of this model, learning is not able to overcome indeterminacy of equilibrium induced by an exogenous interest-rate rule.

[10] Honkapohja and Mitra (2004) also show in their model that all nonfundamentals-based equilibria are unstable under learning dynamics. However, given that no rational expectations equilibria are then learnable in that model, this class of rule has little to recommend itself.

[11] Sargent (1993) provides several examples where learning dynamics provide a criterion for equilibrium selection.

Proposition 1 also implies that to design optimal monetary policy rules, it is generally not enough to specify a rule in terms of exogenous disturbances to implement optimal monetary policy. While such rules are consistent with the desired equilibrium, they are equally consistent with the propagation of self-fulfilling expectations. The challenge to design rules that are immune to such instability is taken up in Preston (2004) and Preston (forthcoming).

#### *4.2.2 The Taylor Rule*

In contrast to interest-rate rules that depend only on the history of exogenous disturbances, Taylor rules can deliver determinacy of rational expectations equilibrium so long as the Taylor principle is satisfied. Under learning dynamics, the Taylor principle is necessary and sufficient for E-stability.

**Proposition 2.** Suppose agents construct forecasts using the perceived law of motion given by (24). Under the Taylor rule (27), the model given by (18) and (19) has minimum-state-variable rational expectations equilibria that are linear in the state variables, $s_t$, for which the Taylor principle

$$\kappa(\psi_\pi - 1) + (1 - \beta)\psi_x > 0$$

is necessary and sufficient for E-stability under least-squares learning dynamics.

**Proof.** A sketch of the proof now follows. Appendix 4 shows that the E-stability mapping implies the associated ordinary differential equation

$$\frac{\partial \phi}{\partial \tau} = \begin{bmatrix} A_3 - I_3 & \mathbf{0} \\ \mathbf{0} & H_{3n} - I_{3n} \end{bmatrix} \phi,$$

where $\phi = \left(a', vec\,(b)'\right)'$, all matrices are square and of indicated dimension, and $I$ is an identity matrix. E-stability requires all $3n + 3$ eigenvalues of this system to have negative real parts. It is immediate that the eigenvalues are determined by the properties of the matrices $A_3 - I_3$ and $H_{3n} - I_{3n}$. The proof establishes that these two matrices have negative real roots so long as the Taylor principle holds.

The following treats the matrix $A_3 - I_3$, which characterizes the stability properties of the constant dynamics, leaving $H_{3n} - I_{3n}$ to appendix 4.

Since the constant dynamics are independent of the dynamics describing the forecast parameters $vec\,(b)$, we can analyze the subsystem

$$\frac{\partial a}{\partial \tau} = [A_3 - I_3]\, a.$$

Noting that an REE implies the coefficient restriction $a_i = \psi_x a_x + \psi_\pi a_\pi$, make a change of variables according to the relation $a_j = \psi_x a_x + \psi_\pi a_\pi - a_i$, where $a_j = b_j = 0$ in an REE. This yields the system

$$\frac{\partial \tilde{a}}{\partial \tau} = \begin{bmatrix} \tilde{A} - I_2 & \tilde{A}_2 \\ \mathbf{0} & -1 \end{bmatrix} \tilde{a},$$

where $\tilde{a} = (a_\pi, a_x, a_j)'$ and all matrices are of dimension $(2 \times 2)$. $\tilde{A}_2$ has elements that are composites of model primitives. The matrix $\tilde{A}$ can be shown to have elements

$$\begin{aligned}
\tilde{a}_{11} &= \left(\kappa\sigma\,(1-\alpha\beta)\,(1-\beta\psi_\pi) + \beta\,(1-\alpha)\,(1-\beta)\,(1+\beta\sigma\psi_x)\right)/\Gamma_1 \\
\tilde{a}_{12} &= \kappa\,(1-\beta(1+(1-\alpha)\,\sigma\psi_x)\,/\Gamma_1 \\
\tilde{a}_{21} &= \sigma\left(1-\alpha\beta\,(1-\psi_\pi) - 2\beta\psi_\pi + \beta^2\psi_\pi\right)/\Gamma_1 \\
\tilde{a}_{22} &= \left((1-\beta)\,(1-\alpha\beta) - \sigma\beta\psi_x\,(1-\alpha\beta) - \alpha\beta\kappa\sigma\psi_\pi\,(1-\beta)\right)/\Gamma_1,
\end{aligned}$$

where

$$\Gamma_1 = (1-\beta)\,(1-\alpha\beta)\,(1+\sigma\psi_x+\sigma\kappa\psi_\pi)$$

and $a_{ij}$ denotes the $(i,j)$ element of the matrix $\tilde{A}$.

For E-stability, the Jacobian $D\frac{\partial \tilde{a}}{\partial \tau}$ must have roots with negative real parts. It is immediate that one root is equal to negative unity. $\tilde{A} - I_2$ must have positive determinant and negative trace for the remaining two eigenvalues to have the desired property. These restrictions imply the inequalities

$$\psi_\pi + \frac{(1-\beta)}{\kappa}\psi_x > 1$$

and

$$\psi_\pi + \frac{\psi_x}{\kappa} \cdot \frac{(1-\alpha\beta)+(1-\beta)^2}{(1-\alpha\beta)+(1-\beta)} > \frac{\kappa\sigma(1-\alpha\beta)-(1-\beta)^2}{\kappa\sigma\left[(1-\alpha\beta)+(1-\beta)\right]},$$

respectively. The first inequality clearly establishes the Taylor principle to be necessary for E-stability. To show that it is sufficient, note that the right-hand side of the second restriction is necessarily less than one. Furthermore, the slope coefficient on the parameter $\psi_x$ is necessarily greater than $(1-\beta)/\kappa$; for it to be less than this value requires $\beta < 0$, contradicting the maintained model assumptions. It follows that any policy parameter pairs $(\psi_x, \psi_\pi)$ satisfying the Taylor principle must also satisfy this second inequality. Appendix 4 applies similar arguments to the restrictions implied by $H_{3n} - I_{3n}$ to establish the desired result.

To give some intuition, particularly for the presence of the eigenvalues equal to negative one, which relate to learning the interest-rate dynamics, consider the following. Suppose that agents, serendipitously, happen to forecast a nominal interest-rate path coinciding with what would be determined by the Taylor rule given the agents' forecasts for the output gap and inflation. It follows that the economy would produce data for the output gap and inflation that are in turn consistent with estimating parameters that would generate forecasts in subsequent periods for the path of the nominal interest rate that would again be obtained under the Taylor rule. It follows that the Taylor rule itself cannot be a source of instability, and agents—by observing the realized values for output, inflation, and the nominal interest rate—can easily discern the restriction between these variables that is required by the Taylor rule in a rational expectations equilibrium. It follows that only the inflation and output gap dynamics are relevant for E-stability. This basic insight is important more generally: Preston (forthcoming) shows that a common property of desirable optimal monetary policies is that they ensure that the instrument rule itself is not a source of instability—that is, the associated eigenvalues are independent of private agents' beliefs as for the Taylor rule examined here.

Bullard and Mitra (2002) show a similar result in a learning analysis based on equations (13) and (14) and assuming the natural rate, $r_t^n$, to be specified as an $AR(1)$ stochastic process. It should be emphasized that this is not obviously to be expected. The framework given by (18) and (19) allows for both significantly more general out-of-equilibrium behavior, with output, inflation, and the nominal interest rate depending on average expectations of these same variables into the indefinite future and a more general stochastic process for the disturbances. The presence of additional expectational variables relative to the analysis of Bullard and Mitra (2002) is a potential source of instability under learning dynamics. That the Taylor principle continues to be the relevant condition for E-stability in the more general framework developed in this paper suggests it to be a robust result for this class of instrument rule.

As discussed by Honkapohja and Mitra (2004), results of this kind also provide an alternative interpretation of the performance of monetary policy in the United States in the 1970s relative to later decades. Clarida, Galí, and Gertler (1998) argue that the inflationary episode of the 1970s was the result of a monetary policy that was not consistent with a determinate price level. As a result, the economy was prone to "sunspot" equilibria and self-fulfilling expectations. In contrast, the results of this paper suggest that an equally consistent interpretation of this episode is that monetary policy, rather than inducing indeterminacy, was conducive to agents expecting ever higher inflation on the basis of their experience with past inflation—and hence to propagating self-fulfilling expectations.

## 5. Conclusion

This paper develops a framework to analyze the robustness of monetary policy rules to an important source of model misspecification—the assumed form of expectations formation. The principal contribution is methodological in nature: the solution to a simple microfounded model under a nonrational expectations assumption. Analysis of the multiperiod decision problems of households and firms under subjective beliefs shows that the predicted aggregate model dynamics are qualitatively different from those obtained under rational expectations. Indeed, the determination of inflation and output depends on the average of agents' long-horizon forecasts of the model's endogenous variables into the indefinite future.

The principal substantive contribution is the analysis of whether instrument rules that have been of particular interest to the monetary policy literature over the past decade are robust to deviations from rational expectations. When policy is specified as a commitment to an exogenous interest-rate rule, agents are unable to learn the associated rational expectations dynamics. Such rules are therefore undesirable both due to inducing multiple equilibria under rational expectations and to being subject to the Friedman (1968) critique that nominal interest-rate rules of this type are subject to self-fulfilling expectations. In contrast, for Taylor-type feedback rules, agents are able to learn the associated rational expectations dynamics so long as the Taylor principle is satisfied. Interestingly, this finding is invariant to a number of different information assumptions on the agent's forecasting model, making it a robust feature of the policy environment in this model. This suggests the Taylor rule to be desirable from the point of view of eliminating instability due to self-fulfilling expectations.

Companion papers, Preston (2004) and Preston (forthcoming), demonstrate for a number of more complicated rules that conclusions differ markedly in the framework developed here as compared to the Euler equation approach to modeling learning. The latter paper shows that forecast-based instrument rules, including the classes of rules proposed by Bullard and Mitra (2002) and Evans and Honkapohja (2002), are frequently prone to self-fulfilling expectations in the present model if the central bank responds to observed private-sector forecasts. However, if the central bank responds to the determinants of these expectations, this instability can be mitigated. The former paper shows that optimal monetary policy can always be implemented using specific targeting rules if the central bank correctly understands agents' behavior. However, without such knowledge, decision procedures that seek to control directly the path of the price level, rather than the inflation rate, tend to perform better under learning dynamics, even though these policies are equivalent in terms of the rational expectations equilibrium they imply.

## Appendix 1. Household Optimality

Defining $W_{t+1}^i = (1 + i_t^m)\, M_t^i + (1 + i_t)\, B_t^i$ as the total beginning-of-period wealth at time $t + 1$ allows the flow budget constraint (2) to be written as

$$P_t C_t^i + \Delta_t M_t^i + \frac{1}{1 + i_t} \cdot W_{t+1}^i \leq W_t^i + \left[ P_t Y_t^i - T_t \right], \qquad (32)$$

where

$$\Delta_t \equiv \frac{i_t - i_m}{1 + i_t}$$

is the opportunity cost of holding wealth in a monetary form. Since (32) must hold in all states, $s$, and dates, $t$, the flow budget constraint can be solved forward recursively, given the appropriate no-Ponzi constraint $\lim_{j\to\infty} R_{t,t+j} W_{t+j+1} = 0$, to give

$$W_t^i \geq \sum_{j=0}^{\infty} R_{t,t+j} \left[ P_{t+j} C_{t+j}^i + \Delta_{t+j} M_{t+j}^i - (P_{t+j} Y_{t+j}^i - T_{t+j}) \right],$$

where

$$R_{t,t+j} = \prod_{s=1}^{j} \left( \frac{1}{1 + i_{t+s-1}} \right).$$

Standard analysis shows that household intertemporal optimality is characterized by the first-order conditions for consumption and labor supply:

$$\frac{1}{1+i_t} = \beta E_t [\frac{P_t}{P_{t+1}} \cdot \frac{U_c(C_{t+1}^i, \xi_{t+1})}{U_c(C_t^i; \xi_t)}] \tag{33}$$

and

$$\frac{v_h(h_t(j); \xi_t)}{u_c(C_t^i; \xi_t)} = \frac{w_t(j)}{P_t} \tag{34}$$

for dates, $t$, and goods $j \in [0, 1]$. Since we are assuming a cashless economy, where the transaction frictions that money is usually held to mitigate are essentially zero, optimization also requires

$$M_t^i = 0$$

or

$$i_t = i_t^m.$$

In each period $t$, households also face the intratemporal problem of allocating expenditures across goods $j$. Optimality for all $j \in [0, 1]$ implies

$$c_t(j) = C_t^i \left( \frac{p_t(j)}{P_t} \right)^{-\theta} \tag{35}$$

so that total expenditure is given by $P_t C_t^i$.[12] To obtain the total consumption demand for good $j$, integrate over $i$ to obtain

$$c_t(j) = C_t \left( \frac{p_t(j)}{P_t} \right)^{-\theta},$$

introducing the notation $\int_i z_t^i di = z_t$ for any variable $z$. In equilibrium, markets must clear for each good and aggregate output. This requires $y_t(j) = c_t(j)$ for all $j$ and $C_t = Y_t$. Substitution of the market-clearing conditions into the above relation gives the demand curve for output produced by firm $j$. Asset market clearing implies $M_t = M_t^s$ and $B_t = B_t^s$, where $\int B_t^i di = B_t$ and similarly for $M_t$. Since $M_t^s > 0$, this implies $i_t = i_t^m$. Finally, the intertemporal budget constraint and the transversality condition must hold with equality. Since we have a zero-debt fiscal policy with bonds in zero net supply, it follows that

$$T_t = (1 + i_{t-1}) M_{t-1} - M_t$$

so that the intertemporal budget constraint can be written as

$$W_t^i = \sum_{j=0}^{\infty} R_{t,t+j} \left[ P_{t+j} C_{t+j}^i - P_{t+j} Y_{t+j}^i \right],$$

redefining $W_t^i \equiv (1 + i_{t-1}) B_{t-1}$.

To obtain a log-linear approximation to the household's decision problem, define the linearization point to be the steady state characterized by $\xi_t = 0$ and $Y_t = \bar{Y}$ (defined in appendix 2) for all $t$.[13] Inspection of the household's first-order conditions implies a solution of the form $\pi_t = P_t/P_{t-1} = 1$, and $\bar{\imath}_t = \beta^{-1} - 1$ for all $t$, where a bar denotes steady-state value. For any variable $z$, define log deviation as $\hat{z}_t \equiv \log(z_t/\bar{z})$, except for the nominal interest rate for which $\hat{\imath} = \log[(1 + i)/(1 + \bar{\imath})]$ is used.[14] The analysis seeks a log-linear

[12]Total expenditure is obtained by multiplying (35) by $p_t(j)$ and integrating over $j$. Applying the definition of the price index delivers the result.

[13]Given that this paper explores a form of bounded rationality that is a minimal deviation from rational expectations, and, moreover, that the analysis will be later concerned with whether an economy under learning dynamics can converge to the associated rational expectations equilibrium, the linearization point is chosen to coincide with that same rational expectations equilibrium.

[14]Thus all hatted variables are interpreted as percentage deviations. The nominal interest rate is treated differently so that it corresponds to percentage point deviations of the continuously compounded nominal interest rate.

solution in which all variables fluctuate forever near these steady-state values.

## Appendix 2. Firm Problem

This appendix characterizes the firm's optimal pricing problem, defines the notion of the natural rate of output, and provides some details of the log-linear relations used in section 1.2. For a thorough analysis, see Woodford (2003, chap. 3). Differentiating (8) with respect to $p_t(i)$ gives firm $i$'s first-order condition,

$$\hat{E}_t^i \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} Y_T P_T^{\theta} \left[\hat{p}_t^*(i) - \mu P_T s_{t,T}(i)\right] = 0, \tag{36}$$

where $\mu = \theta/(\theta - 1)$, $s_{t,T}(i)$ the firm $i$'s real marginal cost function (defined below) in period $T \geq t$ given the optimal price $p_t^*(i)$ determined in period $t$. To derive a log-linear approximation to the firm's optimal pricing condition, recall that the steady state is defined as $\xi_t = 0$ and $Y_t = \bar{Y}$ for all $t$. Inspection of (36) indicates there exists a solution with $p_t^*/P_t = P_t/P_{t-1} = 1$ in each period $t$. Therefore, we look for a log-linear approximation in which $P_t/P_{t-1}$ and $p_t^*/P_t$ remain forever close to one. Before deriving this log-linear approximation, several other useful relations are derived.

Combining the household's optimal labor supply condition (34) with the firm's production function and differentiating with respect to $p_t(i)$ gives the firm's real marginal cost function

$$s(y, Y; \tilde{\xi}) = \frac{v_h(f^{-1}(y/A; \xi))}{u_c(Y; \xi)A} \cdot \frac{1}{f'(f^{-1}(y))}, \tag{37}$$

where $\tilde{\xi} \equiv (A_t, \xi_t)'$ is a composite vector of all preference and technology shocks. Real marginal costs therefore depend on both firm-specific and aggregate conditions.

Now suppose that firms have full information about the current state of the economy and are able to set prices each period—the case of fully flexible price setting. Then, a standard result from a model of monopolistic competition is that prices are optimally set according to the mark-up relation $\frac{p_t(i)}{P_t} = \frac{\theta}{\theta-1} \cdot s(y_t(i), Y_t; \tilde{\xi}_t) = \mu s(y_t(i), Y_t; \tilde{\xi}_t)$. Under this assumption on price-setting behavior, firms—regardless of beliefs—face a symmetric problem. It follows that, in equilibrium, $p_t(i) = P_t$ and $y_t(i) = Y_t$ for all $i$ and $t$, and combining with the optimality condition (36) implies

$$s(Y_t^n, Y_t^n; \tilde{\xi}_t) = \mu^{-1}, \tag{38}$$

where the level of output $Y_t^n$ that satisfies this condition is called the natural rate of output. It is the rate of output that occurs under fully flexible prices and varies in accordance with fundamental shocks $\tilde{\xi}_t$. The quantity of output, $\bar{Y}_t$, used in the definition of the steady state satisfies $s(\bar{Y}, \bar{Y}; 0) = \mu^{-1}$.

To obtain a log-linear approximation to (36), log-linearize equation (37) to give

$$\hat{s}_{t,T}(i) = \omega \hat{y}_T(i) + \sigma^{-1}\hat{Y}_T - (\omega + \sigma^{-1})\hat{Y}_T^n.$$

It follows that the real marginal cost of producing average or aggregate output, $y_t(i) = Y_t$, is

$$\hat{s}_T(i) = (\omega + \sigma^{-1})(\hat{Y}_T - \hat{Y}_T^n) = (\omega + \sigma^{-1})x_T,$$

where the latter equality implicitly defines the output gap $x_t = \hat{Y}_t - \hat{Y}_t^n$. This provides a relationship between the marginal cost of producing output $y_t(i)$ and the average marginal cost of producing total output $\hat{Y}_t$ of the following form:

$$\hat{s}_{t,T}(i) = \hat{s}_T - \omega\theta \left[ p_t(i) - \sum_{\tau=t+1}^{T} \pi_\tau \right], \tag{39}$$

making clear that a firm's marginal cost in producing its good differs from average marginal cost to the extent that its price differs from the aggregate price.

Substituting into the firm's first-order condition, (36), for the discount factor using (33), linearizing and substituting for real marginal costs using (39) gives the prediction that optimal price of firm $i$ satisfies the approximate log-linear relation

$$\widehat{p_t^*}(i) = \hat{E}_t^i \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[ \frac{1-\alpha\beta}{1+\omega\theta} \cdot (\omega + \sigma^{-1})x_T + \alpha\beta\hat{\pi}_{T+1} \right].$$

Thus firm $i$'s optimal price is determined as a linear function of the future expected paths of the output gap and inflation. Variation in the optimal prices set by firms in period $t$ can be due only to differences in beliefs.

## Appendix 3. Conditions for Eigenvalues to Have Negative Real Parts

Consider the matrix $A$ with dimension $(3 \times 3)$. From $|A - \lambda I| = 0$, the characteristic equation is

$$\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0,$$

where $c_1 = \text{Trace}(A)$, $c_2$ is the sum of all second-order principal minors of $A$, and $c_3 = |A|$. The following restrictions on the coefficients $c_i$ must be satisfied for all eigenvalues to have negative real parts:

$$\begin{aligned} c_1 &< 0 \\ c_3 - c_1c_2 &> 0 \\ c_3 &< 0. \end{aligned}$$

For a matrix $A$ with dimension $(2 \times 2)$, $|A - \lambda I| = 0$ implies the characteristic equation is

$$\lambda^2 - c_1\lambda + c_2 = 0,$$

where $c_1 = \text{Trace}(A)$ and $c_2 = |A|$. For both eigenvalues to have negative real parts, $c_1 < 0$ and $c_2 > 0$ must be satisfied.

## Appendix 4. Proof of Proposition 2

Agents are assumed to construct forecasts according to the relation

$$E_t z_{T+1} = a_z + b_z C^{T-t} s_t, \tag{40}$$

where $z_T = (\pi_T,\ x_T,\ i_T)'$and $a_z$ and $b_z$ are estimated coefficient matrices of appropriate dimension. It follows that

$$\hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} z_{T+1} = a_z \left(1 - \beta\right)^{-1} + b_z \left(I_n - \beta C\right)^{-1} s_t$$

and

$$\hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} z_{T+1} = a_z \left(1 - \alpha\beta\right)^{-1} + b_z \left(I_n - \alpha\beta C\right)^{-1} s_t.$$

Denoting these infinite sums by $f_\beta$ and $f_\alpha$, respectively, and substituting into (18) and (19) implies:

$$\begin{aligned} x_t &= -\frac{\sigma\psi_\pi}{w}\left[(1-\alpha)\beta,\ \kappa\alpha\beta,\ \beta\right]\cdot f_\alpha \\ &\quad + \frac{1}{w}\left[\sigma, (1-\beta), -\sigma\right]\cdot f_\beta - \frac{\sigma}{w}\cdot[1\ \ -1]f_\gamma \\ \pi_t &= \left(1-\frac{\kappa\sigma\psi_\pi}{w}\right)\left[(1-\alpha)\beta,\ \kappa\alpha\beta,\ \beta\right]\cdot f_\alpha \\ &\quad + \frac{\kappa}{w}\left[\sigma, (1-\beta), -\sigma\right]\cdot f_\beta - \frac{\kappa\sigma}{w}\cdot[1\ \ -1]f_\gamma, \end{aligned}$$

where $w = 1 + \sigma\psi_x + \sigma\kappa\psi_\pi$ and

$$f_\gamma = f'\left(I_n - \beta C\right)^{-1} s_t.$$

Recalling that the nominal interest rate is given by the Taylor rule

$$i_t = \bar{\imath}_t + \psi_\pi \pi_t + \psi_x x_t,$$

the system can be written compactly as

$$z_t = A_1 f_\alpha + A_2 f_\beta + A_3 f_\gamma,$$

where $A_1$, $A_2$, and $A_3$ collect obvious coefficients and have dimension $(3 \times 3)$. Substituting for $f_\alpha, f_\beta, f_\gamma$ gives

$$\begin{aligned} z_t &= \left[(1-\alpha\beta)^{-1} A_1 + (1-\beta)^{-1} A_2\right] a_z \\ &\quad + \left[A_1 b_z (1-\alpha\beta C)^{-1} + A_2 b_z (1-\beta)^{-1} + A_3 f' (I_n - \beta C)^{-1}\right] s_t. \end{aligned}$$

This expression combined with (40) for $T = t$ defines the E-stability mapping from current private forecast parameters to the optimal forecast coefficients as

$$T\begin{pmatrix} a_z \\ b_z \end{pmatrix} = \begin{pmatrix} \left[(1-\alpha\beta)^{-1} A_1 + (1-\beta)^{-1} A_2\right] a_z \\ \left[A_1 b_z (1-\alpha\beta C)^{-1} + A_2 b_z (1-\beta)^{-1}\right] C \end{pmatrix}.$$

The associated ordinary differential equation can then be written as

$$\frac{\partial\phi}{\partial\tau} = \begin{bmatrix} A - I_3 & \mathbf{0} \\ \mathbf{0} & H - I_{3n} \end{bmatrix}\phi,$$

where $\phi = \left(a_z', vec\left(b_z\right)'\right)'$ and

$$\begin{aligned} A &= (1-\alpha\beta)^{-1} A_1 + (1-\beta)^{-1} A_2 \\ H &= \left[(I_n - \alpha\beta C_n)^{-1} C_n\right]' \otimes A_1 + \left[(I_n - \beta C_n)^{-1} C_n\right]' \otimes A_2 \end{aligned}$$

are $(3 \times 3)$ and $(3n \times 3n)$ matrices, respectively. The Jacobian is then given as

$$D\frac{\partial \phi}{\partial \tau} = \begin{bmatrix} A - I_3 & \mathbf{0} \\ \mathbf{0} & H - I_{3n} \end{bmatrix}.$$

To complete the proof of this proposition, it remains to consider the properties of the eigenvalues of the matrix $H - I_{3n}$, since $A - I_3$ was considered in the main text. Note that $C$ can be diagonalized to give

$$C = S\Lambda S^{-1}, \tag{41}$$

where $\Lambda$ is a diagonal matrix with elements given by the eigenvalues, $\rho_k$, of $C$, and $S$ is a matrix composed of the corresponding eigenvectors, $v_k$. Also note that we can write

$$\begin{aligned} (I_n - \alpha\beta C)^{-1} C &= S\Lambda \left(I_n - \alpha\beta\Lambda\right)^{-1} S^{-1} \\ (I_n - \beta C)^{-1} C &= S\Lambda \left(I_n - \beta\Lambda\right)^{-1} S^{-1}. \end{aligned}$$

The matrix $H$ can therefore be written as

$$\begin{aligned} H &= \left[S\Lambda \left(I_n - \alpha\beta\Lambda\right)^{-1} S^{-1}\right]' \otimes A_1 \\ &\quad + \left[S\Lambda \left(I_n - \beta\Lambda\right)^{-1} S^{-1}\right]' \otimes A_2 \\ &= \left(S^{-1\prime} \otimes I_3\right) \Big[\Lambda \left(I_n - \alpha\beta\Lambda\right)^{-1} \otimes A_1 \\ &\quad + \Lambda \left(I_n - \beta\Lambda\right)^{-1} \otimes A_2\Big] \left(S' \otimes I_3\right). \end{aligned}$$

Note that

$$G \equiv \Lambda \left(I_n - \alpha\beta\Lambda\right)^{-1} \otimes A_1 + \Lambda \left(I_n - \beta\Lambda\right)^{-1} \otimes A_2$$

is block diagonal with elements

$$G_k\left(\rho_k\right) = \rho_k \left(1 - \alpha\beta\rho_k\right)^{-1} \otimes A_1 + \rho_k \left(1 - \beta\rho_k\right)^{-1} \otimes A_2,$$

where each $G_k(\rho_k)$ is $(3 \times 3)$. Let $v_k$ be an eigenvector of $C$ associated with the eigenvalue $\rho_k$ and let $\lambda_i(\rho_k)$ be an eigenvector of the associated diagonal block $G_k(\rho_k)$ (note that there are three such eigenvectors).

Conjecture that the matrix $H$ has eigenvectors of the form $v_k \otimes \lambda_i(\rho_k)$. Then, in the particular case of $v_1 \otimes \lambda_i(\rho_1)$ (where, without loss of generality, assume $\rho_1$ to be the first diagonal element of $\Lambda$ and $v_1$ the first column vector of $S$), we have

$$\begin{aligned}
(v_1 \otimes \lambda_i(\rho_1))' H &= (v_1 \otimes \lambda_i(\rho_1))' \left(S^{-1\prime} \otimes I_2\right) \Big[\Lambda (I_n - \alpha\beta\Lambda)^{-1} \\
&\quad \otimes A_1 + \Lambda (I_n - \beta\Lambda)^{-1} \otimes A_2\Big] \left(S' \otimes I_2\right) \\
&= \left(\left(S^{-1} v_1\right)' \otimes \lambda_i(\rho_1)'\right) \Big[\Lambda (I_n - \alpha\beta\Lambda)^{-1} \otimes A_1 \\
&\quad + \Lambda (I_n - \beta\Lambda)^{-1} \otimes A_2\Big] \left(S' \otimes I_2\right) \\
&= [\lambda_i(\rho_1)' \vdots 0_{1\times(3n-3)}] \Big[\Lambda (I_n - \alpha\beta\Lambda)^{-1} \otimes A_1 \\
&\quad + \Lambda (I_n - \beta\Lambda)^{-1} \otimes A_2\Big] \left(S' \otimes I_2\right) \\
&= [\lambda_i(\rho_1)' G_1(\rho_1) \vdots 0_{1\times(3n-3)}] \left(S' \otimes I_2\right) \\
&= [\lambda_i(\rho_1)' \gamma_i(\rho_1) \vdots 0_{1\times(3n-3)}] \left(S' \otimes I_2\right) \\
&= \gamma_i(\rho_1) (v_1 \otimes \lambda_i(\rho_1))'.
\end{aligned}$$

Thus $v_1 \otimes \lambda_i(\rho_1)$ is in fact an eigenvector of $H$ with associated eigenvalue $\gamma_i(\rho_1)$. Since for each $\rho_k$ there are three eigenvalues $\gamma_i(\rho_k)$ and corresponding eigenvectors $v_k$ and $\lambda_i(\rho_k)$, there are therefore $3n$ eigenvectors of the form $v_k \otimes \lambda_i(\rho_k)$ that span the space of $H$.

