The Natural Rate of Interest in a Non-linear DSGE Model*

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This paper investigates how and to what extent non-linearities, including the zero lower bound on the nominal interest rate, affect the estimate of the U.S. natural rate of interest in a dynamic stochastic general equilibrium model. The estimated natural rate in a non-linear model is substantially different from that in its linear counterpart after the global financial crisis because of the zero lower bound. Other non-linearities such as price and wage dispersion, from which a linear model abstracts, play a negligible role in identifying the natural rate.

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1. Introduction

The natural rate of interest—the equilibrium real interest rate that yields price stability (Wicksell 1898)—has been a key concept for monetary policy analysis. In particular, a modern New Keynesian framework relates the concept of the natural rate to intertemporally optimizing agents and makes it relevant for social welfare (Galí 2008; Woodford 2003). The level of the natural interest rate in this framework is a useful indicator for policymakers because it is a benchmark as to whether policy is too tight or too loose from a welfare perspective.[1] However, the natural rate is unobservable and must be estimated. Whereas the literature has developed various empirical methods to infer the natural rate, an increasing number of researchers have estimated the natural rate measures based on New Keynesian dynamic stochastic general equilibrium (DSGE) models.[2] Examples for the U.S. economy include Andrés, López-Salido, and Nelson (2009), Barsky, Justiniano, and Melosi (2014), Cúrdia (2015), Cúrdia et al. (2015), Del Negro et al. (2017), Edge, Kiley, and Laforte (2008), Justiniano and Primiceri (2010), and Neiss and Nelson (2003).

This paper estimates the natural rate of interest in the U.S. economy using a non-linear New Keynesian DSGE model with a zero lower bound (ZLB) constraint on the nominal interest rate and examines how and to what extent non-linearities affect the estimates of the natural rate and its driving forces. Whereas the previous studies estimate the DSGE-based natural rate only in a linear setting that abstracts from the ZLB, this paper is one of the first to

[1] Closing the gap between the actual real interest rate and the natural rate is not necessarily optimal in the economy where "divine coincidence" (Blanchard and Galí 2007) does not hold. However, Barsky, Justiniano, and Melosi (2014) demonstrate that, even in such a circumstance, a central bank would be able to stabilize both inflation and the welfare-relevant output gap to a considerable degree by tracking the natural rate using an estimated New Keynesian model.

[2] Another stream of the literature estimates the long-run natural interest rate based on semi-structural or reduced-form models. See, for instance, Holston, Laubach, and Williams (2017), Johannsen and Mertens (2021), Kiley (2015), Laubach and Williams (2003, 2016), Lubik and Matthes (2015), Pescatori and Turunen (2016), and Williams (2015).

estimate the natural rate in a fully non-linear and stochastic setting that incorporates the ZLB.[3]

Our analysis is motivated by the following two strands of literature. First, Fernández-Villaverde and Rubio-Ramírez (2005) and Fernández-Villaverde, Rubio-Ramírez, and Santos (2006) demonstrate that the level of likelihood and parameter estimates based on a linearized model can be significantly different from those based on the original non-linear model. The same may be true for the estimation of unobservable state variables, including the natural rate. If substantial differences in the estimates of the natural rate arise between linear and non-linear models, it will cast doubt on the common practice in which the natural rate is estimated based on a linear model. Over- or under-estimation of the natural rate would be misleading in evaluating the stance of monetary policy.

Second, the recent experience of the global financial crisis and the extremely low interest rate period that followed has led researchers to conduct empirical analyses based on non-linear DSGE models in order to take the ZLB into consideration. For instance, Gust et al. (2017) incorporate the ZLB into a medium-scale DSGE model similar to those developed by Christiano, Eichenbaum, and Evans (2005) and Smets and Wouters (2007), and estimate the model in a non-linear setting using U.S. macroeconomic time series. Plante, Richter, and Throckmorton (2018) and Richter and Throckmorton (2016a, 2016b) estimate a non-linear version of a prototypical New Keynesian model with the ZLB for the U.S. economy, and Iiboshi, Shintani, and Ueda (2022) estimate a similar model for the Japanese economy. Aruoba, Cuba-Borda, and Schorfheide (2018) consider Markov switching between the targeted-inflation and deflation regimes in a New Keynesian framework with the ZLB and estimate the probabilities of the U.S. and Japan having been in either the targeted-inflation or deflation regime using a non-linear filtering technique. The present paper contributes to this strand of the literature by focusing on the estimation of the natural rate.

In estimating the natural rate of interest, we follow a two-step approach. First, to parameterize the model, we estimate a piecewise

[3]A contemporaneous paper by Iiboshi, Shintani, and Ueda (2022), which evolved independently from our work, estimates a non-linear small-scale New Keynesian model for Japan and extracts the sequence of the natural rate.

linear version of the model, in which the ZLB constraint is imposed but all the equilibrium conditions are linearized, using the OccBin toolbox developed by Guerrieri and Iacoviello (2015) and the inversion filter following Guerrieri and Iacoviello (2017). Regarding this estimation strategy, Atkinson, Richter, and Throckmorton (2020) demonstrate that piecewise linear and fully non-linear approaches give rise to similar parameter estimates. Thus, the piecewise linear approach enables us not only to avoid a computational burden that would increase exponentially in the estimation of a fully non-linear model, but also to obtain reliable estimates of parameters.

Next, given the estimated parameters, we solve the model in a fully non-linear and stochastic setting with the ZLB and apply a non-linear filter to extract the sequence of the natural interest rate. The literature (e.g., Boneva, Braun, and Waki 2016; Fernández-Villaverde et al. 2015; Gavin et al. 2015; Gust et al. 2017; Nakata 2016, 2017; Ngo 2014; and Richter and Throckmorton 2016a) has emphasized the importance of considering non-linearities and related uncertainty effects in assessing the quantitative implications of New Keynesian models that include the ZLB. The natural rate estimated in the present paper takes these important features into account. Moreover, our analysis is based on an empirically richer DSGE model than the prototypical New Keynesian model. The model features habit persistence in consumption preferences, price and wage stickiness, backward-looking components in price and wage settings, and monetary policy smoothing.

The main results are summarized as follows. Comparing the estimated natural interest rate based on the non-linear model with that based on the linear counterpart, we find that the former is higher than the latter by up to 1.7 percent after the global financial crisis, when the nominal interest rate was constrained by the ZLB. This difference is non-negligible; if we relied on the linear model, we would over-estimate the tightness of monetary policy due to the ZLB to a substantial degree. The difference is ascribed to a contractionary effect arising from the ZLB, which is considered only in the non-linear model. Although such a contractionary effect lowers expected output and inflation, actual output and inflation are pegged to the corresponding observables in the filtering process. Then, shocks to aggregate demand must be identified upward in order to satisfy the household's intertemporal Euler equation that equates the marginal utility of consumption today and the expected discounted one in the future. As a consequence, the estimated natural rate increases in the non-linear setting. Although price and wage dispersion potentially affect the identification of shocks and the estimate of the natural rate, their effects turn out to be negligible.

These findings allure researchers to use a piecewise linear model because such a model is easier to solve than a fully non-linear model. In this