To complete the proof requires demonstrating that all $3n$ eigenvalues $\gamma_i(\rho_k)$ are less than unity. Consider the properties of the matrix

$$G_k(\rho_k) - I_3$$

formed from the $k$th diagonal block of $G$. Using Mathematica, it is easily shown that one eigenvalue (corresponding to the coefficient relevant to the interest-rate dynamics) is equal to negative unity, while the remaining two have properties determined by the quadratic equation

$$a\lambda^2 + b\lambda + c.$$

A sufficient condition for there to be two roots with negative real parts is $a, b, c > 0$. It is easily shown that

$$a = (1 - \beta\rho_k)(1 - \alpha\beta\rho_k)(1 + \sigma\psi_x + \kappa\sigma\psi_\pi).$$

For the remaining two conditions to hold, the restrictions

$$\psi_\pi + \frac{(1 - \beta\rho_k)}{\kappa}\psi_x > \rho_k - \frac{(1 - \rho_k)(1 - \beta\rho_k)}{\kappa\sigma}$$

and

$$\psi_\pi + \frac{\psi_x}{\kappa} \cdot \frac{(1 - \alpha\beta\rho_k) + (1 - \beta\rho_k)^2}{(1 - \alpha\beta\rho_k) + (1 - \beta\rho_k)} > \frac{\kappa\sigma(1 - \alpha\beta\rho_k) - \left[(1 - \alpha\beta\rho_k)(1 - \rho_k) + (1 - \beta\rho_k)^2\right]}{\kappa\sigma\left[(1 - \alpha\beta\rho_k) + (1 - \beta\rho_k)\right]}$$

must be satisfied. To show that satisfaction of the Taylor principle is necessary and sufficient for this first restriction to hold, note that the constant term of first inequality is less than unity by inspection and that the slope coefficient on $\psi_x$ is necessarily greater than $(1 - \beta)/\kappa$ for $\rho_k \in (-1, 1)$. Thus for all positive $(\psi_\pi, \psi_x)$, the Taylor principle ensures satisfaction of this inequality. Finally, consider the second inequality above. The constant is again less than unity by inspection. For the Taylor principle to be sufficient, consider the slope coefficient. If the restriction

$$\frac{(1 - \alpha\beta\rho_k) + (1 - \beta\rho_k)^2}{(1 - \alpha\beta\rho_k) + (1 - \beta\rho_k)} > 1 - \beta$$

holds, then the Taylor principle is indeed sufficient. Rearranging yields the restriction

$$f(\rho_k) \equiv \beta(1 - \alpha\beta\rho_k) + (1 - \beta\rho_k)^2 - (1 - \beta)(1 - \beta\rho_k) > 0,$$

which satisfies $f(-1),\ f(0),\ f(1) > 0$. Since $f'(\rho_k) < 0$ for all $\rho_k \in (-1, 1)$, $f(\rho_k) > 0$ for all $\rho_k \in (-1, 1)$. The Taylor principle is therefore necessary and sufficient for the restrictions $b, c > 0$ to hold. Thus, all eigenvalues of $G_k(\rho_k) - I_3$ have negative real parts and $G_k(\rho_k)$ all have eigenvalues less than unity if and only if the Taylor principle holds. It follows that all eigenvalues of $H$ must be less than unity and, therefore, that the eigenvalues of $H - I_{3n}$ all have negative real parts. The conditions for E-stability are therefore satisfied if and only if the Taylor principle holds.

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# Where Are We Now? Real-Time Estimates of the Macroeconomy*

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This paper describes a method for calculating daily real-time estimates of the current state of the U.S. economy. The estimates are computed from data on scheduled U.S. macroeconomic announcements using an econometric model that allows for variable reporting lags, temporal aggregation, and other complications in the data. The model can be applied to find real-time estimates of GDP, inflation, unemployment, or any other macroeconomic variable of interest. In this paper, I focus on the problem of estimating the current level of and growth rate in GDP. I construct daily real-time estimates of GDP that incorporate public information known on the day in question. The real-time estimates produced by the model are uniquely suited to studying how perceived developments in the macroeconomy are linked to asset prices over a wide range of frequencies. The estimates also provide, for the first time, daily time series that can be used in practical policy decisions.

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Information about the current state of real economic activity is widely dispersed across consumers, firms, and policymakers. While individual consumers and firms know the recent history of their own decisions, they are unaware of the contemporaneous consumption,

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saving, investment, and employment decisions made by other private sector agents. Similarly, policymakers do not have access to accurate contemporaneous information concerning private sector activity. Although information on real economic activity is collected by a number of government agencies, the collection, aggregation, and dissemination process takes time. Thus, while U.S. macroeconomic data are released on an almost daily basis, the data represent official aggregations of past rather than current economic activity.

The lack of timely information concerning the current state of the economy is well recognized among policymakers. This is especially true in the case of GDP, the broadest measure of real activity. The Federal Reserve's ability to make timely changes in monetary policy is made much more complicated by the lack of contemporaneous and accurate information on GDP. The lack of timely information concerning macroeconomic aggregates is also important for understanding private sector behavior and, in particular, the behavior of asset prices. When agents make trading decisions based on their own estimate of current macroeconomic conditions, they transmit information to their trading partners. This trading activity leads to the aggregation of dispersed information and, in the process, affects the behavior of asset prices. Evans and Lyons (2004a) show that the lack of timely information concerning the state of the macroeconomy can significantly alter the dynamics of exchange and interest rates by changing the trading-based process of information aggregation.

This paper describes a method for estimating the current state of the economy on a continual basis using the flow of information from a wide range of macroeconomic data releases. These real-time estimates are computed from an econometric model that allows for variable reporting lags, temporal aggregation, and other complications that characterize the daily flow of macroeconomic information. The model can be applied to find real-time estimates of GDP, inflation, unemployment, or any other macroeconomic variable of interest. In this paper, I focus on the problem of estimating GDP in real time.

The real-time estimates derived here are conceptually distinct from the real-time *data* series studied by Croushore and Stark (1999, 2001), Orphanides (2001), and others. A real-time data series comprises a set of *historical* values for a variable that are known on a particular date. This date identifies the vintage of the real-time data. For example, the March 31 vintage of real-time GDP data would include data releases on GDP growth up to the fourth quarter of the previous year. This vintage incorporates current revisions to earlier GDP releases but does not include a contemporaneous estimate of GDP growth in the first quarter. As such, it represents a *subset* of public information available on March 31. By contrast, the March 31 real-time estimate of GDP growth comprises an estimate of GDP growth in the first quarter based on information available on March 31. The real-time estimates derived in this paper use an information set that spans the history of data releases on GDP and eighteen other macroeconomic variables.

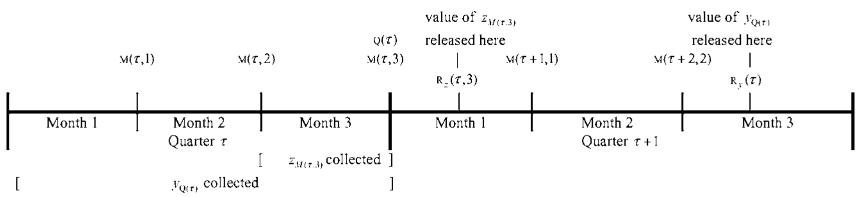
A number of papers have studied the problem of estimating GDP at a monthly frequency. Chow and Lin (1971) first showed how a monthly series could be constructed from regression estimates using monthly data related to GDP and quarterly GDP data. This technique has been subsequently integrated into VAR forecasting procedures (see, for example, Robertson and Tallman 1999). More recently, papers by Liu and Hall (2000) and Mariano and Murasawa (2003) have used state-space models to combine quarterly GDP data with other monthly series. The task of calculating real-time estimates of GDP growth has also been addressed by Kitchen and Monaco (2003). They developed a regression-based method that uses a variety of monthly indicators to forecast GDP growth in the current quarter. The real-time estimates are calculated by combining the different forecasts with a weighting scheme based on the relative explanatory power of each forecasting equation.

I differ from this literature by modeling the growth in GDP as the quarterly aggregate of an unobserved *daily* process for real economy-wide activity. The model also specifies the relationship between GDP, data releases on GDP growth, and data releases on a set of other macroeconomic variables in a manner that accommodates the complex timing of releases. In particular, I incorporate the variable reporting lags that exist between the end of each data collection period (i.e., the end of a month or quarter) and the release day for each variable. This is only possible because the model tracks the evolution of the economy on a daily basis. An alternative approach of assuming that GDP aggregrates an unobserved monthly process for economy-wide activity would result in a simpler model structure (see Liu and Hall [2000] and Mariano and Murasawa [2003]), but it could not accommodate the complex timing of data releases. The structure of the model also enables me to compute real-time estimates of GDP as the solution to an inference problem. In practice, I obtain the real-time estimates as a by-product of estimating the model. First, the model parameters are estimated by (qausi) maximum likelihood using the Kalman Filter algorithm. The real-time estimates are then obtained by applying the algorithm to the model evaluated at the maximum likelihood estimates (MLEs).

My method for computing real-time estimates has several noteworthy features. First, the estimates are derived from a single fully specified econometric model. As such, we can judge the reliability of the real-time estimates by subjecting the model to a variety of diagnostic tests. Second, a wide variety of variables can be computed from the estimated model. For example, the model can provide real-time forecasts for GDP growth for *any* future quarter. It can also be used to compute the precision of the real-time estimates as measured by the relevant conditional variance. Third, the estimated model can be used to construct high-frequency estimates of real economic activity. We can construct a daily series of real-time estimates for GDP growth in the current quarter, or real-time estimates of GDP produced in the current month, week, or even day. Fourth, the method can incorporate information from a wide range of economic indicators. In this paper, I use the data releases for GDP and eighteen other macroeconomic variables, but the set of indicators could be easily expanded to include many other macroeconomic series and financial data. Extending the model in this direction may be particularly useful from a forecasting perspective. Stock and Watson (2002) show that harnessing the information in a large number of indicators can have significant forecasting benefits.

The remainder of the paper is organized as follows. Section 1 describes the inference problem that must be solved in order to compute the real-time estimates. Here I detail the complex timing of data collection and macroeconomic data releases that needs to be accounted for in the model. The structure of the econometric model is presented in section 2. Section 3 covers estimation and the calculation of the real-time estimates. I first show how the model can be written in state-space form. Then I describe how the sample likelihood is constructed with the use of the Kalman Filter. Finally, I describe how various real-time estimates are calculated from the maximum likelihood estimates of the model. Section 4 presents the model estimates and specification tests. Here I compare the forecasting

**Figure 1. Data Collection Periods and Release Times for Quarterly and Monthly Variables**




Note: The reporting lag for "final" GDP growth in quarter $\tau$, $y_{\mathrm{Q}(\tau)}$, is $\mathrm{R}_y(\tau)-\mathrm{Q}(\tau)$. The reporting lag for the monthly series $z_{\mathrm{M}(\tau,j)}$ is $\mathrm{R}_z(\tau,j)-\mathrm{M}(\tau,j)$ for $j = 1, 2, 3$.

performance of the model against a survey of GDP estimates by professional money managers. These private estimates appear comparable to the model-based estimates even though the managers have access to much more information than the model incorporates. Section 5 examines the model-based real-time estimates. First, I consider the relation between the real-time estimates and the final GDP releases. Next, I compare alternative real-time estimates for the level of GDP and examine the forecasting power of the model. Finally, I study how the data releases on other macrovariables are related to changes in GDP at a monthly frequency. Section 6 concludes.

## 1. Real-Time Inference

My aim is to obtain high-frequency real-time estimates on how the macroeconomy is evolving. For this purpose, it is important to distinguish between the arrival of information and data collection periods. Information about GDP can arrive via data releases on any day $t$. GDP data is collected on a quarterly basis. I index quarters by $\tau$ and denote the last day of quarter $\tau$ by $\mathrm{Q}(\tau)$, with the first, second, and third months ending on days $\mathrm{M}(\tau,1)$, $\mathrm{M}(\tau,2)$, and $\mathrm{M}(\tau,3)$, respectively. I identify the days on which data is released in two ways. The release day for variable $\varkappa$ collected over quarter $\tau$ is $\mathrm{R}_{\varkappa}(\tau)$. Thus, $\varkappa_{\mathrm{R}(\tau)}$ denotes the value of variable $\varkappa$, over quarter $\tau$, released on day $\mathrm{R}_{\varkappa}(\tau)$. The release day for monthly variables is identified by $\mathrm{R}_{\varkappa}(\tau,i)$ for $i = 1, 2, 3$. In this case, $\varkappa_{\mathrm{R}(\tau,i)}$ is the value of $\varkappa$, for month $i$ in quarter $\tau$, announced on day $\mathrm{R}_{\varkappa}(\tau,i)$. The relation between data release dates and data collection periods is illustrated in figure 1.

The Bureau of Economic Analysis (BEA) at the U.S. Commerce Department releases data on GDP growth in quarter $\tau$ in a sequence of three announcements: the "advanced" growth data are released during the first month of quarter $\tau+1$; the "preliminary" data are released in the second month; and the "final" data are released at the end of quarter $\tau+1$. The "final" data release does not represent the last official word on GDP growth in the quarter. Each summer, the BEA conducts an "annual" or comprehensive revision that generally leads to revisions in the "final" data values released over the previous three years. These revisions incorporate more complete and detailed microdata than was available before the "final" data release date.[1]

Let $x_{\mathrm{Q}(\tau)}$ denote the log of real GDP for quarter $\tau$ ending on day $\mathrm{Q}(\tau)$, and $y_{\mathrm{R}(\tau)}$ be the "final" data released on day $\mathrm{R}_y(\tau)$. The relation between the "final" data and actual GDP growth is given by

$$y_{\mathrm{R}(\tau)} = \Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)} + \upsilon_{\mathrm{R}(\tau)}, \tag{1}$$

where $\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)} \equiv x_{\mathrm{Q}(\tau)} - x_{\mathrm{Q}(\tau-1)}$ and $\upsilon_{\mathrm{A}(\tau)}$ represents the effect of the future revisions (i.e., the revisions to GDP growth made after $\mathrm{R}_y(\tau)$). Notice that equation (1) distinguishes between the end of the reporting period $\mathrm{Q}(\tau)$ and the release date $\mathrm{R}_y(\tau)$. I shall refer to the difference $\mathrm{R}_y(\tau){-}\mathrm{Q}(\tau)$ as the reporting lag for quarterly data. (For data series $\varkappa$ collected during month $i$ of quarter $\tau$, the reporting lag is $\mathrm{R}_\varkappa(\tau,i){-}\mathrm{M}(\tau,i)$.) Reporting lags vary from quarter to quarter because data is collected on a calendar basis but announcements are not made on holidays and weekends. For example, "final" GDP data for the quarter ending in March has been released between June 27 and July 3.

Real-time estimates of GDP growth are constructed using the information in a specific information set. Let $\Omega_t$ denote an information set that only contains data that is publicly known at the end of day $t$. The real-time *estimate* of GDP growth in quarter $\tau$ is defined as $E[\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)} | \Omega_{\mathrm{Q}(\tau)}]$, the expectation of $\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)}$ conditional on public information available at the end of the quarter, $\Omega_{\mathrm{Q}(\tau)}$. To see how this estimate relates to the "final" data release, $y$, I combine the definition with (1) to obtain

[1]For a complete description of BEA procedures, see Carson (1987) and Seskin and Parker (1998).

$$y_{\mathrm{R}(\tau)} = E\left[\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)} | \Omega_{\mathrm{Q}(\tau)}\right] + E\left[\upsilon_{\mathrm{R}(\tau)} | \Omega_{\mathrm{Q}(\tau)}\right] + \left(y_{\mathrm{R}(\tau)} - E\left[y_{\mathrm{R}(\tau)} | \Omega_{\mathrm{Q}(\tau)}\right]\right). \quad (2)$$

The "final" data released on day $\mathrm{R}_y(\tau)$ comprises three components: the real-time GDP growth estimate; an estimate of future data revisions, $E\left[\upsilon_{\mathrm{R}(\tau)} | \Omega_{\mathrm{Q}(\tau)}\right]$; and the real-time forecast error for the data release, $y_{\mathrm{R}(\tau)} - E\left[y_{\mathrm{R}(\tau)} | \Omega_{\mathrm{Q}(\tau)}\right]$. Under the reasonable assumption that $y_{\mathrm{R}(\tau)}$ represents the BEA's unbiased estimate of GDP growth, and that $\Omega_{\mathrm{Q}(\tau)}$ represents a subset of the information available to the BEA before the release day, $E\left[\upsilon_{\mathrm{R}(\tau)} | \Omega_{\mathrm{Q}(\tau)}\right]$ should equal zero. In this case, (2) becomes

$$y_{\mathrm{R}(\tau)} = E\left[\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)} | \Omega_{\mathrm{Q}(\tau)}\right] + \left(y_{\mathrm{R}(\tau)} - E\left[y_{\mathrm{R}(\tau)} | \Omega_{\mathrm{Q}(\tau)}\right]\right). \quad (3)$$

Thus, the data release $y_{\mathrm{R}(\tau)}$ can be viewed as a noisy signal of the real-time estimate of GDP growth, where the noise arises from the error in forecasting $y_{\mathrm{R}(\tau)}$ over the reporting lag. By construction, the noise term is orthogonal to the real-time estimate because both terms are defined relative to the same information set, $\Omega_{\mathrm{Q}(\tau)}$. The noise term can be further decomposed as

$$y_{\mathrm{R}(\tau)} - E\left[y_{\mathrm{R}(\tau)} | \Omega_{\mathrm{Q}(\tau)}\right] = \left(E\left[y_{\mathrm{R}(\tau)} | \Omega^{\mathrm{BEA}}_{\mathrm{Q}(\tau)}\right] - E\left[y_{\mathrm{R}(\tau)} | \Omega_{\mathrm{Q}(\tau)}\right]\right) + \left(y_{\mathrm{R}(\tau)} - E\left[y_{\mathrm{R}(\tau)} | \Omega^{\mathrm{BEA}}_{\mathrm{Q}(\tau)}\right]\right), \quad (4)$$

where $\Omega^{\mathrm{BEA}}_t$ denotes the BEA's information set. Since the BEA has access to both private and public information sources, the first term on the right identifies the informational advantage conferred on the BEA at the end of the quarter $\mathrm{Q}(\tau)$. The second term identifies the impact of new information the BEA collects about $x_{\mathrm{Q}(\tau)}$ during the reporting lag. Since both of these terms could be sizable, there is no a priori reason to believe that real-time forecast error is always small.

To compute real-time estimates of GDP, we need to characterize the evolution of $\Omega_t$ and describe how inferences about $\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)}$ can be calculated from $\Omega_{\mathrm{Q}(\tau)}$. For this purpose, I incorporate the information contained in the "advanced" and "preliminary" GDP data releases. Let $\hat{y}_{\mathrm{R}(\tau)}$ and $\tilde{y}_{\mathrm{R}(\tau)}$ respectively denote the values for the "advanced" and "preliminary" data released on days $\mathrm{R}_{\hat{y}}(\tau)$ and $\mathrm{R}_{\tilde{y}}(\tau)$, where

$\mathrm{Q}(\tau) < \mathrm{R}_{\hat{y}}(\tau) < \mathrm{R}_{\tilde{y}}(\tau)$. I assume that $\hat{y}_{\mathrm{R}(\tau)}$ and $\tilde{y}_{\mathrm{R}(\tau)}$ represent noisy signals of the "final" data, $y_{\mathrm{R}(\tau)}$:

$$\hat{y}_{\mathrm{R}(\tau)} = y_{\mathrm{R}(\tau)} + \tilde{e}_{\mathrm{R}(\tau)} + \hat{e}_{\mathrm{R}(\tau)}, \tag{5}$$

$$\tilde{y}_{\mathrm{R}(\tau)} = y_{\mathrm{R}(\tau)} + \tilde{e}_{\mathrm{R}(\tau)}, \tag{6}$$

where $\tilde{e}_{\mathrm{R}(\tau)}$ and $\hat{e}_{\mathrm{R}(\tau)}$ are independent mean zero revision shocks. $\tilde{e}_{\mathrm{R}(\tau)}$ represents the revision between days $\mathrm{R}_{\hat{y}}(\tau)$ and $\mathrm{R}_{y}(\tau)$, and $\hat{e}_{\mathrm{R}(\tau)}$ represents the revision between days $\mathrm{R}_{\hat{y}}(\tau)$ and $\mathrm{R}_{\tilde{y}}(\tau)$. The idea that the provisional data releases represent noisy signals of the "final" data is originally due to Mankiw and Shapiro (1986). It implies that the revisions $\tilde{e}_{\mathrm{R}(\tau)}$ and $\tilde{e}_{\mathrm{R}(\tau)} + \hat{e}_{\mathrm{R}(\tau)}$ are orthogonal to $y_{\mathrm{R}(\tau)}$. I impose this orthogonality condition when estimating the model. The specification of (5) and (6) also implies that the "advanced" and "preliminary" data releases represent unbiased estimates of actual GDP growth. This assumption is consistent with the evidence reported in Faust, Rogers, and Wright (2000) for U.S. data releases between 1988 and 1997. (Adding nonzero means for $\tilde{e}_{\mathrm{R}(\tau)}$ and $\hat{e}_{\mathrm{R}(\tau)}$ is a straightforward extension to accommodate bias that may be present in different sample periods.)[2]

[2] It is also possible to accommodate Mankiw and Shapiro's "news" view of data revisions within the model. According to this view, provisional data releases represent the BEA's best estimate of $y_{\mathrm{R}(\tau)}$ at the time the provision data is released. Hence $\tilde{y}_{\mathrm{R}(\tau)} = E[y_{\mathrm{R}(\tau)}|\Omega^{\mathrm{BEA}}_{\mathrm{R}_{\tilde{y}}(\tau)}]$ and $\hat{y}_{\mathrm{R}(\tau)} = E[y_{\mathrm{R}(\tau)}|\Omega^{\mathrm{BEA}}_{\mathrm{R}_{\hat{y}}(\tau)}]$. If the BEA's forecasts are optimal, we can write $y_{\mathrm{R}(\tau)} = \tilde{y}_{\mathrm{R}(\tau)} + \tilde{w}_{\mathrm{R}(\tau)}$ and $y_{\mathrm{R}(\tau)} = \hat{y}_{\mathrm{R}(\tau)} + \hat{w}_{\mathrm{R}(\tau)}$, where $\tilde{w}_{\mathrm{R}(\tau)}$ and $\hat{w}_{\mathrm{R}(\tau)}$ are the forecast errors associated with $E[y_{\mathrm{R}(\tau)}|\Omega^{\mathrm{BEA}}_{\mathrm{R}_{\tilde{y}}(\tau)}]$ and $E[y_{\mathrm{R}(\tau)}|\Omega^{\mathrm{BEA}}_{\mathrm{R}_{\hat{y}}(\tau)}]$, respectively. We could use these equations to compute the projections of $\tilde{y}_{\mathrm{R}(\tau)}$ and $\hat{y}_{\mathrm{R}(\tau)}$ on $y_{\mathrm{R}(\tau)}$ and a constant:

$$\tilde{y}_{\mathrm{R}(\tau)} = \tilde{\beta}_0 + \tilde{\beta} y_{\mathrm{R}(\tau)} + \tilde{\varepsilon}_{\mathrm{R}(\tau)},$$

$$\hat{y}_{\mathrm{R}(\tau)} = \hat{\beta}_0 + \hat{\beta} y_{\mathrm{R}(\tau)} + \hat{\varepsilon}_{\mathrm{R}(\tau)}.$$

The projection errors $\tilde{\varepsilon}_{\mathrm{R}(\tau)}$ and $\hat{\varepsilon}_{\mathrm{R}(\tau)}$ are orthogonal to $y_{\mathrm{R}(\tau)}$ by construction, so these equations could replace (5) and (6). The projection coefficients, $\tilde{\beta}_0$, $\tilde{\beta}$, $\hat{\beta}_0$, and $\hat{\beta}$, would add to the set of model parameters to be estimated. I chose not to follow this alternative formulation because there is evidence that data revisions are forecastable with contemporaneous information (Dynan and Elmendorf 2001). This finding is inconsistent with the "news" view if the BEA makes rational forecasts. Furthermore, as I discuss below, a specification based on (5) and (6) allows the optimal (model-based) forecasts of "final" GDP to closely approximate the provisional data releases. The model estimates will therefore provide us with an empirical perspective on the "noise" and "news" characterizations of data revisions.

The three GDP releases $\{\hat{y}_{\text{R}(\tau)}, \tilde{y}_{\text{R}(\tau)}, y_{\text{R}(\tau)}\}$ represent a sequence of signals on actual GDP growth that augment the public information set on days $\text{R}_{\hat{y}}(\tau)$, $\text{R}_{\tilde{y}}(\tau)$, and $\text{R}_{y}(\tau)$. In principle, we could construct real-time estimates based only on these data releases as $E[\Delta^{\text{Q}} x_{\text{Q}(\tau)}|\Omega^{y}_{\text{Q}(\tau)}]$, where $\Omega^y_t$ is the information set comprising data on the three GDP series released on or before day $t$:

$$\Omega^y_t \equiv \left\{\hat{y}_{\text{R}(\tau)}, \tilde{y}_{\text{R}(\tau)}, y_{\text{R}(\tau)} : \text{R}(\tau) < t\right\}.$$

Notice that these estimates are only based on data releases relating to GDP growth *before* the current quarter because the presence of the reporting lags excludes the values of $\hat{y}_{\text{R}(\tau)}$, $\tilde{y}_{\text{R}(\tau)}$, and $y_{\text{R}(\tau)}$ from $\Omega_{\text{Q}(\tau)}$. As such, these candidate real-time estimates exclude information on $\Delta^{\text{Q}} x_{\text{Q}(\tau)}$ that is available at the end of the quarter. Much of this information comes from the data releases on other macroeconomic variables like employment, retail sales, and industrial production. Data for most of these variables are collected on a monthly basis[3] and, as such, can provide timely information on GDP growth. To see why this is so, consider the data releases on nonfarm payroll employment, $z$. Data on $z$ for the month ending on day $\text{M}_z(\tau, j)$ are released on $\text{R}_z(\tau, j)$, a day that falls between the third and the ninth of month $j+1$ (as illustrated in figure 1). This reporting lag is much shorter than the lag for GDP releases but it does exclude the use of employment data from the third month in estimating real-time GDP. However, insofar as employment during the first two months is related to GDP growth over the quarter, the values of $z_{\text{R}(\tau,1)}$ and $z_{\text{R}(\tau,2)}$ will provide information relevant to estimating GDP growth at the end of the quarter.

The real-time estimates I construct below will be based on data from the three GDP releases and the monthly releases of other macroeconomic data. To incorporate the information from these other variables, I decompose quarterly GDP growth into a sequence of daily increments:

$$\Delta^{\text{Q}} x_{\text{Q}(\tau)} = \sum_{i=1}^{\text{D}(\tau)} \Delta x_{\text{Q}(\tau-1)+i}, \tag{7}$$

[3]Data on initial unemployment claims are collected week by week.

where $\text{D}(\tau) \equiv \text{Q}(\tau) - \text{Q}(\tau-1)$ is the duration of quarter $\tau$. The daily increment $\Delta x_t$ represents the contribution on day $t$ to the growth of GDP in quarter $\tau$. If $x_t$ were a stock variable, like the log price level on day $t$, $\Delta x_t$ would identify the daily growth in the stock (e.g., the daily rate of inflation). Here $x_{\text{Q}(\tau)}$ denotes the log of the flow of output over quarter $\tau$, so it is not appropriate to think of $\Delta x_t$ as the daily growth in GDP. I will examine the link between $\Delta x_t$ and daily GDP in section 3.3 below.

To incorporate the information contained in the $i$th macrovariable, $z^i$, I project $z^i_{\text{R}(\tau,j)}$ on a portion of GDP growth

$$z^i_{\text{R}(\tau,j)} = \beta_i \Delta^{\text{M}} x_{\text{M}(\tau,j)} + u^i_{\text{M}(\tau,j)}, \tag{8}$$

where $\Delta^{\text{M}} x_{\text{M}(\tau,j)}$ is the contribution to GDP growth in quarter $\tau$ during month $j$:

$$\Delta^{\text{M}} x_{\text{M}(\tau,j)} \equiv \sum_{i=\text{M}(\tau,j-1)+1}^{\text{M}(\tau,j)} \Delta x_i.$$

$\beta_i$ is the projection coefficient and $u^i_{\text{M}(\tau,j)}$ is the projection error that is orthogonal to $\Delta^{\text{M}} x_{\text{M}(\tau,j)}$. Notice that equation (8) incorporates the reporting lag $\text{R}_z(\tau,j) - \text{M}_z(\tau,j)$ for variable $z$, which can vary in length from month to month.

The real-time estimates derived in this paper are based on an information set specification that includes the three GDP releases and eighteen monthly macro series: $z^i = 1, 2, \ldots, 18$. Formally, I compute the end-of-quarter real-time estimates as

$$E[\Delta^{\text{Q}} x_{\text{Q}(\tau)} | \Omega_{\text{Q}(\tau)}], \tag{9}$$

where $\Omega_t = \Omega^z_t \cup \Omega^y_t$, with $\Omega^z_t$ denoting the information set comprising data on the eighteen monthly macrovariables that has been released on or before day $t$:

$$\Omega^z_t \equiv \bigcup_{i=1}^{18} \left\{ z^i_{\text{R}(\tau,j)} : \text{R}(\tau,j) < t \text{ for } j = 1,2,3 \right\}.$$

The model presented below enables us to compute the real-time estimates in (9) using equations (1), (5), (6), (7), and (8) together with a time-series process for the daily increments, $\Delta x_t$. The model will also enable us to compute *daily* real-time estimates of quarterly GDP, and GDP growth:

$$x_{\text{Q}(\tau)|i} \equiv E[x_{\text{Q}(\tau)}|\Omega_i] \tag{10}$$

$$\Delta^{\text{Q}} x_{\text{Q}(\tau)|i} \equiv E[\Delta^{\text{Q}} x_{\text{Q}(\tau)}|\Omega_i] \tag{11}$$

for $\text{Q}(\tau - 1) < i \leq \text{Q}(\tau)$. Equations (10) and (11) respectively identify the real-time estimate of log GDP, and GDP growth in quarter $\tau$, based on information available on day $i$ during the quarter. $x_{\text{Q}(\tau)|i}$ and $\Delta^{\text{Q}} x_{\text{Q}(\tau)|i}$ incorporate real-time forecasts of the daily contribution to GDP in quarter $\tau$ between day $i$ and $\text{Q}(\tau)$. These high-frequency estimates are particularly useful in studying how data releases affect estimates of the current state of the economy, and forecasts of how it will evolve in the future. As such, they are uniquely suited to examining how data releases affect a whole array of asset prices.

## 2. The Model

The dynamics of the model center on the behavior of two partial sums:

$$s_t^{\text{Q}} \equiv \sum_{i=\text{Q}(\tau)+1}^{\min\{\text{Q}(\tau),t\}} \Delta x_i, \tag{12}$$

$$s_t^{\text{M}} \equiv \sum_{i=\text{M}(\tau,j-1)+1}^{\min\{\text{M}(\tau,j),t\}} \Delta x_i. \tag{13}$$

Equation (12) defines the cumulative daily contribution to GDP growth in quarter $\tau$, ending on day $t \leq \text{Q}(\tau)$. The cumulative daily contribution between the start of month $j$ in quarter $\tau$ and day $t$ is defined by $s_t^{\text{M}}$. Notice that when $t$ is the last day of the quarter, $\Delta^{\text{Q}} x_{\text{Q}(\tau)} = s_{\text{Q}(\tau)}^{\text{Q}}$, and when $t$ is the last day of month $j$, $\Delta^{\text{M}} x_{\text{M}(\tau,j)} = s_{\text{M}(\tau,j)}^{\text{M}}$. To describe the daily dynamics of $s_t^{\text{Q}}$ and $s_t^{\text{M}}$, I introduce the following dummy variables:

$$\lambda_t^{\mathrm{M}} = \begin{cases} 1 & \text{if } t = \mathrm{M}(\tau, j) + 1, \text{ for } j = 1, 2, 3, \\ 0 & \text{otherwise,} \end{cases}$$

$$\lambda_t^{\mathrm{Q}} = \begin{cases} 1 & \text{if } t = \mathrm{Q}(\tau) + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\lambda_t^{\mathrm{M}}$ and $\lambda_t^{\mathrm{Q}}$ take the value of one if day $t$ is the first day of the month or quarter, respectively. We may now describe the daily dynamics of $s_t^{\mathrm{Q}}$ and $s_t^{\mathrm{M}}$ with the following equations:

$$s_t^{\mathrm{Q}} = \left(1 - \lambda_t^{\mathrm{Q}}\right) s_{t-1}^{\mathrm{Q}} + \Delta x_t, \tag{14}$$

$$s_t^{\mathrm{M}} = \left(1 - \lambda_t^{\mathrm{M}}\right) s_{t-1}^{\mathrm{M}} + \Delta x_t. \tag{15}$$

The next portion of the model accommodates the reporting lags. Let $\Delta^{\mathrm{Q}(j)} x_t$ denote the quarterly growth in GDP ending on day $\mathrm{Q}(\tau - j)$, where $\mathrm{Q}(\tau)$ denotes the last day of the most recently completed quarter and $t \geq \mathrm{Q}(\tau)$. Quarterly GDP growth in the last (completed) quarter is given by

$$\Delta^{\mathrm{Q}(1)} x_t = \left(1 - \lambda_t^{\mathrm{Q}}\right) \Delta^{\mathrm{Q}(1)} x_{t-1} + \lambda_t^{\mathrm{Q}} s_{t-1}^{\mathrm{Q}}. \tag{16}$$

When $t$ is the first day of a new quarter, $\lambda_t^{\mathrm{Q}} = 1$, so $\Delta^{\mathrm{Q}(1)} x_{\mathrm{Q}(\tau)+1} = s_{\mathrm{Q}(t)}^{\mathrm{Q}} = \Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)}$. On all other days, $\Delta^{\mathrm{Q}(1)} x_t = \Delta^{\mathrm{Q}(1)} x_{t-1}$. On some dates, the reporting lag associated with a "final" GDP data release is more than one quarter, so we will need to identify GDP growth from two quarters back, $\Delta^{\mathrm{Q}(2)} x_t$. This is achieved with a similar recursion:

$$\Delta^{\mathrm{Q}(2)} x_t = \left(1 - \lambda_t^{\mathrm{Q}}\right) \Delta^{\mathrm{Q}(2)} x_t + \lambda_t^{\mathrm{Q}} \Delta^{\mathrm{Q}(1)} x_{t-1}. \tag{17}$$

Equations (14), (16), and (17) enable us to define the link between the daily contributions to GDP growth $\Delta x_t$ and the three GDP data releases $\{\hat{y}_t, \tilde{y}_t, y_t\}$. Let us start with the "advanced" GDP data releases. The reporting lag associated with these data is always less than one quarter, so we can combine (1) and (5) with the definition of $\Delta^{\mathrm{Q}(1)} x_t$ to write

$$\hat{y}_t = \Delta^{\mathrm{Q}(1)} x_t + \upsilon_{\mathrm{R}(\tau)} + \tilde{e}_{\mathrm{R}(\tau)} + \hat{e}_{\mathrm{R}(\tau)}. \tag{18}$$

It is important to recognize that (18) builds in the variable reporting lag between the release day, $\text{R}_{\hat{y}}(\tau)$, and the end of the last quarter $\text{Q}(\tau)$. The value of $\Delta^{\text{Q}(1)}x_t$ does not change from day to day after the quarter ends, so the relation between the data release and actual GDP growth is unaffected by within-quarter variations in the reporting lag. The reporting lag for the "preliminary" data is also always less than one quarter. Combining (1) and (6) with the definition of $\Delta^{\text{Q}(1)}x_t$, we obtain

$$\tilde{y}_t = \Delta^{\text{Q}(1)}x_t + \upsilon_{\text{R}(\tau)} + \tilde{e}_{\text{R}(\tau)}. \tag{19}$$

Data on "final" GDP growth is released around the end of the following quarter, so the reporting lag can vary between one and two quarters. In cases where the reporting lag is one quarter,

$$y_t = \Delta^{\text{Q}(1)}x_t + \upsilon_{\text{R}(\tau)}, \tag{20}$$

and when the lag is two quarters,

$$y_t = \Delta^{\text{Q}(2)}x_t + \upsilon_{\text{R}(\tau)}. \tag{21}$$

I model the links between the daily contributions to GDP growth and the monthly macrovariables in a similar manner. Let $\Delta^{\text{M}(i)}x_t$ denote the monthly contribution to quarterly GDP growth ending on day $\text{M}(\tau, j-i)$, where $\text{M}(\tau, j)$ denotes the last day of the most recently completed month and $t \geq \text{M}(\tau, j)$. The contribution to GDP growth in the last (completed) month is given by

$$\Delta^{\text{M}(1)}x_t = (1-\lambda_t^{\text{M}})\,\Delta^{\text{M}(1)}x_{t-1} + \lambda_t^{\text{M}} s_{t-1}^{\text{M}}, \tag{22}$$

and the contribution from $i$ $(> 1)$ months back is

$$\Delta^{\text{M}(i)}x_t = (1-\lambda_t^{\text{M}})\,\Delta^{\text{M}(i)}x_t + \lambda_t^{\text{M}}\Delta^{\text{M}(i-1)}x_{t-1}. \tag{23}$$

These equations are analogous to (16) and (17). If $t$ is the first day of a new month, $\lambda_t^{\text{M}} = 1$, so $\Delta^{\text{M}(1)}x_{\text{M}(\tau,j)+1} = s^{\text{Q}}_{\text{M}(\tau,j)} = \Delta^{\text{M}}x_{\text{M}(\tau,j)}$ and $\Delta^{\text{M}(i)}x_{\text{M}(\tau,j)+1} = \Delta^{\text{M}(i-1)}x_{\text{M}(\tau,j)}$ for $j = 1, 2, 3$. On all other days, $\Delta^{\text{M}(i)}x_t = \Delta^{\text{M}(i)}x_{t-1}$. The $\Delta^{\text{M}(i)}x_t$ variables link the monthly data releases, $z_t^i$, to quarterly GDP growth. If the reporting lag for macro series $i$ is less than one month, the value released on day $t$ can be written as

$$z_t^i = \beta_i \Delta^{\mathrm{M}(1)} x_t + u_t^i. \tag{24}$$

In cases where the reporting lag is two months,

$$z_t^i = \beta_i \Delta^{\mathrm{M}(2)} x_t + u_t^i. \tag{25}$$

As above, both equations allow for a variable within-month reporting lag, $\mathrm{R}_{z^i}(\tau, j) - \mathrm{M}_{z^i}(\tau, j)$.

Equations (24) and (25) accommodate all the monthly data releases I use except for the index of consumer confidence, $i = 18$. This series is released *before* the end of the month in which the survey data are collected. These data are potentially valuable for drawing real-time inferences because they represent the only monthly release before $\mathrm{Q}(\tau)$ that relates to activity during the last month of the quarter. I incorporate the information in the consumer confidence index ($i = 18$) by projecting $z_t^{18}$ on the partial sum $s_t^{\mathrm{M}}$:

$$z_t^{18} = \beta_{18} s_t^{\mathrm{M}} + u_t^{18}. \tag{26}$$

To complete the model, we need to specify the dynamics for the daily contributions, $\Delta x_t$. I assume that

$$\Delta x_t = \sum_{i=1}^{k} \phi_i \Delta^{\mathrm{M}(i)} x_t + e_t, \tag{27}$$

where $e_t$ is an i.i.d.$N(0, \sigma_e^2)$ shock. Equation (27) expresses the growth contribution on day $t$ as a weighted average of the monthly contributions over the last $k$ (completed) months, plus an error term. This specification has two noteworthy features. First, the daily contribution on day $t$ only depends on the history of $\Delta x_t$ insofar as it is summarized by the monthly contributions, $\Delta^{\mathrm{M}(i)} x_t$. Thus, forecasts for $\Delta x_{t+h}$ conditional $\left\{\Delta^{\mathrm{M}(i)} x_t\right\}_{i=1}^{k}$ are the same for horizons $h$ within the current month. The second feature of (27) is that the process aggregates up to an AR($k$) process for $\Delta^{\mathrm{M}} x_{\mathrm{M}(\tau,j)}$ at the monthly frequency. As I shall demonstrate, this feature enables us to compute real-time forecasts of future GDP growth over monthly horizons with comparative ease.

## 3. Estimation

Finding the real-time estimates of GDP and GDP growth requires a solution to two related problems. First, there is a pure inference problem of how to compute $E[x_{Q(\tau)}|\Omega_i]$ and $E[\Delta^{Q} x_{Q(\tau)}|\Omega_i]$ using the quarterly signaling equations (18)–(21), the monthly signaling equations (24)–(26), and the $\Delta x_t$ process in (27), given values for all the parameters in these equations. Second, we need to estimate these parameters from the three data releases on GDP and the eighteen other macro series. This problem is complicated by the fact that individual data releases are irregularly spaced and arrive in a nonsynchronized manner: on some days there is one release, on others there are several, and on some there are none at all. In short, the temporal pattern of data releases is quite unlike that found in standard time-series applications.

The Kalman Filtering algorithm provides a solution to both problems. In particular, given a set of parameter values, the algorithm provides the means to compute the real-time estimates $E[x_{Q(\tau)}|\Omega_i]$ and $E[\Delta^{Q} x_{Q(\tau)}|\Omega_i]$. The algorithm also allows us to construct a sample likelihood function from the data series, so that the model's parameters can be computed by maximum likelihood. Although the Kalman Filtering algorithm has been used extensively in the applied time-series literature, its application in the current context has several novel aspects. For this reason, the presentation below concentrates on these features.[4]

### *3.1 The State-Space Form*

To use the algorithm, we must first write the model in state-space form comprising a state and observation equation. For the sake of clarity, I shall present the state-space form for the model where $\Delta x_t$ depends only on last month's contribution (i.e., $k = 1$ in equation [27]). Modifying the state-space form for the case where $k > 1$ is straightforward.

[4]For a textbook introduction to the Kalman Filter and its uses in standard time-series applications, see Harvey (1989) or Hamilton (1994).

The dynamics described by equations (14)–(17), (22), (23), and (27) with $k = 1$ can be represented by the matrix equation:

$$\begin{bmatrix} s_t^{\mathrm{Q}} \\ \Delta^{\mathrm{Q}(1)} x_t \\ \Delta^{\mathrm{Q}(2)} x_t \\ s_t^{\mathrm{M}} \\ \Delta^{\mathrm{M}(1)} x_t \\ \Delta^{\mathrm{M}(2)} x_t \\ \Delta x_t \end{bmatrix} = \begin{bmatrix} 1-\lambda_t^{\mathrm{Q}} & 0 & 0 & 0 & 0 & 0 & 1 \\ \lambda_t^{\mathrm{Q}} & 1-\lambda_t^{\mathrm{Q}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_t^{\mathrm{Q}} & 1-\lambda_t^{\mathrm{Q}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\lambda_t^{\mathrm{M}} & 0 & 0 & 1 \\ 0 & 0 & 0 & \lambda_t^{\mathrm{M}} & 1-\lambda_t^{\mathrm{M}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_t^{\mathrm{M}} & 1-\lambda_t^{\mathrm{M}} & 0 \\ 0 & 0 & 0 & 0 & \phi_1 & 0 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} s_{t-1}^{\mathrm{Q}} \\ \Delta^{\mathrm{Q}(1)} x_{t-1} \\ \Delta^{\mathrm{Q}(2)} x_{t-1} \\ s_{t-1}^{\mathrm{M}} \\ \Delta^{\mathrm{M}(1)} x_{t-1} \\ \Delta^{\mathrm{M}(2)} x_{t-1} \\ \Delta x_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ e_t \end{bmatrix},$$

or, more compactly,

$$\mathbb{Z}_t = \mathbb{A}_t \mathbb{Z}_{t-1} + \mathbb{V}_t. \tag{28}$$

Equation (28) is known as the state equation. In traditional time-series applications, the state transition matrix $\mathbb{A}$ is constant. Here elements of $\mathbb{A}_t$ depend on the quarterly and monthly dummies, $\lambda_t^{\mathrm{Q}}$ and $\lambda_t^{\mathrm{M}}$, and so $\mathbb{A}_t$ is time varying.

Next, we turn to the observation equation. The link between the data releases on GDP and elements of the state vector are described by (18), (19), (20), and (21). These equations can be rewritten as

$$\begin{bmatrix} \hat{y}_t \\ \tilde{y}_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0 & \text{QL}_t^1(\hat{y}) & \text{QL}_t^2(\hat{y}) & 0 & 0 & 0 & 0 \\ 0 & \text{QL}_t^1(\tilde{y}) & \text{QL}_t^2(\tilde{y}) & 0 & 0 & 0 & 0 \\ 0 & \text{QL}_t^1(y) & \text{QL}_t^2(y) & 0 & 0 & 0 & 0 \end{bmatrix} \mathbb{Z}_t + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{e}_t \\ \tilde{e}_t \\ \upsilon_t \end{bmatrix}, \tag{29}$$

where $\text{QL}_t^i(\varkappa)$ denotes a dummy variable that takes the value of one when the reporting lag for series $\varkappa$ lies between $i-1$ and $i$ quarters, and zero otherwise. Thus, $\text{QL}_t^1(y) = 1$ and $\text{QL}_t^2(y) = 0$ when "final" GDP data for the first quarter are released before the start of the third quarter, while $\text{QL}_t^1(y) = 0$ and $\text{QL}_t^2(y) = 1$ in cases where the release is delayed until the third quarter. Under normal circumstances, the "advanced" and "preliminary" GDP data releases have reporting lags that are less than a month. However, there was one occasion in the sample period where all the GDP releases were delayed, so that the $\text{QL}_t^i(\varkappa)$ dummies are also needed for the $\hat{y}_t$ and $\tilde{y}_t$ equations.

The link between the data releases on the monthly series and elements of the state vector is described by (24)–(26). These equations can be written as

$$z_t^i = \begin{bmatrix} 0 & 0 & 0 & \beta_i\text{ML}_t^0(z^i) & \beta_i\text{ML}_t^1(z^i) & \beta_i\text{ML}_t^2(z^i) & 0 \end{bmatrix} \mathbb{Z}_t + u_t^i, \tag{30}$$

for $i = 1, 2, \ldots, 18$. $\text{ML}_t^i(\varkappa)$ is the monthly version of $\text{QL}_t^i(\varkappa)$. $\text{ML}_t^i(\varkappa)$ is equal to one if the reporting lag for series $\varkappa$ lies between $i-1$ and $i$ months ($i = 1, 2$), and zero otherwise. $\text{ML}_t^0(\varkappa)$ equals one when the release day is before the end of the collection month (as is the case with the index of consumer confidence). Stacking (29) and (30) gives

$$
\begin{bmatrix} \hat{y}_t \\ \tilde{y}_t \\ y_t \\ z_t^1 \\ \vdots \\ z_t^{18} \end{bmatrix}
= \begin{bmatrix}
0 & \mathrm{QL}_t^1(\hat{y}) & \mathrm{QL}_t^2(\hat{y}) & 0 & 0 & 0 & 0 \\
0 & \mathrm{QL}_t^1(\tilde{y}) & \mathrm{QL}_t^2(\tilde{y}) & 0 & 0 & 0 & 0 \\
0 & \mathrm{QL}_t^1(y) & \mathrm{QL}_t^2(y) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \beta_1 \mathrm{ML}_t^0(z^1) & \beta_i \mathrm{ML}_t^1(z^1) & \beta_1 \mathrm{ML}_t^2(z^1) & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \beta_{18} \mathrm{ML}_t^0(z^{18}) & \beta_{18} \mathrm{ML}_t^1(z^{18}) & \beta_{18} \mathrm{ML}_t^2(z^{18}) & 0
\end{bmatrix} \mathbb{Z}_t
+ \begin{bmatrix} \hat{e}_t + \tilde{e}_t + \upsilon_t \\ \tilde{e}_t + \upsilon_t \\ \upsilon_t \\ u_t^1 \\ \vdots \\ u_t^{18} \end{bmatrix},
$$

or

$$
\mathbb{X}_t = \mathbb{C}_t \mathbb{Z}_t + \mathbb{U}_t. \tag{31}
$$

This equation links the vector of *potential* data releases for day $t$, $\mathbb{X}_t$, to elements of the state vector. The elements of $\mathbb{X}_t$ identify the value that would have been released for each series given the current state, $\mathbb{Z}_t$, if day $t$ was in fact the release day. Of course, on a typical day, we would only observe the elements in $\mathbb{X}_t$ that correspond to the actual releases that day. For example, if data on "final" GDP and monthly series $i = 1$ are released on day $t$, we would observe the values in the third and fourth rows of $\mathbb{X}_t$. On days when there are no releases, none of the elements of $\mathbb{X}_t$ are observed.

The observation equation links the data releases for day $t$ to the state vector. The vector of actual data releases for day $t$, $\mathbb{Y}_t$, is related to the vector of *potential* releases by

$$\mathbb{Y}_t = \mathbb{B}_t \mathbb{X}_t,$$

where $\mathbb{B}_t$ is an $n \times 21$ selection matrix that "picks out" the $n \geq 1$ data releases for day $t$. For example, if data on monthly series $i = 1$ are released on day $t$, $\mathbb{B}_t = [\ 0 \quad 0 \quad 0 \quad 1 \quad 0, \quad \ldots\ldots, \quad 0\ ]$. Combining this expression with (31) gives the observation equation:

$$\mathbb{Y}_t = \mathbb{B}_t \mathbb{C}_t \mathbb{Z}_t + \mathbb{B}_t \mathbb{U}_t. \tag{32}$$

Equation (32) differs in several respects from the observation equation specification found in standard time-series applications. First, the equation only applies on days for which at least one data release takes place. Second, the link between the observed data releases and the state vector varies through time via $\mathbb{C}_t$ as $\text{QL}_t^i(\varkappa)$ and $\text{ML}_t^i(\varkappa)$ change. These variations arise because the reporting lag associated with a given data series change from release to release. Third, the number and nature of the data releases vary from day to day (i.e., the dimension of $\mathbb{Y}_t$ can vary across consecutive data-release days) via the $\mathbb{B}_t$ matrix. These changes may be a source of heteroskedasticity. If the $\mathbb{U}_t$ vector has a constant covariance matrix $\Omega_u$, the vector of noise terms entering the observation equation will be heteroskedastic with covariance $\mathbb{B}_t \Omega_u \mathbb{B}_t'$.

### *3.2 The Kalman Filter and Sample Likelihood Function*

Equations (28) and (32) describe a state-space form that can be used to find real-time estimates of GDP in two steps. In the first, I obtain the maximum likelihood estimates of the model's parameters. The second step calculates the real-time estimates of GDP using the maximum likelihood parameter estimates. Below, I briefly describe these steps, noting where the model gives rise to features that are not seen in standard time-series applications.

The parameters of the model to be estimated are $\theta = \{\beta_1, \ldots, \beta_{21}, \phi_1, \ldots, \phi_k, \sigma_e^2, \sigma_{\tilde{e}}^2, \sigma_{\hat{e}}^2, \sigma_v^2, \sigma_1^2, \ldots, \sigma_{18}^2\}$, where $\sigma_e^2$, $\sigma_{\tilde{e}}^2$, $\sigma_{\hat{e}}^2$, and $\sigma_v^2$ denote the variances of $e_t$, $\tilde{e}_t$, $\hat{e}_t$, and $\upsilon_t$, respectively. The variance of $u_t^i$ is $\sigma_i^2$ for $i = 1, \ldots, 18$. For the purpose of estimation, I assume that all variances are constant, so the covariance matrices for $\mathbb{V}_t$ and $\mathbb{U}_t$ can be written as $\Sigma_v$ and $\Sigma_u$, respectively. The sample likelihood function is built up recursively by applying the Kalman Filter to (28) and (32). Let $n_t$ denote the number of data releases on day $t$. The sample log likelihood function for a sample spanning $t = 1, \ldots, T$ is

$$\mathcal{L}(\theta) = \sum_{t=1, n_t>0}^{T} \left\{ -\frac{n_t}{2} \ln(2\pi) - \frac{1}{2} \ln |\omega_t| \; - \frac{1}{2} \eta_t' \omega_t^{-1} \eta_t \right\}, \qquad (33)$$

where $\eta_t$ denotes the vector of innovations on day $t$ with $n_t > 0$, and $\omega_t$ is the associated conditional covariance matrix. The $\eta_t$ and $\omega_t$ sequences are calculated as functions of $\theta$ from the filtering equations:

$$\mathbb{Z}_{t|t} = \mathbb{A}_t \mathbb{Z}_{t-1|t-1} + \mathbb{K}_t \eta_t, \qquad (34a)$$

$$\mathbb{S}_{t+1|t} = \mathbb{A}_t \left( I - \mathbb{K}_t \mathbb{B}_t \mathbb{C}_t \right) \mathbb{S}_{t|t-1} \mathbb{A}_t' + \Sigma_v, \qquad (34b)$$

where

$$\eta_t = \mathbb{Y}_t - \mathbb{B}_t \mathbb{C}_t \mathbb{A}_t \mathbb{Z}_{t-1|t-1}, \qquad (35a)$$

$$\mathbb{K}_t = \mathbb{S}_{t|t-1} \mathbb{C}_t' \mathbb{B}_t' \omega_t^{-1}, \qquad (35b)$$

$$\omega_t = \mathbb{B}_t \mathbb{C}_t \mathbb{S}_{t|t-1} \mathbb{C}_t' \mathbb{B}_t' + \mathbb{B}_t \Sigma_u \mathbb{B}_t', \qquad (35c)$$

if $n_t > 0$, and

$$\mathbb{Z}_{t|t} = \mathbb{A}_t \mathbb{Z}_{t-1|t-1}, \qquad (36a)$$

$$\mathbb{S}_{t+1|t} = \mathbb{A}_t \mathbb{S}_{t|t-1} \mathbb{A}_t' + \Sigma_v, \qquad (36b)$$

when $n_t = 0$. The recursions are initialized with $\mathbb{S}_{1|0} = \Sigma_v$ and $\mathbb{Z}_{0|0}$ equal to a vector of zeros. Notice that (34)–(36) differ from the standard filtering equations because the structure of the state-space form in (28) and (32) changes via the $\mathbb{A}_t$, $\mathbb{C}_t$, and $\mathbb{B}_t$ matrices. The filtering equations also need to account for the days on which no data is released.

As in standard applications of the Kalman Filter, we need to ensure that all the elements of $\theta$ are identified. Recall that equation

(1) includes an error term $\upsilon_t$ to allow for annual revisions to the "final" GDP data that take place after the release day $\mathrm{R}_y(\tau)$. The variance of $\upsilon_t$, $\sigma_\upsilon^2$, is not identified because the state-space form excludes data on the annual revisions. Rather than amend the model to include these data, I impose the identifying restriction: $\sigma_\upsilon^2 = 0$.[5] This restriction limits the duration of uncertainty concerning GDP growth to the reporting lag for the "final" GDP release. In section 5, I show that most of the uncertainty concerning GDP growth in quarter $\tau$ is resolved by the end of the first month in quarter $\tau + 1$, well before the end of the reporting lag. Limiting the duration of uncertainty does not appear unduly restrictive.

### *3.3 Calculating the Real-Time Estimates of GDP*

Once the maximum likelihood estimates of $\theta$ have been found, the Kalman Filtering equations can be readily used to calculate real-time estimates of GDP. Consider, first, the real-time estimates at the end of each quarter $\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)|\mathrm{Q}(\tau)}$. By definition, $\mathbb{Z}_{t|j}$ denotes the expectation of $\mathbb{Z}_t$ conditioned on data released by the end of day $j$, $E[\mathbb{Z}_t|\Omega_j]$. Hence, the real-time estimates of quarterly GDP growth are given by

$$\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)|\mathrm{Q}(\tau)} = E[s^{\mathrm{Q}}_{\mathrm{Q}(\tau)}|\Omega_{\mathrm{Q}(\tau)}] = h_1 \hat{\mathbb{Z}}_{\mathrm{Q}(\tau)|\mathrm{Q}(\tau)}, \tag{37}$$

for $\tau = 1, 2, \ldots$, where $h_i$ is a vector that selects the $i$th element of $\mathbb{Z}_t$. $\hat{\mathbb{Z}}_{t|t}$ denotes the value of $\mathbb{Z}_{t|t}$ based on the MLE of $\theta$ computed from (34)–(36). The Kalman Filter allows us to study how the estimates of $\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)}$ change in the light of data releases after the quarter has ended. For example, the sequence $\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)|t} = h_2 \hat{\mathbb{Z}}_{t|t}$, for $\mathrm{Q}(\tau) < t \leq \mathrm{Q}(\tau + 1)$, shows how data releases between the end of quarters $\tau$ and $\tau + 1$ change the real-time estimates of $\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)}$.

[5] In principle, the state-space form could be augmented to accommodate the revision data, but the resulting state vector would have forty-odd elements because revisions can take place up to three years after the "final" GDP data is released. Estimating such a large state-space system would be quite challenging. Alternatively, one could estimate $\sigma_\upsilon^2$ directly from the various vintages of "final" growth rates for each quarter, and then compute the maximum likelihood estimates of the other parameters conditioned on this value.

We can also use the model to find real-time estimates of GDP growth before the end of the quarter. Recall that quarterly GDP growth can be represented as the sum of daily increments:

$$\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)} = \sum_{i=1}^{\mathrm{D}(\tau)} \Delta x_{\mathrm{Q}(\tau-1)+i}. \tag{7}$$

Real-time estimates of $\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)}$ based on information $\Omega_t$, where $\mathrm{Q}(\tau-1) < t \leq \mathrm{Q}(\tau)$, can be found by taking conditional expectations on both sides of this equation:

$$\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)|t} = E[s_t^{\mathrm{Q}}|\Omega_t] + \sum_{h=1}^{\mathrm{Q}(\tau)} E[\Delta x_{t+h}|\Omega_t]. \tag{38}$$

The first term on the right-hand side is the real-time estimate of the partial sum $s_t^{\mathrm{Q}}$ defined in (12). Since $s_t^{\mathrm{Q}}$ is the first element in the state vector $\mathbb{Z}_t$, a real-time estimate of $s_t^{\mathrm{Q}}$ can be found as $E[s_t^{\mathrm{Q}}|\Omega_t] = h_1 \hat{\mathbb{Z}}_{t|t}$. The second term in (38) contains real-time *forecasts* for the daily increments over the remaining days in the month. These forecasts can be easily computed from the process for the increments in (27):

$$E[\Delta x_{t+h}|\Omega_t] = \sum_{i=1}^{k} \phi_i E[\Delta^{\mathrm{M}(i)} x_t|\Omega_t]. \tag{39}$$

Notice that the real-time estimates of $\Delta^{\mathrm{M}(i)} x_t$ on the right-hand side are also elements of the state vector $\mathbb{Z}_t$, so the real-time forecasts can be easily found from $\hat{\mathbb{Z}}_{t|t}$. For example, for the state-space form with $k = 1$ described above, the real-time estimates can be computed as $\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)|t} = \left[h_1 + h_5 \hat{\phi}_1 (\mathrm{Q}(\tau) - t)\right] \hat{\mathbb{Z}}_{t|t}$, where $\hat{\phi}_1$ is the MLE of $\phi_1$.

The model can also be used to calculate real-time estimates of log GDP, $x_{\mathrm{Q}(\tau)|i}$. Once again, it is easiest to start with the end-of-quarter real-time estimates, $x_{\mathrm{Q}(\tau)|\mathrm{Q}(\tau)}$. Iterating on the identity $\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)} \equiv x_{\mathrm{Q}(\tau)} - x_{\mathrm{Q}(\tau-1)}$, we can write

$$x_{\mathrm{Q}(\tau)} = \sum_{i=1}^{\tau} \Delta^{\mathrm{Q}} x_{\mathrm{Q}(i)} + x_{\mathrm{Q}(0)}. \tag{40}$$

Thus, log GDP for quarter $\tau$ can be written as the sum of quarterly GDP growth from quarters 1 to $\tau$, plus the initial log level of GDP for quarter 0. Taking conditional expectations on both sides of this expression gives

$$x_{\mathrm{Q}(\tau)|\mathrm{Q}(\tau)} = \sum_{i=1}^{\tau} \mathbb{E}[\Delta^{\mathrm{Q}} x_{\mathrm{Q}(i)} | \Omega_{\mathrm{Q}(\tau)}] + \mathbb{E}[x_{\mathrm{Q}(0)} | \Omega_{\mathrm{Q}(\tau)}]. \qquad (41)$$

Notice that the terms in the sum on the right-hand side are *not* the real-time estimates of GDP growth. Rather, they are current estimates (i.e., based on $\Omega_{\mathrm{Q}(\tau)}$) of past GDP growth. Thus, we cannot construct real-time estimates of log GDP by simply aggregating the real-time estimate of GDP growth from the current and past quarters.

In principle, $x_{\mathrm{Q}(\tau)|\mathrm{Q}(\tau)}$ could be found using estimates of $\mathbb{E}[\Delta^{\mathrm{Q}} x_{\mathrm{Q}(i)} | \Omega_{\mathrm{Q}(\tau)}]$ computed from the state-space form with the aid of the Kalman Smoother algorithm (see, for example, Hamilton 1994). An alternative approach is to apply the Kalman Filter to a modified version of the state-space form:

$$\mathbb{Z}_t^a = \mathbb{A}_t^a \mathbb{Z}_t^a + \mathbb{V}_t^a, \qquad (42a)$$

$$\mathbb{Y}_t = \mathbb{B}_t \mathbb{C}_t^a \mathbb{Z}_t^a + \mathbb{B}_t \mathbb{U}_t, \qquad (42b)$$

where

$$\mathbb{Z}_t^a \equiv \begin{bmatrix} \mathbb{Z}_t \\ x_t \end{bmatrix}, \quad \mathbb{A}_t^a \equiv \begin{bmatrix} \mathbb{A}_t & \mathbf{0} \\ h_7 & 1 \end{bmatrix}, \quad \mathbb{V}_t^a = \begin{bmatrix} I_7 \\ h_7 \end{bmatrix} \mathbb{V}_t, \text{ and}$$

$$\mathbb{C}_t^a \equiv \begin{bmatrix} \mathbb{C}_t & \mathbf{0} \end{bmatrix}.$$

This modified state-space form adds the cumulant of the daily increments, $x_t \equiv \sum_{i=1}^{t} \Delta x_i + x_{\mathrm{Q}(0)}$, as the eighth element in the augmented state vector $\mathbb{Z}_t^a$. At the end of the quarter when $t = \mathrm{Q}(\tau)$, the cumulant is equal to $x_{\mathrm{Q}(\tau)}$. So a real-time end-of-quarter estimate of log GDP can be computed as $x_{\mathrm{Q}(\tau)|\mathrm{Q}(\tau)} = h_8^a \hat{\mathbb{Z}}^a_{\mathrm{Q}(\tau)|\mathrm{Q}(\tau)}$, where $\hat{\mathbb{Z}}^a_{t|t}$ is the estimate of $\mathbb{Z}_t^a$ derived by applying that Kalman Filter to (42), and $h_i^a$ is a vector that picks out the $i$th element of $\mathbb{Z}_t^a$.

Real-time estimates of log GDP in quarter $\tau$ based on information available on day $t < \mathrm{Q}(\tau)$ can be calculated in a similar fashion. First, we use (7) and the definition of $x_t$ to rewrite (40) as

$$x_{\mathrm{Q}(\tau)} = x_t + \sum_{h=t+1}^{\mathrm{Q}(\tau)} \Delta x_h.$$

As above, the real-time estimate is found by taking conditional expectations:

$$x_{\mathrm{Q}(\tau)|t} = \mathbb{E}[x_t|\Omega_t] + \sum_{h=t+1}^{\mathrm{Q}(\tau)} \mathbb{E}[\Delta x_h|\Omega_t]. \tag{43}$$

The real-time estimate of log GDP for quarter $\tau$, based on information available on day $t \leq \mathrm{Q}(\tau)$, comprises the real-time estimate of $x_t$ and the sum of the real-time forecasts for $\Delta x_{t+h}$ over the remainder of the quarter. Notice that each component on the right-hand side was present in the real-time estimates discussed above, so finding $x_{\mathrm{Q}(\tau)|t}$ involves nothing new. For example, in the $k = 1$ case, $x_{\mathrm{Q}(\tau)|t} = [h_8^a + h_5^a \hat{\phi}_1 (\mathrm{Q}(\tau) - t)]\hat{\mathbb{Z}}_{t|t}^a$.

To this point, I have concentrated on the problem of calculating real-time estimates for GDP and GDP growth measured on a quarterly basis. We can also use the model to calculate real-time estimates of output flows over shorter horizons, such as a month or week. For this purpose, I first decompose quarterly GDP into its daily components. These components are then aggregated to construct estimates of output measured over any horizon.

Let $d_t$ denote the log of output on day $t$. Since GDP for quarter $\tau$ is simply the aggregate of daily output over the quarter,

$$x_{\mathrm{Q}(\tau)} \equiv \ln\left(\sum\nolimits_{i=1}^{\mathrm{D}(\tau)} \exp\left(d_{\mathrm{Q}(\tau-1)+i}\right)\right), \tag{44}$$

where $\mathrm{D}(\tau) \equiv \mathrm{Q}(\tau) - \mathrm{Q}(\tau - 1)$ is the duration of quarter $\tau$. Equation (44) describes the exact nonlinear relation between log GDP for quarter $\tau$ and the log of daily output. In principle, we would like to use this equation and the real-time estimates of $x_{\mathrm{Q}(\tau)}$ to identify the sequence for $d_t$ over each quarter. Unfortunately, this is a form of nonlinear filtering problem that has no exact solution. Consequently, to make any progress, we must work with either an approximate solution to the filtering problem or a linear approximation of (44).

I follow the second approach by working with a first-order Taylor approximation to (44) around the point where $d_t = x_{(\tau)} - \ln \mathrm{D}(\tau)$:

$$x_{\mathrm{Q}(\tau)} \cong \frac{1}{\mathrm{D}(\tau)} \sum_{i=1}^{\mathrm{D}(\tau)} \left\{ d_{\mathrm{Q}(\tau-1)+i} + \ln \mathrm{D}(\tau) \right\}. \tag{45}$$

Combining (45) with the identity $\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)} \equiv x_{\mathrm{Q}(\tau)} - x_{\mathrm{Q}(\tau-1)}$ gives

$$\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)} \cong \frac{1}{\mathrm{D}(\tau)} \sum_{i=1}^{\mathrm{D}(\tau)} \left\{ d_{\mathrm{Q}(\tau-1)+i} - \left( x_{\mathrm{Q}(\tau-1)} - \ln \mathrm{D}(\tau) \right) \right\}. \tag{46}$$

This expression takes the same form as the decomposition of quarterly GDP growth in (7) with $\Delta x_t \cong \left\{ d_t - x_{\mathrm{Q}(\tau-1)} + \ln \mathrm{D}(\tau) \right\} / \mathrm{D}(\tau)$. Rearranging this expression gives us the following approximation for log daily output:

$$d_t \cong x_{\mathrm{Q}(\tau-1)} + \mathrm{D}(\tau) \Delta x_t - \ln \mathrm{D}(\tau). \tag{47}$$

According to this approximation, all the within-quarter variation in the log of daily output is attributable to daily changes in the increments $\Delta x_t$. Thus, changes in $x_t$ within each quarter provide an approximate (scaled) estimate of the volatility in daily output.

The last step is to construct the new output measure based on (47). Let $x_t^h$ denote the log flow of output over $h$ days ending on day $t$: $x_t^h \equiv \ln \left( \sum_{i=0}^{h-1} \exp\left(d_{t-i}\right) \right)$. As before, I avoid the problems caused by the nonlinearity in this definition by working with a first-order Taylor approximation to $x_t^h$ around the point where $d_t = x_t^h - \ln h$. Combining this approximation with (47) and taking conditional expectations gives

$$\begin{aligned} x_{t|t}^h \cong \frac{1}{h} \sum_{i=1}^{h-1} \big\{ & E[x_{\mathrm{Q}(\tau_{t-i}-1)} | \Omega_t] + \mathrm{D}(\tau_{t-i}) E[\Delta x_{t-i} | \Omega_t] \\ & - \ln \mathrm{D}(\tau_{t-i}) + \ln h \big\}, \end{aligned} \tag{48}$$

where $\tau_t$ denotes the quarter in which day $t$ falls. Equation (48) provides us with an approximation for the real-time estimates of $x_t^h$ in terms that can be computed from the model. In particular, if we augment the state vector to include $x_{\mathrm{Q}(\tau_{t-i}-1)}$ and $\Delta x_{t-i}$ for

$i = 1, \ldots, h-1$, and apply the Kalman Filter to the resulting modified state space, the estimates of $E[x_{\mathrm{Q}(\tau_{t-i}-1)}|\Omega_t]$ and $E[\Delta x_{t-i}|\Omega_t]$ can be constructed from $\hat{\mathbb{Z}}^a_{t|t}$.

## 4. Empirical Results

### 4.1 Data

The macroeconomic data releases used in estimation are from International Money Market Services (MMS). These include real-time data on both expected and announced macrovariables. I estimate the model using the three quarterly GDP releases and the monthly releases on eighteen other variables from April 11, 1993, through June 30, 1999. In specification tests described below, I also use market expectations of GDP growth based on surveys conducted by MMS of approximately forty money managers on the Friday of the week before the release day. Many earlier studies have used MMS data to construct proxies for the news contained in data releases (see, for example, Urich and Watchel 1984; Balduzzi, Elton, and Green 2001; and Andersen et al. 2003). This is the first paper to use MMS data in estimating real-time estimates of macroeconomic variables.

The upper panel of table 1 lists the data series used in estimation. The right-hand columns report the number of releases and the range of the reporting lag for each series during the sample period. The lower panel shows the distribution of data releases. The sample period covers 1,682 workdays (i.e., all days excluding weekends and national holidays).[6] On approximately 55 percent of these days, there was at least one data release. Multiple data releases occurred much less frequently, on approximately 16 percent of the workdays in the sample. There were no occasions when more than four data releases took place.

The release data were transformed in two ways before being incorporated in the model. First, I subtracted the sample mean from

[6] Although economic activity obviously takes place on weekends and holidays, I exclude these days from the sample for two reasons. First, they contain no data releases. This means that the contribution to GDP on weekends and holidays must be exclusively derived from the dynamics of (27). Second, by including only workdays, we can exactly align the real-time estimates with days on which U.S. financial markets were open. This feature will be very helpful in studying the relation between the real-time estimates and asset prices.

**Table 1. Data Series (April 11, 1993–June 30, 1999)**

| | Release | Obs. | Reporting Lag |
|---|---|---|---|
| Quarterly | Advanced GDP | 26 | 1–2 Months |
| | Preliminary GDP | 25 | 2–3 Months |
| | Final GDP | 26 | 3–4 Months |
| Monthly | | | |
| Real Activity | Nonfarm Payroll Employment | 78 | 3–9 Days |
| | Retail Sales | 78 | 12–15 Days |
| | Industrial Production | 78 | 15–18 Days |
| | Capacity Utilization | 78 | 15–18 Days |
| | Personal Income | 76 | 30–33 Days |
| | Consumer Credit | 78 | 33–40 Days |
| Consumption | Personal Consumption Expenditures | 76 | 30–33 Days |
| | New Home Sales | 77 | 27–33 Days |
| Investment | Durable Goods Orders | 77 | 24–29 Days |
| | Construction Spending | 77 | 31–34 Days |
| | Factory Orders | 76 | 29–35 Days |
| | Business Inventories | 78 | 38–44 Days |
| Government | Government Budget Deficit | 78 | 15–21 Days |
| Net Exports | Trade Balance | 78 | 44–53 Days |
| Forward Looking | Consumer Confidence Index | 78 | –8–0 Days |
| | NAPM Index | 78 | 0–6 Days |
| | Housing Starts | 77 | 14–20 Days |
| | Index of Leading Indicators | 78 | 27–45 Days |

Distribution of Data Releases

| Releases per Day | Fraction of Sample | Observations |
|---|---|---|
| 0 | 45.48% | 765 |
| 1 | 38.76% | 652 |
| 2 | 10.46% | 176 |
| 3 | 4.34% | 73 |
| 4 | 0.95% | 16 |
| $> 0$ | 54.52% | 917 |

each of the GDP releases. This transformation implies that the real-time estimates presented below are based on the assumption that long-run GDP growth remained constant over the sample period. If the span of my data were considerably longer, I could identify how the long-run rate of GDP growth has varied by estimating a modified form of the model that replaced (27) with a process that decomposed $\Delta x_t$ into short- and long-run components. I leave this extension of the model for future work.

The second transformation concerns the monthly data. Let $\tilde{z}^i_{\mathrm{R}(\tau,j)}$ denote the raw value for series $i$ released on day $t = \mathrm{R}(\tau, j)$. The model incorporates transformed series $z^i_{\mathrm{R}(\tau,j)} = (\tilde{z}^i_{\mathrm{R}(\tau,j)} - \bar{z}^i) - \alpha_i(\tilde{z}^i_{\mathrm{R}(\tau,j-1)} - \bar{z}^i)$, where $\bar{z}^i$ is the sample mean of $\tilde{z}^i$. Recall that in the model, the monthly series provide noisy signals on the monthly contribution to GDP growth (see equations [24]–[26]). Quasi-differencing in this manner allows each of the raw data series to have a differing degree of persistence than the monthly contribution to GDP growth without inducing serial correlation in the projection errors shown in (24)–(26). The degree of quasi-differencing depends on the $\alpha_i$ parameters which are jointly estimated with the other model parameters.

### *4.2 Estimates and Diagnostics*

The maximum likelihood estimates of the model are reported in table 2. There are sixty-three parameters in the model, and all are estimated with a great deal of precision. T-tests based on the asymptotic standard errors (reported in parentheses) show that all the coefficients are significant at the 1 percent level. Panel A of the table shows the estimated parameters of the daily contribution process in (27). Notice that the reported estimates and standard errors are multiplied by twenty-five. With this scaling, the reported values for the $\phi_i$ parameters represent the coefficients in the time-aggregated AR(6) process for $\Delta^{\mathrm{M}} x_{\mathrm{M}(\tau,j)}$ in a typical month (i.e., one with twenty-five workdays). I shall examine the implications of these estimates for forecasting GDP below.

Panel B reports the estimated standard deviations of the difference between the "advanced" and "final" GDP releases, $\omega_a \equiv \hat{y}_t - y_t$, and the difference between the "preliminary" and "final" releases, $\omega_p \equiv \tilde{y}_t - y_t$. According to equations (5) and (6) of the model, $\mathbb{V}(\omega_a) = \mathbb{V}(\omega_p) + \mathbb{V}(\hat{e}_{\mathrm{R}(\tau)})$, so the standard deviation of $\omega_a$ should be at least as great as that of $\omega_p$. By contrast, the estimates in panel B imply that $\mathbb{V}(\omega_a) < \mathbb{V}(\omega_p)$.[7] This suggests that revisions the BEA made between releasing the "preliminary" and "final" GDP data were negatively correlated with the revisions between the "advanced"

[7]To check robustness, I also estimated the model with the $\mathbb{V}(\omega_a) = \mathbb{V}(\omega_p) + \mathbb{V}(\hat{e}_{\mathrm{R}(\tau)})$ restriction imposed. In this case, the MLE of $\mathbb{V}(\hat{e}_{\mathrm{R}(\tau)})$ is less than 0.0001.

**Table 2. Model Estimates**

| | **A. Process for $\Delta x_t$** | $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_4$ | $\phi_5$ | $\phi_6$ | $\sigma_e$ |
|---|---|---|---|---|---|---|---|---|
| | estimate** | –0.384 | 0.296 | 0.266 | –0.289 | –0.485 | 0.160 | 3.800 |
| | standard error** | (0.004) | (0.003) | (0.003) | (0.003) | (0.003) | (0.004) | (0.010) |
| | **B. Quarterly Data Releases** | | $\mathbb{V}(\omega_i)$ | std$(\omega_i)^*$ | | | | |
| $a$ | Advanced GDP Growth | | 0.508 | (0.177) | | | | |
| $p$ | Preliminary GDP Growth | | 1.212 | (0.312) | | | | |
| | **C. Monthly Data Releases** | | $\alpha_i$ | std$(\alpha_i)^*$ | $\beta_i$ | std$(\beta_i)^*$ | $\sigma_i$ | std$(\sigma_i)^*$ |
| 1 | Nonfarm Payroll Employment | | 0.007 | (0.218) | 0.656 | (0.301) | 0.932 | (0.171) |
| 2 | Retail Sales | | –0.047 | (0.282) | 0.285 | (0.136) | 0.381 | (0.082) |
| 3 | Industrial Production | | –0.028 | (0.145) | 0.189 | (0.090) | 0.229 | (0.035) |
| 4 | Capacity Utilization | | 0.924 | (0.088) | 0.125 | (0.114) | 0.382 | (0.020) |
| 5 | Personal Income | | –0.291 | (0.219) | 0.038 | (0.126) | 0.227 | (0.040) |
| 6 | Consumer Credit | | 0.389 | (0.300) | –0.160 | (0.966) | 2.961 | (0.494) |
| 7 | Personal Cons. Expenditures | | –0.405 | (0.206) | 0.133 | (0.074) | 0.111 | (0.029) |
| 8 | New Home Sales | | 0.726 | (0.170) | –0.011 | (0.171) | 0.473 | (0.071) |
| 9 | Durable Goods Orders | | –0.224 | (0.258) | 0.989 | (0.753) | 1.999 | (0.413) |
| 10 | Construction Spending | | 0.312 | (0.197) | –0.135 | (0.233) | 0.655 | (0.123) |
| 11 | Factory Orders | | –0.194 | (0.288) | 0.997 | (0.489) | –0.856 | (0.306) |
| 12 | Business Inventories | | 0.128 | (0.277) | –0.019 | (0.061) | 0.228 | (0.032) |
| 13 | Government Budget Deficit | | –0.359 | (0.418) | –0.992 | (1.423) | 3.262 | (0.508) |
| 14 | Trade Balance | | 0.819 | (0.189) | 0.361 | (0.602) | 1.585 | (0.344) |
| 15 | Consumer Confidence Index | | 0.977 | (0.076) | 0.208 | (0.136) | –0.482 | (0.084) |
| 16 | NAPM Index | | 0.849 | (0.115) | –0.008 | (0.047) | 0.151 | (0.024) |
| 17 | Housing Starts | | 0.832 | (0.175) | 0.002 | (0.026) | 0.071 | (0.014) |
| 18 | Index of Leading Indicators | | 0.107 | (0.240) | 0.212 | (0.077) | 0.231 | (0.033) |

Note: * and ** indicate that the estimate or standard error is multiplied by 100 and 25, respectively.

and "preliminary" releases. It is hard to understand how this could be a feature of an optimal revision process within the BEA. However, it is also possible that the implied correlation arises simply by chance because the sample period only covers twenty-five quarters.

Estimates of the parameters linking the monthly data releases to GDP growth are reported in panel C. The first column shows that there is considerable variation across the eighteen series in the estimates of $\alpha_i$. In all cases, the estimates of $\alpha_i$ are statistically significant, indicating that the quasi-differenced monthly releases are more informative about GDP growth than the raw series. The $\alpha_i$

estimates also imply that the temporal impact of a change in growth varies across the different monthly series. For example, changes in GDP growth will have a more persistent effect on the consumer confidence index ($\hat{\alpha}_{15} = 0.977$) than on nonfarm payroll employment ($\hat{\alpha}_1 = 0.007$). The $\beta_i$ estimates reported in the third column show that twelve of the eighteen monthly releases are procyclical (i.e., positively correlated with contemporaneous GDP growth). Recall that all the coefficients are significant at the 1 percent level, so the $\beta_i$ estimates provide strong evidence that all the monthly releases contain incremental information about current GDP growth beyond that contained in past GDP data releases.

The standard method for assessing the adequacy of a model estimated by the Kalman Filter is to examine the properties of the estimated filter innovations, $\hat{\eta}_t$ defined in (35a) above. If the model is correctly specified, all elements of the innovation vector $\hat{\eta}_t$ should be uncorrelated with any elements of $\Omega_{t-1}$, including past innovations. To check this implication, table 3 reports the autocorrelation coefficients for the innovations associated with each data release. For example, the estimated innovation associated with the "final" GDP release for quarter $\tau$ on day $\text{R}(\tau)$ is $\eta^y_{\text{R}(\tau)} \equiv y_{\text{R}(\tau)} - \mathbb{E}[y_{\text{R}(\tau)}|\hat{\Omega}_{\text{R}(\tau)-1}]$. For the quarterly releases, the table shows the correlation between $\eta^y_{\text{R}(\tau)}$ and $\eta^y_{\text{R}(\tau-n)}$ for $n = 1$ and 6. In the case of monthly release $i$, the innovation is $\eta^i_{\text{R}(\tau,j)} \equiv z^i_{\text{R}(\tau,j)} - \mathbb{E}[z^i_{\text{R}(\tau,j)}|\hat{\Omega}_{\text{R}(\tau,j)-1}]$ and the table shows the correlation between $\eta^i_{\text{R}(\tau,j)}$ and $\eta^i_{\text{R}(\tau,j-n)}$ for $n = 1$ and 6. Under the BPQ($j$) headings, the table also reports p-values computed from the Box-Pierce Q statistic for joint significance of the correlations from lag 1 to $j$. Overall, there is little evidence of serial correlation in the innovations. Exceptions arise only in the case of "preliminary" GDP at the six-quarter lag, and in the cases of consumer credit and business inventories at the six-month lag.

Panel A of table 4 (shown on page 158) compares model-based forecasts for "final" GDP against the provisional data releases. Under the Data Revision columns, I report the mean and mean squared error (MSE) for the data revisions associated with the "advanced" and "preliminary" data releases (i.e., $y_{\text{R}(\tau)} - \hat{y}_{\text{R}(\tau)}$ and $y_{\text{R}(\tau)} - \tilde{y}_{\text{R}(\tau)}$). The mean and MSE for the difference between "final" GDP, $y_{\text{R}(\tau)}$, and the estimates of $\mathbb{E}[y_{\text{R}(\tau)}|\Omega_{\text{DR}(\tau)}]$, where $\text{DR}(\tau)$ denotes the date of the day of either the "advanced" or "preliminary" release (i.e.,

**Table 3. Model Diagnostics**

| **Innovation Autocorrelations** | | $\rho_1$ | BPQ(1) | $\rho_6$ | BPQ(6) |
|---|---|---|---|---|---|
| Quarterly Releases | | | | | |
| | Advanced GDP | 0.058 | (0.766) | –0.061 | (0.889) |
| | Preliminary GDP | –0.364 | (0.069) | –0.034 | (0.012) |
| | Final GDP | 0.001 | (0.996) | –0.172 | (0.729) |
| Monthly Releases | | | | | |
| $i = 1$ | Nonfarm Payroll Employment | –0.023 | (0.841) | 0.051 | (0.902) |
| 2 | Retail Sales | 0.005 | (0.966) | –0.028 | (0.789) |
| 3 | Industrial Production | 0.005 | (0.963) | 0.003 | (0.981) |
| 4 | Capacity Utilization | –0.029 | (0.800) | 0.147 | (0.885) |
| 5 | Personal Income | –0.069 | (0.687) | 0.057 | (0.770) |
| 6 | Consumer Credit | –0.091 | (0.422) | 0.310 | (0.040) |
| 7 | Personal Consumption Expenditures | 0.122 | (0.477) | –0.021 | (0.427) |
| 8 | New Home Sales | –0.219 | (0.084) | –0.056 | (0.220) |
| 9 | Durable Goods Orders | –0.094 | (0.418) | –0.121 | (0.650) |
| 10 | Construction Spending | 0.064 | (0.699) | 0.131 | (0.798) |
| 11 | Factory Orders | –0.161 | (0.327) | –0.113 | (0.483) |
| 12 | Business Inventories | –0.068 | (0.552) | 0.339 | (0.000) |
| 13 | Government Budget Deficit | –0.091 | (0.421) | –0.137 | (0.100) |
| 14 | Trade Balance | –0.203 | (0.077) | 0.087 | (0.578) |
| 15 | Consumer Confidence Index | 0.047 | (0.678) | –0.111 | (0.624) |
| 16 | NAPM Index | –0.067 | (0.556) | –0.017 | (0.639) |
| 17 | Housing Starts | –0.160 | (0.161) | –0.127 | (0.518) |
| 18 | Index of Leading Indicators | 0.021 | (0.850) | 0.043 | (0.525) |

Note: $\rho_i$ denotes the sample autocorrelation at lag $i$. p-values are calculated for the null hypothesis of $\rho_i = 0$.

**Table 4. Forecast Comparisons**

| A. | Data Revision | | Model | |
|---|---|---|---|---|
| | Mean | MSE | Mean | MSE |
| Advanced | 0.246 | (0.446) | 0.090 | (0.441) |
| Preliminary | 0.038 | (0.067) | 0.040 | (0.066) |
| Combined | 0.142 | (0.257) | 0.065 | (0.254) |
| **B.** | **MMS** | | **Model** | |
| | Mean | MSE | Mean | MSE |
| In-Sample | | | | |
| Advanced | 0.729 | (1.310) | 0.190 | (1.407) |
| Preliminary | 0.160 | (0.249) | 0.096 | (0.418) |
| Final | 0.042 | (0.062) | 0.080 | (0.395) |
| Combined | 0.310 | (0.540) | 0.122 | (0.740) |
| Out-of-Sample | | | | |
| Advanced | 0.985 | (1.464) | 0.380 | (1.500) |
| Preliminary | 0.046 | (0.178) | 0.178 | (0.801) |
| Final | –0.015 | (0.057) | 0.099 | (0.208) |
| Combined | 0.338 | (0.566) | 0.219 | (0.836) |

$\mathrm{R}_{\hat{y}}(\tau)$ or $\mathrm{R}_{\tilde{y}}(\tau)$), are reported under the Model columns. Note that the provisional data are part of the information set $\Omega_{\mathrm{DR}(\tau)}$ used to compute the model-based forecasts. If $\hat{y}_{\mathrm{R}(\tau)}$ and $\tilde{y}_{\mathrm{R}(\tau)}$ are close to the best forecasts of "final" GDP on days $\mathrm{R}_{\hat{y}}(\tau)$ and $\mathrm{R}_{\tilde{y}}(\tau)$, there should be little difference between the data revisions and the estimated forecast errors, $y_{\mathrm{R}(\tau)} - \mathbb{E}[y_{\mathrm{R}(\tau)}|\Omega_{\mathrm{DR}(\tau)}]$.[8] The table shows that the mean

[8]The model does not impose the assumption that the provisional data are equal to the best forecast of "final" GDP on day $\mathrm{DR}(\tau)$. Rather, equation (34a) of the Kalman Filter implies that the model forecasts for "final" GDP on these days are given by

$$\mathbb{E}[y_{\mathrm{R}(\tau)}|\Omega_{\mathrm{DR}(\tau)}] = \mathbb{E}[y_{\mathrm{R}(\tau)}|\Omega_{\mathrm{DR}(\tau)-1}] + \mathbb{K}^{y}_{\mathrm{DR}(\tau)}\left(\bar{y}_{\mathrm{R}(\tau)} - \mathbb{E}[\bar{y}_{\mathrm{R}(\tau)}|\Omega_{\mathrm{DR}(\tau)-1}]\right) \\ + \mathbb{K}^{\varkappa}_{\mathrm{DR}(\tau)}\left(\varkappa_{\mathrm{R}(\tau)} - \mathbb{E}[\varkappa_{\mathrm{R}(\tau)}|\Omega_{\mathrm{DR}(\tau)-1}]\right),$$

where $\bar{y}_{\mathrm{R}(\tau)} = \{\hat{y}_{\mathrm{R}(\tau)}, \tilde{y}_{\mathrm{R}(\tau)}\}$ and $\varkappa_{\mathrm{R}(\tau)}$ denotes the vector of other data releases on $\mathrm{DR}(\tau)$. $\mathbb{K}^{y}_{\mathrm{DR}(\tau)}$ and $\mathbb{K}^{\varkappa}_{\mathrm{DR}(\tau)}$ are elements of the Kalman Gain matrix on release days. Inspection of this equation reveals that $\mathbb{E}[y_{\mathrm{R}(\tau)}|\Omega_{\mathrm{DR}(\tau)}] = \bar{y}_{\mathrm{R}(\tau)}$ if $\mathbb{K}^{y}_{\mathrm{DR}(\tau)} = 1$ and $\mathbb{K}^{\varkappa}_{\mathrm{DR}(\tau)}$ is a vector of zeros. These conditions cannot be exactly satisfied when there is any noise in (5) and (6), but they could hold approximately if the noise variance and MSE of revision errors based on the "preliminary" data releases are comparable to those based on the model forecasts. In the case of the "advanced" releases, by contrast, the mean revision error is roughly two and one-half times the size of the forecast error. This finding suggests that the "advanced" releases contain some "noise" and do not represent the best forecasts for "final" GDP that can be computed using publicly available data. It is also consistent with the regression findings reported by Dynan and Elmendorf (2001).

Panel B of table 4 compares the forecasting performance of the model against the survey responses collected by MMS. On the Friday before each scheduled data release, MMS surveys approximately forty professional money managers on their estimate for the upcoming release. Panel B compares the median estimate from the surveys against the real-time estimate of GDP growth implied by the model on survey days. For example, in the first row under the MMS columns, I report the mean and MSE for the difference between $y_{\text{R}(\tau)}$, and the median response from the survey conducted on the last Friday before the "advanced" GDP release on day $\text{S}(\tau)$. The mean and MSE of the difference between $y_{\text{R}(\tau)}$ and the estimate of $\mathbb{E}[y_{\text{R}(\tau)}|\Omega_{\text{S}(\tau)}]$ derived from the model are reported under the Model columns. As above, all the survey and model estimates are compared against the value for the "final" GDP release. This means that the forecasting horizon, (i.e., the difference between $\text{R}(\tau)$ and $\text{S}(\tau)$) falls from approximately eleven weeks in the first row, to five weeks in the second, and less than one week in the third. The fourth row reports the mean and MSE at all three horizons.

The upper portion of panel B compares the survey responses against model-based forecasts computed from parameter estimates reported in table 2. These estimates are derived from the full data sample and so contain information that was not available to the money managers at the time they were surveyed. The lower portion of panel B reports on a pseudo out-of-sample comparison. Here the model-based forecasts are computed from model estimates obtained from the first half of the sample (April 11, 1993–March 31, 1996). These estimates are then used to compute model-based forecasts of "final" GDP on the survey days during the second half of the sample (April 1, 1996–June 30, 1999). The table compares the mean and

is small relative to the variance of other shocks. Under these circumstances, the provisional data could closely approximate the model's forecasts for "final" GDP (i.e., $\mathbb{E}[y_{\text{R}(\tau)}|\Omega_{\text{DR}(\tau)}] \simeq \bar{y}_{\text{R}(\tau)}$).

MSE of these out-of-sample forecasts against the survey responses during this latter period.

The table shows that both the mean and MSE associated with both the survey and model forecasts fall with the forecasting horizon. In the case of the in-sample statistics, the median survey response provides a superior forecast than the model in terms of mean and MSE at short horizons. Both the mean and MSE are smaller for survey responses than the model forecasts in the third row. Moving up a row, the evidence is ambiguous. The model produces a smaller mean but larger MSE than the median survey. In the first row, the balance of the evidence favors the model; the MSE is slightly higher but the mean is much lower than the survey estimates. This general pattern is repeated in the out-of-sample statistics. The strongest support for the model again comes from a comparison of the survey and model forecasts conducted one week before the "advanced" release. In this case, the mean forecast error from the model is approximately 60 percent smaller than the mean survey error. Over shorter forecasting horizons, the survey measures dominate the model-based forecasts.

Overall, these forecast comparisons provide rather strong support for the model. It is clear that the in-sample comparisons use model estimates based in part on information that was not available to the money managers at the time. But it is much less clear whether this puts the managers at a significant informational disadvantage. Remember that the money managers had access to private information and other contemporaneous data that is absent from the model. Moreover, we are comparing a model-based forecast against the median forecast from a forty-manager survey. In the out-of-sample comparisons, the informational advantage clearly lies with the managers. Here the model forecasts are based on a true subset of the information available to managers, so the median forecast from a forty-manager survey should outperform the model. This is what we see when the forecasting horizon is less than five weeks. At longer horizons, the use of private information imparts less of a forecasting advantage to money managers.[9] In fact, the results suggest that as

[9]This advantage might be further reduced if each of the model forecasts were computed using parameter estimates that utilized all the data available on the survey date rather than a single set of estimates using data from the first half of the sample. A full-blown real-time forecasting exercise of this kind would be

we move the forecasting day back toward the end of quarter $\tau$, both the in- and out-of-sample model-based forecasts outperform the surveys. When the forecasting day is pushed all the way back to the end of the quarter, the model-based forecast gives us the real-time estimates of GDP growth. Thus, the results in panel B indicate that real-time estimates derived from the model should be at least comparable to private forecasts based on much richer information sets.

## 5. Analysis

This section examines the model estimates. First, I consider the relation between the real-time estimates and the "final" GDP releases. Next, I compare alternative real-time estimates for the level of GDP and examine the forecasting power of the model. Finally, I study how the monthly releases are related to changes in GDP at a monthly frequency.

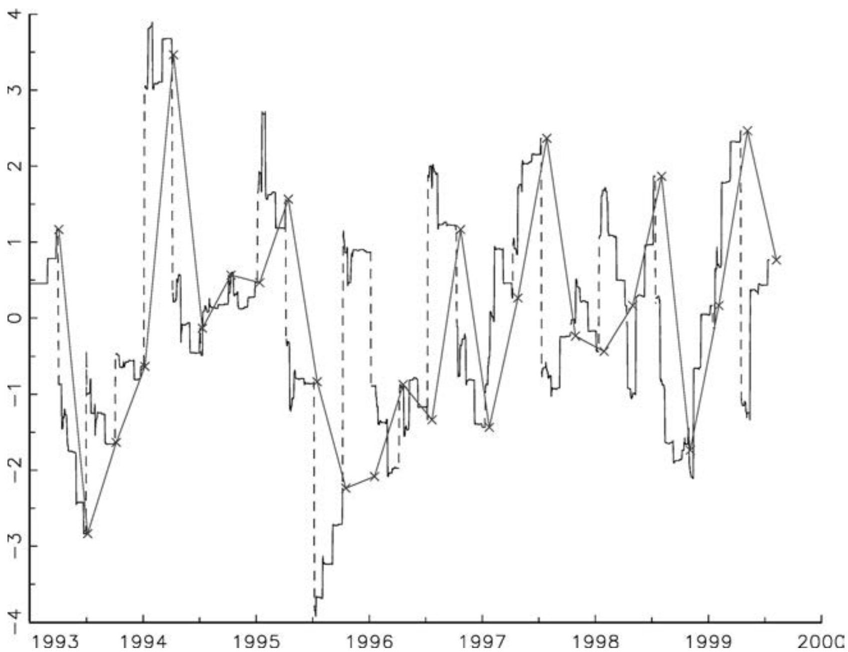
### *5.1 Real-Time Estimates over the Reporting Lag*

Figure 2 allows us to examine how the real-time estimates of GDP growth change over the reporting lag. The solid line with stars plots the "final" GDP growth for quarter $\tau$ released on day $\mathrm{R}_y(\tau)$. The intermittent line plots the real-time estimates of the GDP growth last month, $\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)|t}$, where $\mathrm{Q}(\tau) < t \leq \mathrm{R}_y(\tau)$ for each quarter. The vertical dashed portion represents the discontinuity in the series at the end of each quarter (i.e., on day $\mathrm{Q}(\tau)$).[10] Several features of the figure stand out. First, the real-time estimates vary considerably in the days immediately after the end of the quarter. For example,

computationally demanding because the model would have to be repeatedly estimated, but it should also give superior model-based forecasts. For this reason, the out-of-sample exercise undertaken here probably understates the true real-time forecasting potential of the model.

[10]In cases where the reporting lag is less than one quarter, the discontinuity occurs (a couple of days) after $\mathrm{R}_y(\tau)$, so the end of each solid segment meets the turning point "x" identifying the "final" GDP release. When the reporting lag is longer than one quarter, there is a horizontal gap between the end of a solid segment and the next turning point "x" equal to $\mathrm{R}_y(\tau) - \mathrm{Q}(\tau + 1)$ days.

**Figure 2. Real-Time Estimates of Quarterly GDP Growth**

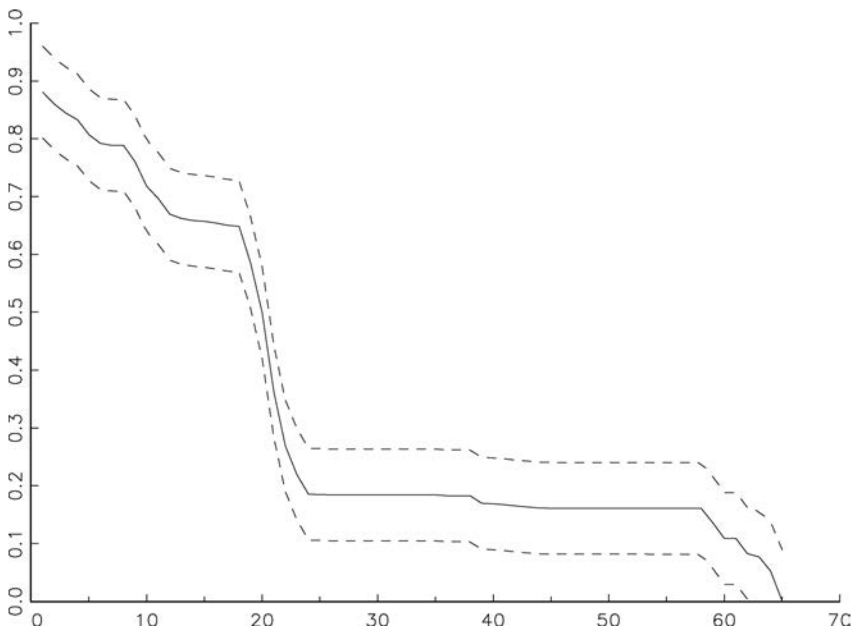



Note: The intermittent solid line is the real-time estimate of quarterly GDP growth, $\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)|t}$, where $\mathrm{Q}(\tau) < t \leq |\mathrm{R}_y(\tau)$, and the solid line with stars is the "final" release for GDP, $y_{\mathrm{R}(\tau)}$.

at the end of 1994, the real-time estimates of GDP growth in the fourth quarter change from approximately 1.25 percent to 2.25 percent and then to 1.5 percent in the space of a few days. Second, in many cases there is very little difference between the value for "final" GDP and the real-time estimate immediately prior to the release (i.e., $y_{\mathrm{R}(\tau)} \simeq \Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)|\mathrm{R}(\tau)-1}$). In these cases, the "final" release contains no new information about GDP growth that was not already inferred from earlier data releases. In cases where the "final" release contains significant new information, the intermittent-line plot "jumps" to meet the solid-line-with-stars plot on the release day.

Figure 3 provides further information on the relation between the real-time estimates and the "final" data releases. Here I plot the variance of $\Delta^{\mathrm{Q}} x_{\mathrm{Q}(\tau)}$ conditioned on information available over

**Figure 3. Relation between Real-Time Estimates and the "Final" Data Releases**




Note: The solid line is the sample average of $\mathbb{V}(\Delta^{\text{Q}} x_{\text{Q}(\tau)} | \Omega_{\text{Q}(\tau)+i})$ for $0 < i \leq \text{R}_y(\tau) - \text{Q}(\tau)$, and the dashed lines denote the 95 percent confidence band. The horizontal axis marks the number of days $i$ past the end of quarter $\text{Q}(\tau)$.

the reporting lag: $\mathbb{V}(\Delta^{\text{Q}} x_{\text{Q}(\tau)} | \Omega_{\text{Q}(\tau)+i})$ for $0 < i \leq \text{R}_y(\tau) - \text{Q}(\tau)$. Estimates of this variance are identified as the second diagonal element in $\mathbb{S}_{t+1|t}$ obtained from the Kalman Filter evaluated at the maximum likelihood estimates. Figure 3 plots the sample average of $\mathbb{V}(\Delta^{\text{Q}} x_{\text{Q}(\tau)} | \Omega_{\text{Q}(\tau)+i})$ together with a 95 percent confidence band. Although the path for the conditional variance varies somewhat from quarter to quarter, the narrow confidence band shows that the average pattern displayed in the figure is in fact quite representative of the variance path seen throughout the sample.

Figure 3 clearly shows how the flow of data releases during the reporting lag provides information on $\Delta^{\text{Q}} x_{\text{Q}(\tau)}$. In the first twenty days or so, the variance falls by approximately 25 percent as information from the monthly releases provides information on the behavior of

GDP during the previous month (month 3 of quarter $\tau$). The variance then falls significantly following the "advanced" GDP release. The timing of this release occurs between nineteen and twenty-three working days after the end of the quarter, so the averaged variance path displayed by the figure spreads the fall across these days. Thereafter, the variance falls very little until the end of the reporting lag when the "final" value for GDP growth is released.[11] This pattern indicates that the "preliminary" GDP release provides little new information about GDP growth beyond that contained in the "advanced" GDP release and the monthly data. The figure also shows that most of the uncertainty concerning GDP growth in the last quarter is resolved well before the day when the "final" data is released.

### *5.2 Real-Time Estimates of GDP and GDP Growth*

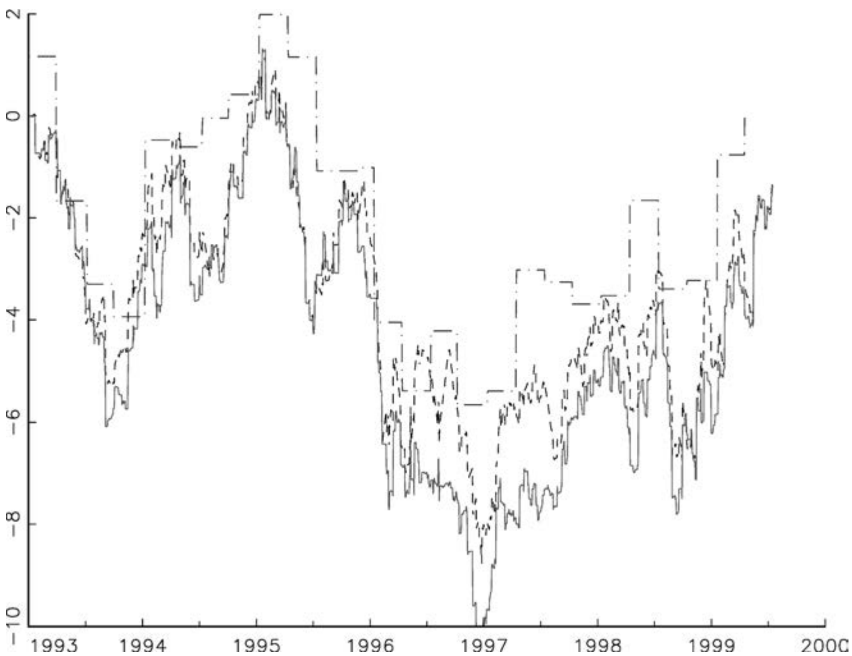
Figure 4 compares the real-time estimates of log GDP against the values implied by the "final" GDP growth releases. The solid line plots the values of $x_{\mathrm{Q}(\tau)|t}$ computed from (43):

$$x_{\mathrm{Q}(\tau)|t} = \mathbb{E}[x_t|\Omega_t] + \sum_{h=1}^{\mathrm{Q}(\tau)-t} \mathbb{E}[\Delta x_{t+h}|\Omega_t]. \tag{43}$$

Recall that $x_{\mathrm{Q}(\tau)}$ represents log GDP for quarter $\tau$, so $x_{\mathrm{Q}(\tau)|t}$ includes forecasts for $\Delta x_{t+h}$ over the remaining days in the quarter when $t < \mathrm{Q}(\tau)$. To assess the importance of the forecast terms, figure 4 also plots $\mathbb{E}[x_t|\Omega_t]$ as a dashed line. This series represents a naive real-time estimate of GDP since it assumes $\mathbb{E}[\Delta x_{t+h}|\Omega_t] = 0$ for $1 \leq h \leq \mathrm{Q}(\tau) - t$. The dash-dot line in the figure plots the cumulant of the "final" GDP releases $\sum_{i=1}^{\tau} y_{\mathrm{R}(i)}$ with a lead of sixty days. This plot represents an ex-post estimate of log GDP based on the "final" data releases. The vertical steps identify the values for "final" GDP growth sixty days before the actual release day.

[11] Although the variance falls immediately to zero on the day of the release, the averaged variance falls to zero over several days in the figure because the reporting lag varies from fifty-eight to sixty-five workdays in the sample.

**Figure 4. Real-Time Estimates of Log GDP and "Final" GDP Releases**




Note: The solid line is the real-time estimate of log GDP, $x_{\mathrm{Q}(\tau)|t}$, the dashed line shows $E[x_t|\Omega_t]$, and the dash-dot line is the cumulant of "final" GDP releases.

Figure 4 displays three notable features. First, both sets of the real-time estimates display a much greater degree of volatility than the cumulant series. This volatility reflects how inferences about current GDP change as information arrives in the form of monthly data releases during the current quarter and GDP releases referring to growth in the previous quarter. The second noteworthy feature concerns the relation between the real-time estimates. The vertical difference between the solid-line plots and the dashed-line plots represents the contribution of the $\Delta x_{t+h}$ forecasts to $x_{\mathrm{Q}(\tau)|t}$. As the figure shows, these forecasts contributed significantly to the real-time estimates in 1996 and 1997, pushing the real-time estimates of $x_{\mathrm{Q}(\tau)}$ well below the value for $\mathbb{E}[x_t|\Omega_t]$. The third noteworthy feature of the figure concerns the vertical gap between the solid-line and dash-dot line plots. This represents the difference between the real-time estimates and an ex-post estimate of log GDP based on the "final" GDP releases. This gap should be insignificant if the current level of GDP could be precisely inferred from contemporaneously available data releases. Figure 3 shows this to be the case during the third and fourth quarters of 1995. During many other periods, the real-time estimates were much less precise.

### *5.3 Forecasting GDP Growth*

The model estimates in table 2 show that all of the $\phi_i$ coefficients in the daily growth process are statistically significant. I now examine their implications for forecasting GDP growth. Consider the difference between the real-time estimates of $x_{t+m}$ and $x_{t+n}$ based on information available on days $t+m$ and $t+n$, where $m > n$:

$$x_{t+m|t+m} - x_{t+n|t+n} = \sum_{h=t+n+1}^{t+m} \mathbb{E}\left[\Delta x_h | \Omega_t\right] + \left(x_{t+m|t+m} - x_{t+m|t}\right) - \left(x_{t+n|t+n} - x_{t+n|t}\right). \tag{49}$$

This equation decomposes the difference in real-time estimates of $x_t$ into forecasts for $\Delta x_t$ over the forecast horizon between $t+n$ and $t+m$, the revision in the estimates of $x_{t+m}$ between $t$ and $t+m$ and $x_{t+n}$ between $t$ and $t+n$.

Since both revision terms are uncorrelated with elements of $\Omega_t$, we can use (49) to examine how the predictability of $\Delta x_t$ implied by the model estimates translates into predictability for changes in $x_{t|t}$. In particular, after multiplying both sides of (49) by $x_{t+m|t+m} - x_{t+n|t+n}$ and taking expectations, we obtain

$$\mathbb{R}^2(m,n) = \frac{\sum_{h=t+n+1}^{t+m} \mathbb{CV}\left(\mathbb{E}\left[\Delta x_h|\Omega_t\right], x_{t+m|t+m} - x_{t+n|t+n}\right)}{\mathbb{V}\left(x_{t+m|t+m} - x_{t+n|t+n}\right)}.$$

This statistic measures the contribution of $\Delta x_t$ forecasts to the variance of $x_{t+m|t+m} - x_{t+n|t+n}$. If most of the volatility in $x_{t+m|t+m} - x_{t+n|t+n}$ is due to the arrival of new information between $t$ and $t+m$ (i.e., via the revision terms in [49]), the $\mathbb{R}^2(m,n)$ statistic should be close to zero. Alternatively, if the daily growth process is highly

**Table 5. Forecasting Real-Time GDP**

| $m$ | $\mathbb{R}^2(m,n)$ | (std.) |
|---|---|---|
| Monthly ($n = m - 20$) | | |
| 20 | 0.196 | (0.015) |
| 40 | 0.186 | (0.014) |
| 60 | 0.180 | (0.013) |
| 80 | 0.163 | (0.013) |
| 100 | 0.138 | (0.012) |
| 120 | 0.125 | (0.011) |
| 140 | 0.097 | (0.010) |
| 160 | 0.061 | (0.007) |
| 180 | 0.032 | (0.006) |
| 200 | 0.035 | (0.006) |
| 220 | 0.027 | (0.006) |
| 240 | 0.005 | (0.006) |
| Quarterly ($n = m - 60$) | | |
| 60 | 0.144 | (0.008) |
| 120 | 0.079 | (0.006) |
| 180 | 0.031 | (0.003) |
| 240 | 0.006 | (0.002) |

Note: The table reports estimates of $R^2(m, m-20)$ computed as the slope coefficient $\delta_m$ from the regression $\sum_{h=t+m-19}^{t+m} E\left[\Delta x_h|\Omega_t\right] = \delta_m(x_{t+m|t+m} - x_{t+m-20|t+m-20}) + \xi_{t+m}$ computed in daily data from the maximum likelihood estimates of $x_{t+m|t+m} - x_{t+m-20|t+m-20}$ and $E\left[\Delta x_h|\Omega_t\right]$. OLS standard errors are reported in the right-hand column.

forecastable, much less of the volatility in $x_{t+m|t+m} - x_{t+n|t+n}$ will be attributable to news, and the $\mathbb{R}^2\,(m,n)$ statistic will be positive.

Table 5 reports estimates for $\mathbb{R}^2(m,n)$ for various forecasting horizons. The estimates are computed as the slope coefficient in the regression of $\sum_{h=t+n+1}^{t+m} \mathbb{E}\left[\Delta x_h|\Omega_t\right]$ on $x_{t+m|t+m} - x_{t+n|t+n}$ in daily data. The table also reports OLS (ordinary least squares) standard errors in parentheses.[12] The upper panel shows how predictable the monthly changes in the real-time estimates of $x_t$ are for horizons $m$ of 20 to 240 days. The estimated process for $\Delta x_t$ implies a reasonable high degree of predictability: the $\mathbb{R}^2$ estimates fall from approximately 20 to 10 percent as the horizon rises from 20 to 140 workdays. Estimates of $\mathbb{R}^2(m,n)$ for quarterly changes in the real-time estimates are reported in the lower panel of the table. These are somewhat smaller, but again clearly indicate the presence of some predictability.

The results in table 5 relate to the change in the real-time estimates of $x_t$ rather than GDP growth. Recall from (43) that the real-time estimate of log GDP on day $t$ in quarter $\tau$ is $x_{\mathrm{Q}(\tau_t)|t} = \mathbb{E}[x_t|\Omega_t] + \sum_{h=t+1}^{\mathrm{Q}(\tau_t)} \mathbb{E}[\Delta x_h|\Omega_t]$, so the growth in quarterly GDP between quarters $\tau$ and $\tau+1$ based on the real-time estimates available at $t+m$ and $t+n$ (where $t+n < \mathrm{Q}(\tau) < t+m < \mathrm{Q}(\tau+1)$) is

$$x_{\mathrm{Q}(\tau+1)|t+m} - x_{\mathrm{Q}(\tau)|t+n} = \left(x_{t+m|t+m} - x_{t+n|t+n}\right) + \sum_{h=t+m+1}^{\mathrm{Q}(\tau+1)} \mathbb{E}[\Delta x_h|\Omega_{t+m}] - \sum_{h=t+n+1}^{\mathrm{Q}(\tau)} \mathbb{E}[\Delta x_h|\Omega_{t+n}]. \tag{50}$$

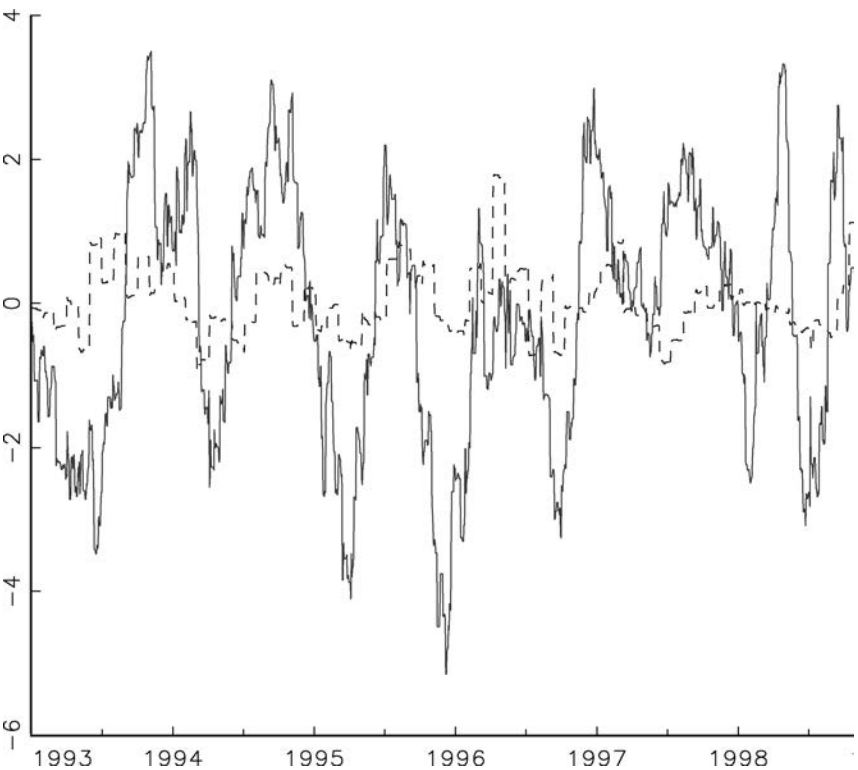
This equation shows that changes in the real-time estimates of $x_t$ are only one component of the estimated quarterly growth in GDP. Moreover, since the second and third terms on the right-hand side of (50) will generally be correlated with the first term, the results in table 5 may be an unreliable guide to the predictability of GDP growth.

We can examine the predictability of GDP growth by combining (49) and (50) to give

$$x_{\mathrm{Q}(\tau+1)|t+m} - x_{\mathrm{Q}(\tau)|t+n} = \mathbb{E}[x_{\mathrm{Q}(\tau+1)} - x_{\mathrm{Q}(\tau)+1}|\Omega_t] + \varsigma_{t+m},$$

[12] These statistics only approximate to the true standard errors for two reasons. First, the regressor is computed from the maximum likelihood estimates and so contains some sampling errors. Second, there is no correction for the moving average process induced by the overlapping forecast horizons in the regression residuals.

**Figure 5. Real-Time Estimates of Quarterly GDP Growth**




Note: The solid line is the real-time estimate of quarterly GDP growth, $x_{\mathrm{Q}(\tau+1)|t+60} - x_{\mathrm{Q}(\tau)|t}$, and the dashed line is the forecast, $\mathbb{E}[x_{\mathrm{Q}(\tau+1)} - x_{\mathrm{Q}(\tau)}|\Omega_t]$.

where $\varsigma_{t+m}$ is an error term that depends on news that arrives between $t$ and $t+m$. This equation shows that GDP growth should be predictable in the model because the process for $\Delta x_t$ implies that $\mathbb{E}[x_{\mathrm{Q}(\tau+1)} - x_{\mathrm{Q}(\tau)+1}|\Omega_t]$ changes over the sample. Figure 5 plots the estimates of $\mathbb{E}[x_{\mathrm{Q}(\tau+1)} - x_{\mathrm{Q}(\tau)+1}|\Omega_t]$ and $x_{\mathrm{Q}(\tau+1)|t+m} - x_{\mathrm{Q}(\tau)|t+n}$ for $m = 60$ and $n = 0$. Clearly, news contributes significantly to the volatility of GDP growth since the $\mathbb{E}[x_{\mathrm{Q}(\tau+1)} - x_{\mathrm{Q}(\tau)+1}|\Omega_t]$ series is much less volatile than $x_{\mathrm{Q}(\tau+1)|t+m} - x_{\mathrm{Q}(\tau)|t+n}$. In fact, only 6 percent of the variance in $x_{\mathrm{Q}(\tau+1)|t+m} - x_{\mathrm{Q}(\tau)|t+n}$ can be attributable to $\mathbb{E}[x_{\mathrm{Q}(\tau+1)} - x_{\mathrm{Q}(\tau)+1}|\Omega_t]$ over the sample. While this implies a rather modest degree of predictability, close inspection of figure 5 suggests that predicting the future direction of GDP growth may be a little more successful. Indeed, the model estimates of $\mathbb{E}[x_{\text{Q}(\tau+1)} - x_{\text{Q}(\tau)+1}|\Omega_t]$ correctly predict the direction of GDP growth 59 percent of the time.

For perspective on these forecasting results, I also estimated an AR(2) model for quarterly GDP growth using the sequence of "final" GDP releases. This model provides a simple time-series forecast for GDP growth (approximately) one quarter ahead, based on the two most recent releases. In contrast to the results presented above, estimates of the AR(2) model do not indicate that quarterly GDP growth is at all predictable: the coefficients are small and statistically insignificant, and the $R^2$ statistic is only 3 percent. These results are hardly surprising. Remember that the sample mean was removed from all the GDP growth releases, so we are attempting to forecast future *deviations* in GDP growth. Figure 2 shows that these deviations display little serial correlation. Thus, it is not surprising that the history of GDP releases has little forecasting power. These observations also point to the significance of the forecasting results displayed in table 5 and figure 5. In particular, they show that monthly data releases contain information that is useful for both estimating the current state of the GDP and for forecasting its future path. This aspect of the model ties in with recent research that harnesses the information in a large number of indicators for forecasting (see, for example, Stock and Watson 2002). The results from these studies suggest that the real-time forecasting performance of this model may be further enhanced by addition of other macroeconomic and financial indicators.

### *5.4 Monthly Estimates of GDP Growth*

One of the unique features of the model is its ability to provide us with high-frequency estimates of log GDP and GDP growth. I now examine how the monthly data releases relate to the changing real-time estimates of log GDP. My aim is to provide a simple description of the complex inference problem solved by the model regarding the current state of GDP.

Let $t$ and $t + 20$ be workdays in quarters $\tau_0$ and $\tau_1$ with $t \leq \text{Q}(\tau_0)$ and $t + 20 \leq \text{Q}(\tau_1)$. (Note that $\tau_0$ and $\tau_1$ can refer to the same quarter.) I consider two regression models. The first relates the monthly change in the real-time estimates of log GDP to all eighteen monthly releases:

$$x_{\mathrm{Q}(\tau_1)|t+20} - x_{\mathrm{Q}(\tau_0)|t} = \sum_{i=1}^{18} a_i(r^i_{t+20} - r^i_t) + \zeta_{t+20}, \qquad (51)$$

where $r^i_t$ denotes the last value released for series $i$ on day $t$. Thus, the value of $r^i_t$ remains the same from day to day unless $t$ is the day on which a data release for series $i$ takes place. This means that $r^i_{t+20} - r^i_t$ identifies the change in the latest value for series $i$ released during the twenty workdays ending on $t + 20$. The second model relates the change in the real-time estimates to each monthly release separately:

$$x_{\mathrm{Q}(\tau_1)|t+20} - x_{\mathrm{Q}(\tau_0)|t} = b_i(r^i_{t+20} - r^i_t) + \zeta_{t+20}, \qquad (52)$$

for $i = 1, 2, \ldots, 18$. Notice that the regressors in both models are available on a daily basis. Estimates of the $a_i$ and $b_i$ coefficients therefore summarize how the complex inference imbedded in the model relates to observable changes in the information set comprised of monthly data releases.

Table 6 reports the estimates of equations (51) and (52). The left-hand columns show that the $a_i$ estimates are statistically significant for data on nonfarm payroll employment, retail sales, industrial production, personal consumption, factory orders, the trade balance, the index of consumer confidence, and housing starts. Each of these data releases provides significant incremental information about the change in the real-time estimates. The eighteen monthly releases account for approximately 57 percent of the variance in $x_{\mathrm{Q}(\tau_1)|t+20} - x_{\mathrm{Q}(\tau_0)|t}$. This means that more than 40 percent of the variation in the real-time estimates is not captured by the simple linear specification. The right-hand columns report the results from estimating equation (52). The most noteworthy aspects of these estimates can be seen for $i = 1, 2, 3$. Changes in both nonfarm payroll employment and retail sales appear strongly linked to changes in the real-time estimates. Based on the $R^2$ statistics (in the right-hand column), these variables account for 23 and 19 percent of the variance in the real-time estimates. The results for $i = 3$ provide some justification for the frequent use of industrial production as a

**Table 6. Monthly Indicator Estimates**

| | Data Release | $a_i$ | std($a_i$) | $b_i$ | std($b_i$) | $R^2$ |
|---|---|---|---|---|---|---|
| 1 | Nonfarm Payroll Employment | 0.224* | (0.033) | 0.330* | (0.050) | 0.233 |
| 2 | Retail Sales | 0.520* | (0.074) | 0.838* | (0.099) | 0.186 |
| 3 | Industrial Production | 0.745* | (0.116) | 1.047* | (0.142) | 0.233 |
| 4 | Capacity Utilization | –0.006 | (0.034) | 0.066 | (0.057) | 0.010 |
| 5 | Personal Income | –0.110 | (0.132) | 0.163 | (0.224) | 0.003 |
| 6 | Consumer Credit | 0.001 | (0.012) | –0.002 | (0.019) | 0.000 |
| 7 | Personal Consumption Expenditures | 0.377* | (0.182) | 0.603* | (0.277) | 0.033 |
| 8 | New Home Sales | 0.016 | (0.069) | 0.181* | (0.088) | 0.039 |
| 9 | Durable Goods Orders | 0.013 | (0.021) | 0.064* | (0.027) | 0.030 |
| 10 | Construction Spending | –0.050 | (0.051) | –0.102 | (0.060) | 0.015 |
| 11 | Factory Orders | 0.097* | (0.038) | 0.160* | (0.043) | 0.066 |
| 12 | Business Inventories | –0.013 | (0.152) | 0.289 | (0.247) | 0.008 |
| 13 | Government Budget Deficit | –0.008 | (0.011) | –0.017 | (0.016) | 0.005 |
| 14 | Trade Balance | –0.071* | (0.021) | –0.068* | (0.030) | 0.042 |
| 15 | Consumer Confidence Index | 0.068* | (0.028) | 0.060 | (0.037) | 0.021 |
| 16 | NAPM Index | –0.001 | (0.145) | 0.514* | (0.216) | 0.046 |
| 17 | Housing Starts | –1.330* | (0.421) | 0.705 | (0.546) | 0.012 |
| 18 | Index of Leading Indicators | 0.164 | (0.179) | 0.476* | (0.199) | 0.038 |

Note: The table reports the OLS estimates of $a_i$ and $b_i$ from equations (51) and (52). Both equations are estimated at the daily frequency, and the standard errors are corrected for the moving average (19) process induced by the overlapping data. The right-hand column reports the $R^2$ statistic from estimating equation (52). *denotes significance at the 5 percent level.

monthly proxy for GDP growth. The estimate for $b_3$ indicates that the real-time estimates of log GDP change approximately one-for-one with change in industrial production between releases. Notice, however, that the $R^2$ statistic is only 0.233. Industrial production does not account for most of the variance in the real-time estimates.

## 6. Conclusion

In this paper, I have presented a method for estimating the current state of the economy on a continual basis using the flow of information from a wide range of macroeconomic data releases. These real-time estimates were computed from an econometric model that allows for variable reporting lags, temporal aggregation, and other complications that characterize the daily flow of macroeconomic information. The model can be applied to find real-time estimates of GDP, inflation, unemployment, or any other macroeconomic variable of interest.

In this paper, I focused on the problem of estimating GDP in real time. This application of the estimation procedure should be of particular interest to policymakers concerned with the lack of timely information about economy-wide real activity. The real-time estimates I calculate have several noteworthy features: First, the estimates of log GDP display a good deal of high-frequency volatility. This volatility reflects how inferences about current GDP change as information arrives in the form of monthly data releases during the quarter. Second, the gaps between the real-time estimates and ex-post GDP data are on occasion both persistent and significant. These findings suggest that the ex-post data should not be viewed as a close approximation to what was known at the time. Third, the model estimates reveal that the monthly data releases contain information that is useful for forecasting the future path of GDP. Finally, my comparison of the real-time estimates with the monthly data series shows that standard proxies for real activity at the monthly frequency capture only a fraction of the variance in the real-time estimates.

These findings give but a flavor of the possible uses for real-time estimates. One obvious topic for the future concerns the ability of the real-time estimates and forecasts to identify turning points in the business cycle. This issue could be readily addressed if the model were reestimated over a much longer time span than was undertaken here. The use of real-time estimates may also bring new perspective to the link between asset prices and macroeconomic fundamentals. Evans and Lyons (2004b) use the methods described here to construct real-time estimates for GDP, inflation, and money supplies for the United States and Germany. With the aid of these estimates, they then show that foreign exchange transactions contain significant information about the future path of fundamentals. Since transaction flows also exert a very strong influence on exchange rate dynamics in the short run, this finding points to a much stronger link between fundamentals and exchange rates than previous research has uncovered. It remains to be seen whether the use of real-time estimates will similarly illuminate the links between macrofundamentals and other asset prices.

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# Dollar Shortages and Crises*

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Emerging markets do not handle adverse shocks well. In this paper, we lay out an argument about why emerging markets are so fragile, and why they may adopt contractual mechanisms—such as a dollarized banking system—that increase their fragility. We draw on this analysis to explain why dollarized economies may be prone to dollar shortages and twin crises. The model of crises described here differs in some important aspects from what are now termed the first-, second-, and third-generation models of crises. We then examine how domestic policies, especially monetary policy, can mitigate the adverse effects of these crises. Finally, we consider the role, potentially constructive, that international financial institutions may undertake both in helping to prevent the crises and in helping to resolve them.

JEL Codes: E5, F3, G2.

There is a strong correlation between the stoppages of capital flows to a country, the extent of dollarization of the country's banking system, and the prevalence of banking crises. Between 1974 and 2003, 56 percent of all episodes where capital flows underwent a "sudden stop" ended in a banking crisis; the same proportion rises to 75 percent in those episodes where the country also had a high level of dollarization, and to 100 percent if, in addition to a high level of dollarization, the country had in place a fixed exchange rate (see Inter-American Development Bank 2005). What accounts for these correlations? Are there domestic policies that can mitigate such risks? How can international financial institutions (IFIs) help

*This paper reflects our opinions and is not meant to represent the views of the International Monetary Fund or its member countries. We thank Douglas Diamond, Olivier Jeanne, Se-Jik Kim, and Ashok Mody for helpful comments. The paper is based on talks Rajan gave at the Central Bank of Argentina and the Central Bank of Chile.

their member countries avoid or diminish the consequences of such a predicament? We address these questions in this paper.

We review, first, why emerging markets may adopt contractual mechanisms—such as a dollarized banking system—that accentuate rather than lessen their vulnerabilities. We argue that weak institutions in emerging markets tend to make it harder for the emerging markets to cope properly with economic adversity. The burden of downturns, instead of being shared in predictable ways, is spread in a haphazard manner through, for example, selective defaults and high inflation. In such an environment, and with limited contract enforcement, the best protection for investors against risks may well be a domestic deposit denominated in foreign currency (following the convention in this literature, we will call the foreign currency "dollars").

Having enough dollars at all times is critical to the functioning of a dollarized banking system. We argue that an incipient dollar shortage, brought about by excessive government borrowing, or an external "liquidity" shock, or an overvalued exchange rate, can be magnified by a dollarized banking system into a total collapse of the financial system, the exchange rate, and other asset prices. Our explanation of crisis differs in some important aspects from what are now termed the first-, second-, and third-generation models of crises.

The links between the government and the banking system can come about simply because both dip into a common pool of dollars. Difficulties for one may create difficulties for the other even if the banking system does not hold significant amounts of government debt or the government does not bear the contingent liabilities of the banking system. Similarly, the collapse in the exchange rate and the collapse in the banking system can occur close together, not just because the corporate or banking system's liabilities explode in value after depreciation, but also because the depreciation is a result of the banking system's desperation for dollars. While dollar shortages can cause banking system crises, the reverse is also possible. By no means do we imply that any of the other channels already identified in the literature are unimportant (see Burnside, Eichenbaum, and Rebelo [2001a] or Aghion, Bachetta, and Banerjee [2001] for models that emphasize these other channels). Rather, we focus on one particular channel, the banking system's need for dollar liquidity, which can tie many of these effects together.

Then, we explore various possible policy interventions to mitigate the effects of dollar shortages, including whether the multilateral financial institutions have a role to play. If dollarization arises primarily from institutional infirmities rather than a distorted incentive to take on risk, it may be costly to legislate it away. Countries may have to learn to live with dollarization for awhile. At the same time, if poor institutions rather than poor incentives are to blame, interventions to mitigate the effects of dollarization need not exacerbate typical sources of moral hazard.

In the rest of the paper, we lay out first the basic argument and provide evidence for some of its assumptions. We then examine various interventions domestic authorities could undertake, and end with a discussion of possible interventions by the multilateral institutions.

## 1. A Framework

### *1.1 Why Are Emerging Markets Different?*

A growing number of economists identify the quality of institutions as producing important differences between emerging markets and developed economies. Broadly speaking, economic institutions may be *basic* or *narrow*. By basic, we mean fundamental institutions, such as those that ensure the security of property, including through the prevention of arbitrary taxation, or those that help enforce contracts. Basic institutions create the broad enabling environment for transactions between private agents and the state, and between private agents themselves. By narrow, we mean more detailed features of the institutional environment, such as whether the central bank is de-facto independent or whether there is a functioning bankruptcy code. Although not without exceptions, a country with weak basic institutions also finds it difficult to build effective narrow institutions.

One important role played by basic institutions is to mediate the process and outcome of social conflicts, particularly in times of adversity. Typically, in a growing economy, differences between social actors may be papered over. A downturn, though, usually brings out or sharpens latent social tensions.

Why growth seems to be easier to share than adversity is no trivial question. If consumption is shaped by habit, an income loss is much harder to swallow, while satisfaction from additional gains is less important to fight for. Individual aversion to losses in wealth is well documented in behavioral science. On the other hand, conflict may dissipate growth opportunities more easily than it may worsen an already stagnant situation. For example, squabbling between workers and management may drive investors away, chasing away the chance to start new projects; however, if there are no new investment opportunities on the horizon, squabbling is less costly, as the existing plant and machinery is already a sunk investment.

Regardless of why conflicts are greater in times of economic adversity, how a society deals with such conflicts depends on the kind of conflict management institutions it has. In a comprehensive study of failed states, Collier et al. (2003) find that years of poor economic growth precede civil war. Even after concluding a peace, the probability of these states lapsing back into war is high. Not surprisingly, these states typically have weak conflict management institutions, such as patchy law enforcement, limited adherence to democratic principles, and few meaningful checks and balances on the government. Similarly, Rodrik (1999) finds that the countries that experienced the sharpest drops in growth after 1975 were those with divided societies and weak conflict management institutions (as proxied for by indicators of the quality of government institutions, rule of law, democratic rights, and social safety nets).

Acemoglu et al. (2003) find that countries with poor institutions have the highest volatility of growth and higher levels of inflation than countries with well-functioning institutions. Satyanath and Subramanian (2004) show that over and above the effect of policies, the quality of political institutions affects the extent of nominal macroeconomic instability in a country.

Societies with well-functioning institutions allocate burden sharing in times of distress in predictable ways. For example, those who suffer the most adversity can fall back on an explicit social safety net—a minimum level of unemployment insurance. Debtors and creditors can appeal to bankruptcy proceedings to determine their relative shares. With an explicit and contingent institutional sharing mechanism dictating the division of pain in place, there is no need to take to the streets, the backrooms, or to the money printing press to settle outcomes.

By contrast, when institutions are too weak to offer predictable and acceptable settlements, or protect existing shares, everyone has an incentive to jockey for a greater share of the pie. Outcomes will be mediated more by the relative bargaining power of actors than by preexisting implicit or explicit contracts.

Often, bargaining will break down. Then, a government without the institutional capacity to allocate the burdens of adversity among its citizenry fairly will be tempted to spread it through the easiest means available—inflation. Nominal instability will accompany real instability in countries with weak institutions, lending support to the view that while the proximate cause for inflation may be monetary expansion, inflation is always and everywhere a political phenomenon!

### *1.2 Evidence for the Link Between Inflation and Poor Growth*

We want to establish two facts here, which are a little different from the work cited so far. First, we want to test whether the inflation "tax" is higher in downturns, and second, whether this phenomenon is particularly acute for countries with poor institutions. To check this, we have data on the value of the inflation tax, which is measured as $\Delta\text{CPI} /(1 + \Delta\text{CPI})$, where $\Delta$CPI is the change in the consumer price index in the country over the year. This is computed every year from 1965 through 2002 for 165 countries. In table 1, we present summary statistics and cross-correlations for the inflation tax, the standard deviation of the inflation tax computed over the preceding five years, the growth rate in GDP, and the quality of institutions measured by four different indices: government efficiency, rule of law, quality of regulation, and control of corruption. These indices are from the Governance Matters III database (see Kaufman, Kraay, and Mastruzzi 2004). We also report an index of institutional quality constructed using the International Country Risk Guide (ICRG) indicators; this second institutional index approximates the one used by Knack and Keefer (1995).

In figure 1 (shown on page 184), we plot the real growth of a country's GDP, averaged over 1980 to 1995, against average inflation tax over the same period. This is plotted separately for countries with below-median levels of government effectiveness and for countries with above-median levels. The negative slope is steeper in the former,

**Table 1. Institutions, Growth, and Inflation in a Panel of 165 Countries, 1965–2002**

| A. Summary Statistics | | | | | | |
|---|---|---|---|---|---|---|
| **Variable** | **Obs.** | **Mean** | **Std. Dev.** | **Min.** | **Max.** | **CV** |
| | | | | | | |
| **Inflation Tax** | 4902 | 0.098 | 0.120 | –0.323 | 0.846 | 1.226 |
| **Standard Deviation of Inflation Tax** | 4859 | 0.042 | 0.044 | 0.00015 | 0.365 | 1.050 |
| **Inflation** | 4902 | 14.871 | 34.683 | –24.430 | 547.534 | 2.332 |
| **Standard Deviation of Inflation** | 4859 | 8.289 | 19.453 | 0.018 | 206.265 | 2.347 |
| **Real GDP Growth** | 6428 | 3.521 | 5.896 | –84.380 | 59.860 | 1.675 |
| | | | | | | |
| **Governance Matters III Institutional Indicators** | | | | | | |
| **Government Efficiency** | 165 | 0.062 | 0.907 | –1.827 | 2.370 | 14.612 |
| **Rule of Law** | 165 | 0.075 | 0.939 | –1.830 | 2.210 | 12.493 |
| **Quality of Regulation** | 165 | 0.110 | 0.809 | –2.593 | 1.957 | 7.372 |
| **Control of Corruption** | 165 | 0.057 | 0.946 | –1.610 | 2.390 | 16.559 |
| | | | | | | |
| **ICRG Institutional Indicators** | | | | | | |
| **Index** | 2486 | 0.551 | 0.190 | 0.025 | 1 | 0.344 |
| **with components:** | | | | | | |
| **Quality of Bureaucracy** | 2486 | 0.534 | 0.304 | 0 | 1 | 0.570 |
| **Law and Order** | 2486 | 0.606 | 0.259 | 0 | 1 | 0.427 |
| **Corruption** | 2486 | 0.540 | 0.231 | 0 | 1 | 0.428 |
| **Investment Protection** | 2487 | 0.538 | 0.187 | 0 | 1 | 0.347 |

**Notes**: Inflation tax is $\pi/(1+\pi)$, with $\pi$ the annual CPI inflation.
The standard deviation of inflation and the inflation tax at year $t$ is calculated over the five-year period from $t$–4 to $t$.
Growth is the annual growth rate of real GDP.
The Governance Matters III indicators of the institutional environment are measured by their respective averages over the years 1996, 1998, and 2000.
The ICRG index is the normalized 0–1 sum (quality of bureaucracy + law and order + corruption + 2* investment protection) similar to Knack and Keefer (1995).
The ICRG individual components have been annualized and normalized 0–1 from available monthly observations for the period 1984–2002.
**Sources**: Inflation and GDP figures are from the IMF's World Economic Outlook 2004 database.
Governance Matters institutional indicators are from Kaufmann, Kraay, and Mastruzzi's 2003 Governance Matters III database.
ICRG institutional indicators are from the International Country Risk Guide database, www.icrgonline.com.

**continued**

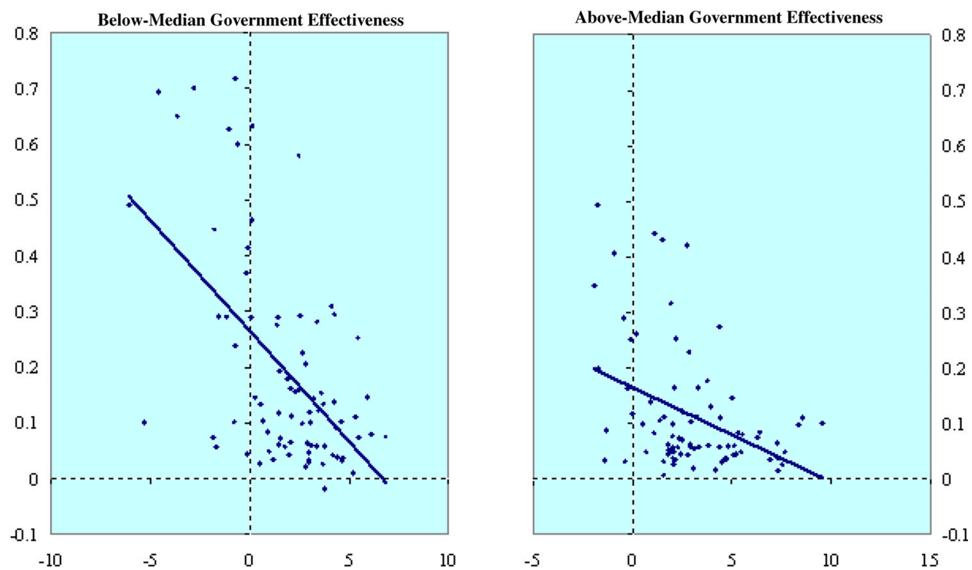
**Table 1 (continued). Institutions, Growth, and Inflation in a Panel of 165 Countries, 1965–2002**

**B. Pairwise Correlation Coefficients** (All correlation coefficients in the table are significant at 1% or less)

| | Inf.Tax | Std. Dev. (I. Tax) | Inflat. | Std. Dev. (Infl.) | Growth | Gov. Eff. | Rule Law | Qlt. Reg. | Ctrl. Corr. | Index | Qu. Bur. | Law/ Order | Corr. | Inv. Prot. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Inflation Tax** | 1.000 | | | | | | | | | | | | | |
| **St. Dev. of Inf. Tax** | 0.453 | 1.000 | | | | | | | | | | | | |
| **Inflation** | 0.876 | 0.386 | 1.000 | | | | | | | | | | | |
| **St. Dev. of Inflation** | 0.547 | 0.802 | 0.595 | 1.000 | | | | | | | | | | |
| **Real GDP Growth** | −0.236 | −0.088 | −0.233 | −0.115 | 1.000 | | | | | | | | | |
| | | | | | | | | | | | | | | |
| **Gov. Matters III Inst. Indicators** | | | | | | | | | | | | | | |
| **Govern. Efficiency** | −0.205 | −0.324 | −0.156 | −0.193 | 0.049 | 1.000 | | | | | | | | |
| **Rule of Law** | −0.227 | −0.350 | −0.173 | −0.215 | 0.038 | 0.940 | 1.000 | | | | | | | |
| **Quality of Regulation** | −0.160 | −0.286 | −0.133 | −0.169 | 0.042 | 0.866 | 0.848 | 1.000 | | | | | | |
| **Control of Corruption** | −0.210 | −0.338 | −0.159 | −0.204 | 0.032 | 0.949 | 0.946 | 0.800 | 1.000 | | | | | |
| | | | | | | | | | | | | | | |
| **ICRG Institutional Indicators** | | | | | | | | | | | | | | |
| **Index** | −0.409 | −0.423 | −0.288 | −0.292 | 0.133 | 0.808 | 0.815 | 0.685 | 0.813 | 1.000 | | | | |
| **with components:** | | | | | | | | | | | | | | |
| **Quality of Bureaucracy** | −0.314 | −0.401 | −0.205 | −0.258 | 0.074 | 0.759 | 0.766 | 0.620 | 0.764 | 0.879 | 1.000 | | | |
| **Law and Order** | −0.351 | −0.338 | −0.255 | −0.239 | 0.112 | 0.705 | 0.740 | 0.569 | 0.715 | 0.839 | 0.684 | 1.000 | | |
| **Corruption** | −0.165 | −0.236 | −0.101 | −0.126 | 0.041 | 0.705 | 0.692 | 0.583 | 0.745 | 0.759 | 0.678 | 0.634 | 1.000 | |
| **Investment Protection** | −0.429 | −0.357 | −0.320 | −0.282 | 0.173 | 0.493 | 0.487 | 0.466 | 0.468 | 0.773 | 0.525 | 0.489 | 0.317 | 1.000 |

**Notes**: Inflation tax is $\pi/(1+\pi)$ with $\pi$ the annual CPI inflation.
The standard deviation of inflation and the inflation tax at year $t$ is calculated over the five-year period from $t$–4 to $t$.
Growth is measured as the annual growth rate of real GDP.
The Governance Matters III indicators of the institutional environment are measured by their respective averages over the years 1996, 1998, and 2000.
The ICRG index is the normalized 0–1 sum (quality of bureaucracy + law and order + corruption + 2* investment protection) similar to Knack and Keefer (1995).
The ICRG individual components have been annualized and normalized 0–1 from available monthly observations for the period 1984–2002.
**Sources**: Inflation and GDP figures are from the IMF's World Economic Outlook 2004 database.
Governance Matters institutional indicators are from Kaufmann, Kraay, and Mastruzzi's 2003 Governance Matters III database.
ICRG institutional indicators are from the International Country Risk Guide database, www.icrgonline.com.

**Figure 1. Real Growth and Inflation Tax When Institutional Quality Is Below and Above Its Median**




Note: Real growth, on the horizontal axis, is measured for each country as the average over 1980 to 1995 of the annual growth rate of real GDP. Inflation tax, $(\pi/(1+\pi))$, with $\pi$ the annual CPI inflation, on the vertical axis, is each country's average inflation tax over 1980 to 1995. In the left (right) panel I group those of the 165 countries in the sample for which government effectiveness (average value for 1996, 1998, and 2000 as in Kaufmann, Kraay, and Mastruzzi 2004) is below (above) the sample median.

suggesting that slower growth is correlated with more inflation in countries with weak institutions.

Rather than average correlations, we are interested in the time-series patterns across countries. In table 2 we use a panel of yearly observations from 1965 through 2002 where the dependent variable is the inflation tax in a year in a country. In column 1, we estimate a random effects GLS (generalized least squares) model where the explanatory variables are a constant and the growth rate in GDP. The coefficient of the GDP growth rate is negative and highly significant, suggesting that the inflation tax is highest in periods of low GDP growth. A standard deviation increase in the growth rate is associated with a reduction in the inflation tax by .0241, which is 20 percent of its sample standard deviation. In column 2, we include the index of government efficiency (the results with the other "Governance Matters" institutional variables are qualitatively similar) and the interaction of GDP growth with the index. As the prior literature has found, countries with a better institutional environment tend to experience lower inflation tax. Particularly interesting is that the positive significant coefficient of the interaction term suggests, as predicted, that the inflation tax in countries with better institutions is less sensitive to growth. In column 3, we estimate the model including country fixed effects, and find no qualitative change in the coefficients of interest.[1]

One problem with the estimated model is that we cannot tell the direction of causality. High inflation may, in fact, cause low growth, though why this should be more pronounced in countries with poor institutions is harder to say. Nevertheless, it is important to examine the effect of the exogenous component of growth on the inflation tax. Typically, a country will be affected by similar exogenous shocks as its neighbors—if not directly, then via trade. So one plausible instrument for a country $i$'s growth is EXTGROWTH, which is the weighted average growth of all other countries $j$, with each country $j$'s growth weighted by that country's log GDP and divided by the square of the distance between $i$ and $j$. In column 4, we reestimate the fixed-effects model, using EXTGROWTH to instrument for growth.

[1]We also cluster by country and include year indicators with no qualitative change in the interaction coefficient.

**Table 2. Determinants of Inflation Tax in a Panel of 165 Countries, 1965–2002**

| Dependent Variable: Inflation Tax | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | RE/GLS | RE/GLS | FE | FE/IV | FE/IV | FE/IV | FE/IV2 | FE/IV |
| | | | Fixed Effects | Instrument for Growth | Small Country Effects | Interaction of Initial GDP and Growth | Instrument for Gov. Eff. and Growth | Alternative Institutional Indicator |
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| **Explanatory Variables:** | | | | | | | | |
| **Constant** | 0.1191 *** (0.0064) | 0.1209 *** (0.0062) | 0.1114 *** (0.0015) | 0.1388 *** (0.0048) | 0.1399 *** (0.0049) | 0.1362 *** (0.0054) | 0.1351 *** (0.0072) | 0.2876 *** (0.0177) |
| **Real GDP Growth Rate** | –0.0041 *** (0.00025) | –0.0048 *** (0.00035) | –0.0047 *** (0.00035) | –0.0224 *** (0.0020) | –0.0233 *** (0.0021) | –0.0079 (0.0068) | –0.0185 *** (0.0055) | –0.0264 *** (0.0070) |
| **Government Efficiency** | | –0.0323 *** (0.0067) | | | | | | |
| **Growth*Institutions (Gov. Eff.)** | | 0.0015 *** (0.0005) | 0.0014 *** (0.0005) | 0.019 *** (0.0031) | 0.0216 *** (0.0046) | 0.0246 *** (0.0031) | 0.0232 ** (0.0104) | |
| **Small*Growth*Institutions** | | | | | –0.0016 (0.0043) | | | |
| **Initial GDP*Growth** | | | | | | –.0019 *** (0.0007) | | |
| **Institutions = ICRG Index** | | | | | | | | –0.2864 *** (0.0397) |
| **Growth*ICRG Index** | | | | | | | | 0.0345 *** (0.0116) |
| **Number of Observations** | 4895 | 4895 | 4895 | 4753 | 4388 | 4387 | 2916 | 2178 |

**Notes:** Standard deviations are in parentheses below the estimated coefficients. *** indicates significance at 1 percent or less, ** at 5 percent or less.
Columns 1 and 2 report the estimates of random effects GLS regressions; columns 3–8 report the estimates of fixed effects.
In columns 4–8 we instrument the growth rate and the interaction of growth with institutions, by the "external" growth rate and its interaction with institutions. For every country $i$ and every year, the "external" growth rate is calculated as the average of every other country's $j \neq i$ growth rate weighted by the ratio of log GDP to the square of the distance between country $j$ and country $i$.
In column 5, we interact (Growth*Institutions) with the dummy Small, which equals 1, if the country's GDP in that year is below the ninetieth percentile of the sample in that year.
In column 6, we interact Growth with the log of the initial level of real GDP. For most countries the initial level is that of 1965; when this is not available, as in the transition economies, for example, we take the first year for which we have an observation for real GDP.
In column 7, we instrument the growth rate, institutions, and the interaction of growth and institutions, by the "external" growth rate, the log of population density in 1500 (see Acemoglu, Johnson, and Robinson 2002), and the interaction of "external" growth with the log of population density, respectively.
In column 8, institutions are proxied by the normalized 0–1 ICRG index = quality of bureaucracy + rule of law + corruption + 2* investment protection.
Investment protection includes expropriation, contract repudiation, and repatriation of profits. It is weighted by 2, so that the index can approximate the one used in Knack and Keefer (1995).

**Sources:** Inflation tax and GDP growth series based on annual CPI and real GDP series from IMF's World Economic Outlook 2004 database.
Government efficiency indicator: Kaufmann, Kraay, and Mastruzzi (2003) Governance Indicators database. Log of Population Density in 1500: Acemoglu, Johnson, and Robinson (2002).
ICRG institutional indicators: International Country Risk Guide, www.icrgonline.org.

The coefficient of the interaction is now larger in magnitude and stronger in significance.

Large countries may affect the growth of their neighbors, so there is a case for arguing the instrument is purer for small countries. One should ask if the coefficient estimate for the interaction differs for small countries. In column 5, we reestimate the fixed-effect instrumented regression with an additional term, the interaction multiplied by an indicator for countries whose GDP is below the ninetieth percentile GDP. The coefficient estimate for the indicator is statistically insignificant and small, suggesting that small countries do not have a different estimated interaction coefficient than large countries. This lends confidence to the instrument for growth.

There could, however, be some concern about our measure for institutional quality. It may be that the proxy for institutions is simply a proxy for per capita GDP. In column 6, we also include the interaction between initial GDP for the country (in 1965 or the first year for which we have GDP) and the country's growth rate. The coefficient of the interaction between institutions and growth is now slightly higher in magnitude, and still strongly significant.

We have assumed that institutions are slow moving, and have thus taken the index of government efficiency from Kaufman, Kraay, and Mastruzzi (2004)—averaged over the years 1996, 1998, and 2000—as the measure of institutional quality in a country for the period 1965–2002. One concern is that this measure is not predetermined and exogenous. There is some controversy about what instruments are appropriate for institutions. Following Acemoglu, Johnson, and Robinson (2002), we use the log of a country's population density in 1500 (countries that had less of a native population were less likely to have an exploitative colonial structure imposed on them and have better institutions today) as an instrument for institutional quality in column 7. While we lose a number of countries, the coefficient of the interaction is still positive, large, and statistically significant.[2]

The opposite concern would be that the measure of institutional quality is too static, that it does not reflect changes that take place in a country over time. The problem is that detailed measures of

[2]Of course, while the instrument for institutional quality is exogenous and predetermined, whether it satisfies exclusion restrictions depends on what else we think might explain the institutions-growth interaction. Since we have put forward no such alternative explanation, we do not pursue this issue.

institutional quality going back to 1965 are simply not available. However, the index we have constructed from ICRG data is one measure of institutional quality that does go back to 1985.[3] In column 8, we use the data from 1985, with the time-varying index as our measure of institutional quality, and find that the interaction variable is positive and significant as predicted.

The bottom line is that the inflation tax is higher when countries experience poor growth (as also in Kaminsky, Reinhart, and Vegh 2005), and it is particularly high when those countries have poor institutions. Poor societies with weak institutions do not share the burden of distress well.[4]

### *1.3 Contractual Adaptation*

If the country's underlying basic and narrow institutions do not permit a contingent, speedy, and predictable sharing of adverse economic circumstances, and the tendency of the government is to spread the burden along the path of least resistance, economic agents will take steps to protect themselves. But without a reliable and effective legal system, what can they do? Clearly, the answer is to use instruments that depend in a very limited way on the legal system for enforcement.

One approach is to use inflexible, noncontingent contracts, whose violation is easily detected. For example, labor contracts in many developing countries effectively do not permit employees to be fired. This is seen as inefficient because it does not allow firms to react quickly to business conditions. Often, these prohibitions are ascribed to overly strong unions that hold the economy to ransom. But if courts are slow and corrupt, so that a worker who is wrongfully fired has no redress, perhaps the prohibition of firing—because violations are so easily and publicly observable and can be responded to through mass protests—is the only way to protect workers from

[3]We try to approximate the index created by Knack and Keefer (1995). The ICRG measures for quality of bureaucracy, rule of law, corruption, and investment protection (including risk of expropriation, contract repudiation, and repatriation of profits) are all normalized to be between zero and one. The index is the weighted sum of these four measures, with the first three having a weight of 0.2, and the last having a weight of 0.4.

[4]This contrasts with the view in Lane and Tornell (1998) where developing countries do not share windfalls well and overspend them.

arbitrary decisions by employers (also see Glaeser and Shleifer 2001). Job tenure may also act as a form of social security because the government does a miserable job providing a safety net, and private insurance markets do not exist. Thus an inflexible contract can protect workers when the preponderance of bargaining power is with firms.

This is not to argue that such contractual arrangements should never be reformed—they may outlive their initial usefulness if the legal system improves, but may continue to be supported by vested interests. The arguments we have made may be trotted out as a defense long after they are valid.

### *1.3.1 Demandable Debt*

Another form of a rigid contract, but one with special features, is a bank demand deposit. Essentially, a demand deposit has two features that make it virtually self-enforcing. First, the bank is required to honor the claim when it is presented at the teller window. If it is slow in doing so, or attempts in any way to renege, the news spreads quickly since the refusal to honor a demand deposit is such a clear and incontrovertible event. Second, the bank honors withdrawals in the order they are presented until no more depositors want to withdraw or the bank fails. "Sequential service" implies that when depositors sense even the slightest hint of potential distress, they have a strong incentive to withdraw their money—if they do, at worst they have the trouble of redepositing if the bank later turns out to be safe; if they don't, they may end up penniless as the bank fails.

The two features ensure that the ordinary depositor has a fairly secure claim, supported by other depositors. The threat of a bank run plays the same role as the threat of a labor strike—if bank management reneges on the commitment to repay the deposit contract, it will face a depositor run, which will close it down. So except in the case where it absolutely cannot pay, bank management will honor deposit contracts (see Calomiris and Kahn [1991] and Diamond and Rajan [2001]). This may be one reason why banks are such an important component of the financial sector in emerging markets.

The broader point is that anticipating little power over outcomes in downturns, weaker agents might demand contractual options that will protect them in those states. For labor, it is the option to keep a job; for depositors, it is the option to get their money. For the economy as a whole, however, the exercise of these options adds to the difficulty of adjustment in downturns, exacerbating the problems created by institutional weakness.

In the rest of the paper, we will examine these problems further, specifically focusing on how demandable debt raises the risks of financing industry in emerging markets. But before we explore that, let us add two more ingredients.

#### *1.3.2 Domestic Liability Dollarization*

Inflation is a greater systematic risk, in the financial sense, in emerging markets. When it is likely to explode in downturns there, depositors will demand an extraordinarily high risk premium for holding inflation risk. Issuers who want to minimize expected debt service—perhaps because of short horizons, or because they are liquidity constrained—will opt to issue real instruments (see Caballero and Krishnamurthy [2003] for a related explanation, and Ize and Levy-Yeyati [2003] and Jeanne [2005] for other theories of why inflation risk could lead to dollarization).

If there is high volatility in inflation (which usually accompanies a high inflation rate) in addition to weak institutions, inflation-indexed instruments may not be attractive to the public. Uncertainty about the measurement of inflation, delays in producing an accurate estimate, and fears that the measurement will be manipulated can increase their risks. The natural alternative to issuing inflation-indexed bonds is to denominate them in a foreign currency. This way, suspicion about the official actions in a downturn may lead quite naturally to domestic liability dollarization.

#### *1.3.3 Evidence on Liability Dollarization*

What evidence do we have for this conjecture? Nicolo, Honohan, and Ize (2003) find that in a cross-section of countries, the extent of dollarization (dollar deposits to total deposits) is positively and significantly correlated with the log of inflation. However, when a proxy for institutional quality is included, inflation no longer enters significantly. The evidence is consistent with weak institutions driving inflation, which in turn leads to greater dollarization.

Again, however, we want to test a more nuanced version. We also want to see if there is a relationship between the sensitivity of inflation tax to growth (which, we have seen, appears to reflect the ability of a country to allocate the costs of economic adversity) and the level of dollarization. We also want to see if the extent of dollarization is related to the volatility of inflation, over and above its correlation with the level of inflation. In table 3, we present summary statistics and cross-correlations. The extent of liability dollarization is measured by the ratio of foreign currency deposits to total deposits (FCDTD) in a country's banking system averaged over the 1990s, using the Nicolo, Honohan, and Ize (2003) data. The sensitivity of inflation tax to growth for a country (henceforth "sensitivity") is the coefficient estimate on GDP growth in a regression of the inflation tax on GDP growth for that country for the period 1965–2002. The standard deviation of inflation tax is measured for every period $t$ by its standard deviation during the five years from $t-4$ to $t$; then for the cross-section, we take the average of standard deviation over the period 1965–2002.

In table 4, the dependent variable is liability dollarization in a country in the 1990s. In column 1 we include the sensitivity of inflation tax to GDP growth and a constant as explanatory variables. The coefficient estimate for the sensitivity is negative and significant. Since the sensitivity is typically negative (lower growth, more inflation tax), countries with a higher magnitude of the sensitivity have greater deposit dollarization, as expected. In figure 2 (shown on page 194), we plot the extent of dollarization against sensitivity. As the graph suggests, the relationship is likely to be nonlinear. So in column 2, we allow for a nonlinear specification of sensitivity by including the square of sensitivity. The coefficient of the squared term is positive and strongly significant. Greater sensitivity again is correlated with greater dollarization. If sensitivity changes from 0 to its lower 1 percentile threshold (–0.029), dollarization increases by 33 percent, which is 140 percent of its standard deviation.

We check that this relationship persists even when we include the "usual suspects." In column 3, we include the average inflation tax in the country, and in column 4 we add the standard deviation of the inflation tax. While the coefficients for the nonlinear specification for sensitivity are positive and statistically significant in both columns, the coefficient for inflation tax is positive and significant only when

**Table 3. Growth, Inflation, and Dollarization**

| **A. Summary Statistics** | **Obs.** | **Mean** | **Std.Dev.** | **Min.** | **Max.** | **CV** |
|---|---|---|---|---|---|---|
| **Foreign Currency Deposits as** | | | | | | |
| **Percentage of Total Deposits** | 91 | 0.2801 | 0.2352 | 0.0014 | 0.9156 | 0.840 |
| **Sensitivity of Inflation Tax on Growth** | 91 | –0.0070 | 0.0106 | –0.0453 | 0.0167 | –1.524 |
| **Inflation Tax** | 91 | 0.1350 | 0.1008 | 0.0263 | 0.5666 | 0.746 |
| **Standard Deviation of Inflation Tax** | 91 | 0.0574 | 0.0409 | 0.0106 | 0.1706 | 0.712 |
| **Log of Per Capita GDP** | 91 | 7.2451 | 1.2306 | 4.2402 | 9.6286 | 0.170 |
| **Legal Restrictions on Dollarization** | 83 | 0.5542 | 1.1609 | 0 | 5 | 2.095 |
| **B. Pairwise Correlations** (* indicates significance at 5% or less) | | | | | | |
| | **FCD/TD** | **Sensitivity** | **Inf. Tax** | **SD(Inf.Tax)** | **LnPcapGDP** | **Leg.Rest.** |
| **Foreign Currency Deposits as** | | | | | | |
| **Percentage of Total Deposits** | 1 | | | | | |
| **Sensitivity of Inflation Tax on Growth** | 0.3718* | 1 | | | | |
| **Inflation Tax** | 0.5581* | –0.4713* | 1 | | | |
| **Standard Deviation of Inflation Tax** | 0.6807* | –0.5859* | 0.7604* | 1 | | |
| **Log of Per Capita GDP** | –0.3011* | 0.0649 | –0.2345* | –0.2784* | 1 | |
| **Legal Restrictions on Dollarization** | –0.2097 | 0.2172* | –0.2493* | –0.1658 | –0.2274* | 1 |

**Notes:** Foreign currency deposits to total deposits (0–1) for each country is the average of available observations over 1990–2001.
Inflation tax and its standard deviation for each country is the average for the period 1965–2002.
The sensitivity of inflation tax on growth is the estimated coefficient of growth as a regressor on inflation tax as the dependent variable. The regressions were estimated by country for the period 1965–2002.
The measure of legal restrictions on dollarization (0–5, meaning no legal impediments) is based on IMF's Exchange Arrangements and Restrictions for 2001.

**Sources:** Foreign currency deposits/total deposits and legal restrictions on dollarization: Nicolo, Honohan, and Ize (2003).
Inflation tax and GDP growth calculations were based on annual CPI and GDP series in IMF's World Economic Outlook 2004 database.

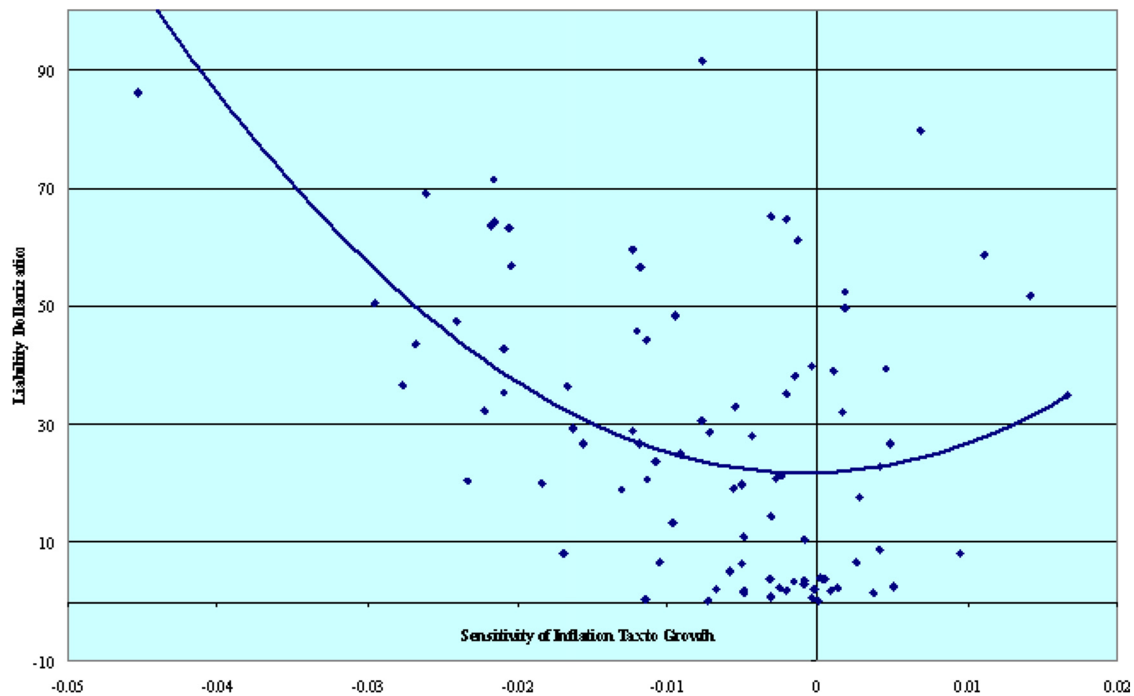
## Table 4. Determinants of Liability Dollarization

| Dependent Variable: 1990–2001 Average of Foreign Currency Deposits to Total Deposits, (0–1) | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| **Explanatory Variables:** | | | | | |
| **Constant** | 0.2227 *** (0.0275) | 0.2184 *** (0.0262) | 0.1038 *** (0.0324) | 0.0546 * (0.0318) | 0.3652 *** (0.1366) |
| **Sensitivity of Inflation Tax to Growth** | –8.2546 *** (2.1845) | 0.7657 (3.4375) | 5.0396 * (3.1545) | 6.4243 ** (2.9051) | 5.6295 ** (2.9516) |
| **Square of Sensitivity** | | 420.6529 *** (127.874) | 388.5883 *** (113.2333) | 283.8134 *** (106.5391) | 254.0754 *** (105.2839) |
| **Inflation Tax** | | | 1.1070 *** (0.2188) | 0.3170 (0.2738) | 0.2584 (0.2926) |
| **Standard Deviation of Inflation Tax** | | | | 3.1707 *** (0.7495) | 2.7209 *** (0.7769) |
| **Log of Per Capita GDP** | | | | | –0.0356 ** (0.0163) |
| **Legal Restrictions on Dollarization** | | | | | –0.0311 * (0.0169) |

**Notes:** Estimates based on a cross-section of 91 countries except for column 5, where availability of legal restrictions limits the sample to 83.
The standard deviations are in parentheses under the estimated coefficients. *** indicates significance at 1 percent, ** at 5 percent, and * at 10 percent.
The sensitivity of inflation tax to growth is for each country the estimated coefficient of the growth rate of real GDP as a regressor on the inflation tax as the dependent variable; the regressions for the estimation of sensitivity have been estimated for each country separately for the period 1965 to 2002.
Square of sensitivity is the square of the above variable.
Inflation tax and its standard deviation are measured here by their averages over 1965–2002.
The measure of legal restrictions on dollarization (0–5, 0 meaning no legal impediments) is based on IMF's Exchange Arrangements and Restrictions for 2001.

**Sources:** Inflation tax, sensitivity of inflation tax to growth, standard deviation of inflation tax: calculations based on CPI and real GDP series in IMF's World Economic Outlook, 2004 database. Log of per capita GDP: World Bank World Development Indicators, 2004.
Foreign currency deposits as percentage of total deposits, and index of legal restrictions on dollarization: Nicolo, Honohan, and Ize (2003).

**Figure 2. Dollarization as a Function of the Sensitivity of Inflation Tax to Real Growth**




Note: The sensitivity of inflation tax to growth is the estimated coefficient of the real GDP growth rate as a regressor on inflation tax as the dependent variable; regressions by country were based on 1965–2002 samples. Liability dollarization is measured as the ratio of foreign currency deposits to total deposits in percent; for each country we take the average for the period 1990–2001.

included alone, but becomes insignificant when the standard deviation of inflation tax is included. The estimates for sensitivity are qualitatively similar if we include squared terms for inflation tax and the standard deviation of inflation tax (estimates not reported). Finally, in column 5, we include both the log of per capita GDP and the index of legal restrictions on dollarization compiled by Nicolo, Honohan, and Ize (2003), which is available for only eighty-three of the countries, and find qualitatively similar results.

One should not read too much into these last few "kitchen sink" regressions since sensitivity, inflation tax, and the standard deviation of the inflation tax measure various aspects of the same thing. All we want to show the reader is that both sensitivity and the standard deviation of inflation tax seem to be correlated with the extent of dollarization as predicted by the earlier discussion, and seem to capture something more than just the level of the inflation tax, which the prior literature has identified.

The evidence thus far is consistent with the following conclusions: countries with weak institutions have greater sensitivity of inflation to growth. In countries with higher sensitivity, investors have a higher demand for real deposits. Because inflation is also very volatile, they may prefer deposits denominated in foreign exchange rather than deposits that are indexed.[5]

### *1.4 Aggregate Dollar Constraints/Sudden Stops*

Let us now add the final ingredient to the model. Since emerging markets with the weakest institutions for conflict management (and the most divided societies) have the hardest time spreading the burdens of distress, they are also likely to have the most difficulty raising resources to continue to service external debt. The tendency of some countries to default repeatedly (Reinhart, Rogoff, and Savastano 2003) may reflect the weakness of their capacity to manage

[5]There is a sense in which this argument runs counter to the "Original Sin" thesis (for example, see Eichengreen and Hausmann [2005] and Eichengreen, Hausmann, and Panizza [2005a, 2005b]) because we attribute financial fragilities to weak institutions rather than to other factors like country size. But Eichengreen and Hausmann (2005) and Eichengreen, Hausmann, and Panizza (2005a, 2005b) focus on the currency denomination of public debt rather than on the currency denomination of bank debt. For yet another view of institutional explanations of financial system fragilities, see Mody (2004).

economic adversity rather than any inherent lack of honesty on the part of their governments. But this means that these countries are likely to face aggregate constraints on external borrowing sooner than other countries. Since in periods of adversity creditors will reduce their expectations of what the country will be able to repay, they will also reduce what they are willing to lend. Such a "vertical" constraint on dollars the country can borrow (as in Caballero and Krishnamurthy [2000, 2004], or as a sudden stop in Calvo and Reinhart [2002]), will interact with liability dollarization to produce unfortunate consequences, which we now document.

## 2. Consequences: Overshooting, Liquidation, and Contagion

Now that we have the ingredients, dollarized bank deposits and the possibility of aggregate constraints on borrowing, let us sketch the consequences.

### *2.1 The Sources of Dollar Shortage*

In the normal course, dollar depositors will want to withdraw some of their deposits. The reasons for this can range from normal liquidity needs (such as importing foreign goods) to good dollar investment opportunities outside the country. Clearly, if their bank has fewer dollar reserves than the amount of withdrawals, it will buy dollars on the market. Summing across banks, there will be an aggregate demand for dollars, which will have to be met out of the country's reserves, dollar repatriation by exporters, and, if necessary, additional external borrowing. It does not really matter which domestic entity (government or banks) does the external borrowing since the aggregate available pool of dollar resources will determine whether the aggregate domestic demand can be satisfied.

Problems arise when the aggregate demand exceeds the aggregate supply (not including external borrowing) and the country has difficulty borrowing the shortfall. One such situation is one where the economy is booming but the (fixed) exchange rate is overvalued. Exporters may not earn enough and, far from bringing foreign exchange into the country to repay loans, they may seek to draw down their deposits to continue operations. Importers may have a huge demand for dollars because foreign goods appear cheap. When added to the normal liquidity needs of depositors, the demand may be so high that it even exceeds the willingness of foreign investors to lend the shortfall. Another situation arises when the excess demand is relatively small but the economy is in a bad way, or the government has overborrowed, so foreign investors are unwilling even to lend meager amounts of extra dollars needed. In fact, the government can contribute to the private sector dollar shortage by adding its own external financing needs.

Regardless of how the dollar shortage emerges (and we will shortly see some examples), the dollarized banking system can exacerbate it (see Diamond and Rajan [2005] for a detailed model). Since the banks have issued a nonrenegotiable promise to pay dollars, they either have to convince their own depositors not to withdraw, by hiking the interest rates paid on dollar deposits, or they have to attract dollars away from other banks in the spot market. Higher rates may quell some depositor demand, but a core liquidity demand that cannot be deterred with higher rates will remain. If this still exceeds the available dollars, the banks will compete with each other for scarce dollars. Given that a bank fails if it does not come up with the needed dollars, it will be willing to pay what it must for additional dollars. With an overall shortage in place, however, banks can competitively drive each other into failure.

Short banks will sell nondollar spot assets and long-term assets for dollars. Thus the exchange rate (dollars per domestic currency) will tend to fall and interest rates (both for long-term dollar assets and for long-term domestic currency assets) will rise. In principle, because the quantity of dollar demand and supply cannot adjust readily, these prices can move very far from any notion of fundamental value. Both the exchange rate and the interest rate can overshoot during the scramble for dollar liquidity. Real decisions will be affected during this scramble, with lasting consequences. Let us go systematically through them.

### *2.2 Real Consequences*

The first place banks will look for additional dollars is among those who generate additional dollars and those who use them. Exporters will be squeezed in an attempt to get them to speed up their own dollar receipts and hasten repayment of dollar borrowings to banks (on average, across emerging markets, approximately 30 percent of domestic loans made by banks are denominated in foreign currency). To raise these amounts quickly, exporters will sell finished goods inventories at steep discounts and reduce near-term sales prices. They will shelve exports that are highly import intensive and abandon long-term projects, especially those that require capital goods imports.

Clearly, all these actions will impair the economy's medium-run ability to export and thus its ability to generate dollars in the future. The weaker a country's institutions, the greater will be the discount banks place on a future dollar generated by an exporter relative to a current dollar (foreign investors will be willing to lend less against the future), and the greater the long-run destructive consequences of a scramble for dollars.

We will see these effects not only in the tradable sector, but also in the nontradable sector. As domestic interest rates rise (because long-run domestic assets are being sold for dollars), more and more domestic projects will be shelved as they have to meet an impossible hurdle rate.

As bank assets fall in value, some banks—typically the ones with the greatest asset liability currency mismatch (though see later)—will become insolvent. This will trigger a generalized run on the banks' assets, causing even those who had no desire to withdraw to add to dollar demand.[6] The horizon of failing banks will be even shorter, causing them to be even more indiscriminate in the squeeze they put on borrowers. Even projects that could produce substantial dollar revenues in the near term may be sacrificed for the immediate need—for example, banks may stop offering working capital loans and export credit even if these are essential for the exporter to generate revenues. As a result, the aggregate pool of dollars available over the near term could fall as banks fail, and the aggregate excess demand for dollars could increase, putting pressure on other banks.[7] This form of contagion could imperil the entire banking system.

[6]Note that if the exchange rate is fixed but there are no capital controls, domestic currency depositors have an even greater incentive to withdraw (and convert) than dollar depositors because they will fear a devaluation.

[7]Clearly, a bank that fails will refuse to honor some of its dollar depositors. The unsatisfied demand of these depositors will reduce aggregate demand. Therefore, the effect of bank failure on the excess demand for dollars depends on whether supply falls faster or slower than demand. See Diamond and Rajan (2005) for conditions under which each is true.

To summarize, when bank depositors demand repayment in dollars but the economy cannot generate enough dollars to satisfy them, the consequences can be very serious. Domestic dollar interest rates will rise to draw in dollars and choke off depositor demands. If, however, there is a core group of depositors who absolutely want to withdraw dollars, and a limit to which outsiders are willing to lend to the country, the country's banking system can face an excess demand for dollars that it cannot meet. If so, other asset prices will fall precipitously as banks scramble to capture enough dollars from the common pool to save themselves. Domestic currency interest rates will spike up, while the exchange rate will plummet. Banks will squeeze borrowers, and aggregate activity will fall. Some banks may become insolvent and such failures could be contagious.

Of course, in any such model, we could get multiple equilibria, where outside lenders impose a sudden stop, which leads to the dollar shortage, which leads to bank actions that reduce future dollar receipts, which justify the stop. We do not need, however, to appeal to multiple equilibria to explain the crisis—a spike upward in dollar demand or downward in dollar supply, coupled with a "normal" demand for liquidity, is sufficient to produce the effects.

### *2.3 Related Literature*

Consider now how this model differs from earlier work. In a comprehensive survey, Frankel and Wei (2004) attempt to distinguish between the three "generations" of crisis models on the basis on their explanation of why the crisis occurs:

> "Whose fault is the crisis? Generation I says domestic macroeconomic policy, Generation II says volatile financial markets, and Generation III says financial structure. In neutral language, the explanations are, respectively, excessive macroeconomic expansion, 'multiple equilibria,' and moral hazard. In finger-pointing language, the respective culprits are undisciplined domestic policymakers, crazy international investors, and crony capitalists."

The model in this paper is related to the third-generation models in that it focuses on structural problems associated with lending to emerging-market countries. However, in our paper, crises are not necessarily caused by willful misbehavior. Instead, they stem from adverse liquidity shocks that jolt a system that is necessarily rigid, given the institutional inadequacies of the economy. Put another way, better regulation and supervision may not necessarily eliminate the possibility of a crisis. What is really needed is deep-rooted institutional reform: susceptibility to crises in our framework ultimately rests not in an incentive problem but a collective action problem.

A closely related paper is that of Calvo, Izquierdo, and Mejia (2004), who also focus on a link between sudden stops, dollarization, and banking crises. In their paper, sudden stops lead to a devaluation—in order to maintain external balance—which then causes problems in the dollarized banking system through liability mismatches. In other words, macrocauses have microconsequences. In our model, the channel is not the need to maintain external balance but, rather, bank liquidity. The sudden stop creates a dollar shortage, which leads banks to dump assets, causing the exchange rate (and interest rates) to overshoot fundamentals, which then create balance sheet problems for the banking system. Microcauses aggregate up to have macroconsequences.

While we think both explanations have merit, there are differences. For instance, to the extent that a devaluation gives exporters the ability to earn more (expansionary devaluation), there is no reason for it to hurt the solvency of a dollarized banking system—since banks typically make dollar loans to the exporters (see Nicola, Honohan, and Ize 2003). But to the extent that the capacity to earn future dollars does not translate into current dollars, a liquidity mismatch could persist, and banks could still go under in our framework.

## *2.4 Some Examples*

Consider some examples.

### *2.4.1 Argentina (2001)*

By the end of 2000, the Argentinean banking system had approximately $72 billion in foreign-currency-denominated assets and about the same amount in liabilities. By most standards, it seemed to have matched exposures. However, $25 billion of its assets were government securities, issued by a government that was increasingly strapped for financing. Another $41 billion were foreign-currency-denominated loans and securities issued by Argentinean corporations, which clearly did not have the ability to repay quickly, as exports amounted to only $31 billion. Of the liabilities, $48.5 billion were foreign currency deposits.

In this fragile situation, depositor runs could start for two related reasons. First, if the government could not draw in more external resources to meet its own external debt service needs, or its new borrowing requirements, the anticipated available dollar pool would be severely constrained. The banking system's liquidity needs would compete with the government's needs, pushing up interest rates and leading, perhaps, to a devaluation.[8] Second, given the extensive bank holdings of government assets, a government default could render banks insolvent (though see below).

Depositor runs started in 2001. Bank liabilities fell by $24 billion (approximately 9 percent of GDP). In fact, Argentina lost more as a result of the bank run than as a result of the inability of the government to access external markets to meet financing needs. Interestingly, the fall in domestic-currency-denominated deposits was far greater than the fall in foreign currency deposits, suggesting that depositors feared a devaluation, perhaps from the liquidity shortage, more than a bank default. Since bank holdings of government debt could not be reduced—in fact, they increased—the run was financed by curtailing private lending ($12 billion), running down bank liquid assets ($5 billion), and borrowing from the central bank ($9 billion).

Ultimately, the entire banking system was affected, deposits were frozen, then loans and deposits were "pesified" at different rates. The consequences are still being dealt with. The point to take away is that a government may affect the dollarized domestic banking system simply by crowding out access to dollars.

[8]This would not necessarily lead to a default by dollar borrowers. For instance, Bleakley and Cowan (2002) find that the negative balance sheet effects of devaluation are outweighed by the competitiveness gains for a sample of Latin American firms.

#### *2.4.2 Uruguay (2002)*

Uruguay experienced an almost reverse sequence of events: liquidity problems in the banking sector triggered a crisis, a devaluation, and problems for the government, which then had to restructure debt. Let us examine how this happened.

Uruguay also had a highly dollarized banking system. Bank deposits were about 90 percent of GDP by the end of 2001; 90 percent of these deposits were denominated in U.S. dollars. About half these deposits were held by nonresidents, mostly Argentineans.

As the Argentineans saw their deposits in Argentina frozen, they started withdrawing from Uruguayan banks. Their liquidity need could have been met by Uruguay's domestic holdings of liquid foreign currency assets. However, anticipating a shortage, Uruguayan residents also began withdrawing deposits. With over 45 percent of the foreign currency deposits withdrawn, the currency depreciated precipitously, prompting further concerns about bank solvency. The government declared a bank holiday to stop the run; eventually, it successfully reopened the banking system with the help of a standby arrangement from the International Monetary Fund (IMF) and a rescheduling of deposits.

As a result of the depreciation, public debt, which was largely denominated in foreign currency, ballooned from about 45 percent of GDP by the end of 2001, to 100 percent of GDP by the end of 2002. Eventually, it had to be restructured. In this case, liquidity problems in the banking sector created problems for the government in servicing its public debt.

#### *2.4.3 South Korea (1997–98)*

It is generally accepted that the trigger for the Korean financial crisis was a decline in export growth, especially in key areas like semiconductors. The weakening demand in importing partners and the appreciation of the real exchange rate as the dollar—to which many Asian currencies were implicitly pegged—strengthened against the yen, were behind the pressure on the real external sector. In the case of Korea, such initial pressure, emanating in the real external sector, was compounded by a banking system that had issued a significant amount of short-term external debt and thus was vulnerable to a liquidity shock. Contrast this with the liquidity shock that resulted from the government's losing access to external borrowing, in the case of Argentina, or the liquidity shock set in motion by Argentineans withdrawing their deposits abroad, in the case of Uruguay.

We will not describe the details of the crisis, which resembled in many ways what we have described above (see Ghosh et al. [2002, 210] and Lindgren et al. [1999, 188] for details). Korean banks initially started facing difficulties in mid-1997. The government announced a guarantee of foreign borrowings by Korean banks, and the central bank attempted to help foreign branches and subsidiaries of Korean banks roll over their foreign currency borrowings. But this depleted reserves, leaving the central bank with little to fight domestic bank runs. The government simply did not have the necessary dollars to back the guarantee it had announced. The won fell sharply.

In early December, the IMF announced a standby arrangement with Korea equivalent to $21 billion, with additional financing from others of $37 billion. Yet this massive package was insufficient, and the won continued falling. It was only when foreign private banks agreed to maintain their exposure to Korean banks by exchanging their interbank loans for short-term government-guaranteed bonds, and when the IMF accelerated disbursement of the loan, that pressure on the won abated. In terms of our framework, the shortage was eliminated by reducing dollar demand and increasing dollar supply and thus alleviating pressure on both the exchange rate and the interest rate.

Interestingly, in the case of Korea, a liquidity crisis was averted because the government had spare borrowing capacity and could draw in dollars (with some help from the IFIs and developed country governments), which it then lent out to the banks. With a few examples behind us, we can now discuss policy interventions in general terms.

## 3. Interventions

Let us recapitulate what happens if no intervention takes place. Obviously, the only way to eliminate a dollar shortage is to increase supply or reduce demand for dollars. If dollar depositors who seek to withdraw are not tempted to stay in the bank by higher dollar interest rates (for the same reason, perhaps, that higher interest rates do not draw fresh foreign investors in), then banks will start competing for scarce dollars. Since a bank has to satisfy every one of its withdrawing dollar depositors in order to stay in business, it will be willing to pay any feasible price for dollars if it is falling short. This is why prices of nondollar assets can deviate so far from fundamentals—the bank essentially faces a classic short squeeze where it has to deliver a specific asset in short supply, so it is willing to sell all other assets, almost regardless of price.

The dollar shortage is a form of liquidity shortage for the banking system. However, it can affect the solvency of firms, banks, and even the government, so it is extremely difficult to tell it apart from a solvency problem. There are many ways solvency and liquidity can interact. Consider two. First, the dollar shortage may stem from an insolvent government not being able to roll over dollar liabilities (a sudden stop). Too few dollars relative to demand can cause banks to dump domestic assets, pushing up interest rates and depressing the exchange rate, thus rendering firms with dollar liabilities insolvent. Second, some banks may be insolvent because of bad loans. A run on them may cause a squeeze on credit to exporters, a shortage of dollars, and contagion in the banking system, rendering the government insolvent through its contingent liabilities to the banking system.

Put another way, the practical challenge in the midst of a crisis is to make a judgment call on whether the crisis is one of illiquidity. If the judgment is in the affirmative, the proper intervention is to lend freely to the healthiest banks the commodity that is in short supply, in this case dollars (or alternatively, convince dollar demanders to hold off pressing their claims). For, if the crisis is truly one of illiquidity, prevailing market rates will fall, and weaker institutions will regain market access rapidly. If the crisis is one of insolvency, market rates will stay high, and the banking system will continue to be fragile. Thus, except when failures are isolated or when asset values collapse for other reasons than high interest rates or low exchange rates, it is typically only after intervening in a crisis and seeing the consequences that one can tell whether the crisis was one of illiquidity or insolvency.

Given the difficulty of telling illiquidity from insolvency, and given the cost of a banking system meltdown, if the government has dollar reserves, spare external borrowing capacity, or support from international financial institutions, it would sell dollars into the banking system hoping to alleviate pressure. However, we have defined a dollar shortage as one where the government itself has too few resources to contribute. So let us turn to other interventions.

## *3.1 Ex-post Intervention by Country Authorities*

### *3.1.1 Recapitalization*

The authorities can recapitalize specific banks by offering them additional domestic assets or guarantees (backed by domestic assets). Often, what is termed "liquidity support" are simply loans by the central bank to distressed banks without adequate collateral backing the loans—in short, they are partial recapitalizations.

While targeted recapitalizations can prevent specific banks from failing, they leave, on the aggregate, a dollar gap that has to be closed somehow. Unless other banks are allowed to fail, the aggregate dollar demand cannot be satisfied. This implies that a bank recapitalization without any attempt to bridge the dollar gap only forces other, potentially healthier, banks to fail. A blanket recapitalization or guarantee of all banks simply allows all banks to bid more for dollars (that is, it increases the interest or exchange overshooting) without reducing the eventual extent of bank failures. This is why it is best to close down some banks and thus resolve the dollar shortage before offering indiscriminate guarantees.

### *3.1.2 Monetary Policy*

The monetary authorities could be accommodative and buy long-term domestic assets in exchange for domestic reserves (or do the opposite). Monetary accommodation will reduce the extent to which the burden of adjustment falls on the interest rate, and increase the downward pressure on the exchange rate. If not reversed later, it will increase inflationary pressures.

However, the proximate effect will be to shift the burden among banks—the survival chances of banks with relatively more holdings of long-term domestic assets will improve, while the chances of those with more net dollar liabilities will weaken. Whether the new pattern of failure reduces the overall dollar shortage depends on whether the newly failing banks subtract more dollar liquidity in failing than the banks that would fail absent the intervention.

An exchange rate defense (keeping the exchange rate high by tightening monetary policy) and an interest rate defense (keeping the interest rate low by being accommodative) are different in this simple framework only in that they select different sets of banks for failure. The choice between them rests on which one allows the banking system to come through the dollar shortage creating the minimum long-term damage to the real economy—through the damage the failing banks and their clients sustain. We are, of course, abstracting from any issues of credibility here, though it would be hard to unambiguously relate monetary authority credibility gains to a particular form of defense.

Caballero and Krishnamurthy (2004) also examine monetary policy in a situation of a dollar shortage. However, while our focus is on the effects of monetary policy on the extent of failure and economic damage (an ex-post analysis), their focus is on the ex-ante effects on the incentive to hold dollars. An accommodative monetary policy is unambiguously preferred in their analysis because it increases agent incentives to hold dollars before the crisis. Since the main friction in their model is that the price of dollars is too low ex-post (agents who need them do not have the collateral to buy them), any policy that enhances the price of dollars during the crisis enhances ex-ante incentives.

If, however, such a policy did considerable damage to the banking system (because, for example, moderately illiquid banks had many dollar liabilities), an effect they abstract from, our analysis suggests the policy would be dominated. In fact, such a preannounced policy might have limited credibility ex-ante and thus limited incentive effects. On the other hand, unlike their paper, we assume no frictions in trading dollars ex-post, so we abstract from the effects they focus on. The two papers should therefore be seen as complementary.

Before proceeding to other interventions, we should note that the monetary authority also has the ability to select banks that will fail by allocating its limited foreign exchange reserves only to some banks (i.e., at a subsidized price) and not to others. While such an intervention is fraught with political difficulties (who will be chosen and will the process be transparent), ultimately, it is an optimization problem where regulators allocate scarce resources to minimize the overall cost of bank failures. Thus it is similar in consequence to the interventions just discussed.

#### *3.1.3 Forced Conversion/Suspension of Convertibility/Capital Controls*

Finally, consider even stronger interventions that violate the rights of the depositors. These include forced conversion into domestic currency at a predetermined (typically below market) rate, the freezing of foreign currency deposits, and the imposition of capital controls. Clearly, such interventions can be implemented only by the country authorities and not by the banks alone.

While these interventions do solve, to differing extents, the problem of dollar shortage, they do so at the expense of a substantial loss in future credibility. Moreover, it is not clear that they can be implemented effectively and for the long term. For instance, capital controls tend to leak, and the longer they are in place, the more they leak. Therefore, the authorities had better be confident that the liquidity shortage is temporary, or else the breathing space these measures give them will be insufficient to rectify the problem; the problem will return with a vengeance with the added difficulty that the authorities then have no credibility.

### *3.2 Ex-ante Intervention by Country Authorities*

Thus far we have discussed measures that could be taken in the midst of a crisis. Consider now measures that could be taken up front by the economy.

#### *3.2.1 Reserves*

One way to bullet-proof an economy against dollar shortages is for the authorities to build foreign reserves. Of course, there are costs to holding reserves and to building them, including the fiscal costs and possible distortions in the exchange rate. Furthermore, it is possible that the level of dollarization in the economy increases as reserves, and confidence, grow. As a result, the authorities may lose all control over monetary policy and the transmission mechanism. Building a moderate amount of reserves is clearly warranted, but the welfare effects of building a hoard large enough to buffer against most crises are ambiguous.

A second question that arises with reserves is whether the country should use them to prepay debt. In other words, is spare debt capacity not the same as holding reserves, and less costly to boot? For the riskiest countries, though, prepaying debt may be dominated by holding reserves: spare debt capacity is less fungible than reserves, and may also disappear in a crisis. Also, by prepaying debt, the country loses the option to force a restructuring, which may be valuable in times of stress.

#### *3.2.2 De-dollarization and Shifting Dollarization*

Given the risks associated with dollar shortages, some countries, including Mexico and Bolivia in 1982 and Peru in 1985, have opted to ban dollarization. But if the proximate cause, monetary instability, is not eliminated, investors will demand significantly higher interest rates to hold domestic currency deposits, and some may simply take the money out of the country. Consistent with this, countries that today have significant restrictions on dollarization—such as Brazil, Colombia, and Venezuela—have particularly high loan spreads (see Inter-American Development Bank 2005).

Also, domestic currency depositors are not passive. With less-than-effective monetary authorities, banks could be subject to stress even if they only issue domestic deposits. For instance, suppose the authorities maintain an overvalued but fixed exchange rate. Fearing an eventual return to equilibrium, depositors have an incentive to withdraw and convert into foreign currency. This puts enormous stress on the banking system, forcing it to pay high interest rates to keep depositors in, with the level of interest rates being determined by the degree of overvaluation rather than more typical determinants like the return on investment and expected inflation. As described earlier, domestic currency depositors were prominent in the Argentinean bank runs in 2001.

The point is that dollarization is not necessarily an aberration in the environment that gives birth to it. Instead, it may be a reasonable adaptation. As Savastano (1996) and Baliño, Bennet, and Borensztein (1999) document, the consequence of banning dollarization in Mexico, Bolivia, and Peru was typically a severe contraction of intermediation that was reversed in Bolivia and Peru only when dollar deposits were allowed again. Similarly, Nicolo, Honohan, and Ize (2003) show that economies with high inflation tend to have more monetary depth with dollarization than without.

Rather than banning liability dollarization altogether, authorities may want to focus on removing the distortions that lead it to its excessive practice, such as the dollar-liability issuers not internalizing all the risks. More useful, of course, is to focus on changing the underlying conditions that lead to dollarization in the first place, a point we will touch on shortly.

Before concluding this section, note two points. First, the transition from an economy with liability dollarization to one where dollarization is banned implies either violating existing dollar contracts and prohibiting new ones, or shifting dollar liabilities to another domestic entity. The Brazilian government essentially took the latter route by taking on the dollar liabilities of its banking system—through the issuance of dollar-denominated bonds to banks in 1998. As a result, even though the Brazilian real depreciated substantially in 1998–99, the banks were relatively immunized. Of course, government debt ballooned as a result.

From a theoretical perspective, the government could improve welfare by taking on the dollar liabilities of the banking sector. When individual banks fail during a dollar shortage, we have seen that they can worsen the aggregate shortage. When the government takes over the liabilities of the banking sector, these individual failures are eliminated, so the dollar shortage need not be as severe. Against this, one should weigh the increased moral hazard if the government is expected to step in every time banks anticipate trouble.

Second, as argued above, with a fixed exchange rate and full convertibility, even domestic-currency-denominated liabilities may become a source of vulnerability. This suggests that the choice of exchange regime is not without consequence (also see, for example, Burnside, Eichenbaum, and Rebelo [2001b] or Edwards [2004]). But unfortunately, the very institutional requirements needed to maintain a monetary anchor with a floating exchange rate regime may be missing in countries where fixed exchange regimes create vulnerabilities. Therefore, there are trade-offs involved in the choice of exchange regime, and as suggested by Calvo and Mishkin (2003), it may be more useful to focus on changing the underlying institutions rather than on choosing a specific regime.

#### *3.2.3 Institutional Reform*

The root cause of deposit dollarization, we have argued, is weak basic institutions for conflict management. The more proximate causes are inadequate fiscal and monetary institutions. Of course, it is easier—though not easy—to reform the narrow institutions rather than the basic ones. But without reforming the deeper basic institutions, which typically requires deep-rooted political change, how successful can reform of narrow institutions be? We do not know much about the process of institutional reform; countries like Chile, Mexico, and South Korea that have improved their basic institutions over a relatively short time, aided by good policies and rapid economic growth, offer only a few successful examples. Understanding what ingredients in this mix are essential, and what simply are coincidental, is a topic of ongoing research on which, hopefully, researchers will have more to say in the near future. For now, let us turn to the role the international financial institutions can play.

### *3.3 What Can IFIs Do?*

Clearly, the international financial institutions (IFIs) can provide the technical support that will help countries adopt good policies and improve their narrow institutions (such as their fiscal framework or their inflation-targeting framework). They can also provide the bilateral and multilateral economic surveillance that can alert countries to possible sources of shocks. The International Monetary Fund does all this. The million-dollar question, of course, is whether IFIs should lend in such situations.

#### *3.3.1 "Liquidity" Loans*

A dollar shortage seems precisely the kind of temporary need that certain IFIs like the International Monetary Fund were set up to meet. By creating a common reserve pool of dollars, the IFIs can substitute for costly reserve hoarding by countries.

The most persuasive case for lending is when the IFI alleviates what is essentially a market-driven short squeeze on the country. It tides the country over its temporary exchange shortage, preventing more destructive domestic sector real adjustment, and gets repaid once the reasons for the temporary need vanish (e.g., exports recover).

The difficulty, of course, even with this simple scenario, is that the ultimate cause for a dollar shortage has to be that the country loses access to international markets. Thus the IFI has to make the judgment call of whether the loss of access is due to irrational/rational uncoordinated behavior by market participants, or whether it stems from genuine fears. If the former is the case, most would agree that the IFI should act as a lender of last resort. The only remaining concern would be whether this role creates bad incentives for market participants, for the government, and for banks—the issue of moral hazard, to which we will come in a moment.

If, however, the adverse shock precipitating the dollar shortage reflects a genuine institutional infirmity in the country—for instance, that the government has no fiscal discipline, it has reached borrowing limits, and thus it is shut off from international capital markets—matters become more difficult. It may well be that the country could undertake reforms that would help it regain access. In this case, the country is illiquid but solvent contingent on undertaking reforms. Solvency, however, will not be restored until the markets gain confidence that the reforms are irreversible. This implies that the lending may well not be temporary.

If the alternative is a banking system crisis coupled with a burden to restructure public debt, both of which will set back the country's economy considerably, it may well make sense to lend even when reforms are highly probable but not fully assured. The IFI bears some risk here that it will not be repaid, but it does so in the larger interest of the member country facing distress (and it should impose conditionality as well as charge an adequate premium for the risk).

The problem critics have is with the assumption that the IFI has a better ability to gauge willingness to reform than market participants. Two arguments have been put forward to justify this. First, the IFI may have better information about the country. This may have been true in the past, but given the development of financial markets, we see little reason to believe it to be true today. Second, the IFI may have a better sense of its own ability (and willingness) to coax the reform process forward, and may in fact have to show some success (or put its money at stake) before the market is persuaded. The IFI may also be able to put in place incentives for the country to reform. We find the second argument more persuasive, but one should not rule out the possibility that the IFI has an incentive to find a role for itself where none exists (see below).

A final situation where IFI lending may be warranted is when the country's public debt is too high given its underlying fundamentals, so it cannot borrow, but, as a result, it also faces an immediate dollar shortage that threatens its banking system. Rather than stand back and watch the banking system implode, the IFI may want to offer a bridge loan targeted at the banking system, to be repaid when the country regains market access after restructuring its external public debt. Again, this is a form of liquidity lending but one compounded by the problem that the excessive public debt will prolong the eventual resolution.

All this, however, raises two questions. First, does IFI intervention distort incentives among participants? Second, are there better ways to provide assurance of liquidity support to member countries?

#### *3.3.2 Incentive Distortion and Tough Love*

At least three types of incentive distortions are possible: (1) an unwillingness on the part of countries to take adequate precautions or to avoid excessively risky situations, (2) an unwillingness on the part of investors to take all risks into account, knowing they will be "bailed out," and (3) an unwillingness on the part of domestic corporations and banks to insure themselves adequately.

Reams and reams have been written on the issue of moral hazard, and we have little to add. Some argue that country moral hazard is not an issue because finance ministers and central bank governors lose their jobs in a financial crisis. Others argue that investor moral hazard is not a problem because investors lose their shirts in a crisis. These arguments are reasonable but miss the point. No finance minister will take an action that is certain to create a crisis. At the margin, however, concerned about budget deficits, a finance minister may prefer to borrow cheaply in dollars than borrow more expensively and for a longer term in domestic currency—especially if the IFI is there to help if things go wrong. At the margin, interventions do distort incentives to take risk.

The question is, how much? Unfortunately, the empirical evidence does not offer a reasonable assessment of magnitudes (see Jeanne and Zettelmeyer [2004] for an excellent exposition of the issues). Our reading of the current consensus is that country and investor moral hazard is small in most situations; in a few cases, though, it could be really big. We need more research identifying the circumstances where moral hazard is really a problem.

What seems clearer is that domestic corporations and banks may have too little incentive to prepare themselves for possible shocks, knowing that there are ways they can force the system to share that risk with them. This, however, is a case for better domestic regulation and supervision rather than limiting IFI intervention.

In sum, then, the moral hazard rationale against IFI intervention may well exist in some cases, but we need to be able to identify those cases better. If these cases are indeed few in number, as a reasonable judgment would suggest, then it may well make sense to accept the risks of inducing moral hazard through intervention while trying harder to identify when it is a mistake.

If, however, the reasons for dollarization lie primarily in poor institutions rather than in gaming—a collective action problem rather than an incentive problem—the greater concern should not be about distorting individual incentives but about altering collective actions. Sometimes external discipline forces a country to reform in ways, and at a speed, that the domestic constellation of political forces will simply not allow, if left to its own devices. Put another way, bailouts may help governments shift the burden of a crisis off the shoulders of the domestic business elite and onto domestic taxpayers (see Jeanne and Zettelmeyer 2004). To the extent that the latter do not have an adequate voice, they bear the brunt of excessive intervention, and everyone else who has a voice is willing to go along. The knowledge that bailouts may be hard to come by—a policy of "tough love"—could pressure domestic forces to compromise and effect much-needed reform.

We simply do not understand the political economy of deep institutional reform, or of crisis, well enough to offer a categorical answer on whether external financial support is a good thing or a bad thing on net. Clearly, if there was an assurance that the pain would be short and borne by those best able to absorb it, that the country would undertake genuine reforms, and that the future would be much brighter, "tough love" is certainly an argument worth considering. But what if the pain is prolonged, the economy degenerates into warring factions, and much of the pain is borne by weaker sections of society? Again, further research is needed here. What seems unquestionable is that if this route is chosen, there is a need to apply steady external pressure long before a crisis, even conditioning the extent of crisis assistance on past willingness to reform or maintain good policies.

#### *3.3.3 A Better Way to Intervene?*

IFIs like the International Monetary Fund typically agree to lend large amounts only when the member country is experiencing conditions of distress. Since intervention, let alone adequate assistance, is not assured, and the political considerations of large shareholders as well as the economic situation of the member country can affect these decisions, countries face uncertainty—which reduces the effectiveness of intervention in warding off the crisis. Moreover, countries fear that they will be forced to accept unwarranted conditionality even if assistance is forthcoming, because they really have no alternatives in a moment of crisis. These are understandable concerns: countries with a strong policy regime seem to want insurance, not uncertain loans laden with further uncertainty about conditions.

There is a second problem with leaving assistance to the complete discretion of the IFI. As we have argued, it is hard to tell, even in the midst of a crisis, whether the underlying factors are temporary or more structural, i.e., liquidity or solvency. The facts that emerge are unlikely to help decision making. In such a case, discretion can be harmful, as it exposes the IFI to internal and external political pressures to intervene. Not only will the decision be biased as a result of discretion, it will also be noisy, as it will not be based on underlying fundamentals.

One way to change this is through rules. For instance, access to IFI lending could be tied to a country's policies and reforms in normal times, as suggested by Jeanne and Zettelmeyer (2004). If a country follows sound policies and undertakes needed reforms, there should be a presumption that if and when it faces a crisis, it is likely to be a liquidity crisis, or a solvency problem (such as a permanent terms of trade shock) that is not of its own making. The IFI should intervene in the former, and will be providing insurance in the latter case (with the country then making needed adjustments on its own accord), not an entirely bad use of IFI resources.

These access limits could be set in the regular annual consultations between the IFI and a member country, where they would be based on an in-depth analysis of the country's policies. To the extent that a country's policy environment changes significantly, interim assessments could also be undertaken. The assessments will be a clear signal to the market about the IFI's view of a country's policies, putting steady pressure outside normal IFI programs on the country to stay the course of reforms (a "nonborrowing" program), and putting more pressure on IFI staff to do a good analysis because inadequate assessments will be contested.

Two immediate concerns arise. First, if the IFI is to intervene successfully in a liquidity crisis, it usually makes sense to pump in enough funds to stop the panic. Smaller amounts may not do the job and access limits, if set too low, may inhibit successful intervention. While no one may be able to determine whether a crisis is one of liquidity or solvency, IFI staff can certainly judge how much is needed from the facts of the crisis. So would not the rules on access be overly constraining?

The answer, of course, is yes, but that is the point. To the extent that the access limit is deemed insufficient for the crisis at hand, the IFI will have to convince bilateral or private parties to join, which will limit excessive intervention. Otherwise, it will have to stay out. Thus the access limit will effectively translate into a probability of intervention.

An alternative, though, is to link the probability of intervention, but not the quantum of assistance, to the country's policies and reforms in normal times. For instance, for countries in good standing, a decision to help might need approval by only a minority of the board, while for countries in poor standing, it might need approval by a supermajority. A politically independent minority of the members could then block loans to countries that have not shown much ownership of reform policies in the past.

This then leads us to the second concern. Ex-ante conditionality, which is effectively what rules amount to, is more intrusive than anything the IFIs currently do—even members not under programs will be subject to greater scrutiny. Members will be rightly concerned about the kind of policies the IFI's staff will encourage and the possibility of political interference in setting access limits (or voting requirements). This implies that reforms to the governance of the IFIs, ensuring that they are seen as legitimate by all the members, will be critical to any such change in lending policies.

## 4. Conclusion

We examine liquidity or dollar shortages in dollarized economies in this paper and explore how they precipitate and exacerbate crisis. Unfortunately, the obvious solution—ban liability dollarization—may not be appropriate. Liability dollarization is a response to institutional infirmities. It will not diminish unless those infirmities are fixed. In the meantime, we have to, as Guillermo Calvo suggested, learn to "live with dollarization."

In particular, this means stepped-up regulation and supervision up front to ensure that dollarization does not become excessive. It also implies that the government has the responsibility to maintain a reasonable fiscal position so it does not crowd out liquidity, and to maintain adequate reserves. It means developing tools for crisis resolution that recognize the nature of the problem; a banking crisis driven by a dollar shortage has to be dealt with in a different way from a banking crisis driven by bad loans. IFIs can play a role in all this, but the precise way to circumscribe that role has to be worked out.

Finally, we have to pay more attention to deep-rooted institutional reform. Giving central banks more independence and adopting inflation-targeting frameworks are good steps, but if not accompanied by serious fiscal reform, they are unlikely to persuade the public to forego dollarization.[9] It may not be surprising that the level of dollarization has increased over the 1990s despite a fall in inflation, perhaps because monetary reforms still lack credibility. Fiscal reform itself may be difficult unless political reform creates better basic institutions for allocating burden sharing in the economy. This suggests much work needs to be done.

[9] We agree in many ways with the analysis of Goldstein and Turner (2004), who also focus on institutional reform as a way of dealing with dollarization. However, we think it will be more difficult than they seem to suggest.

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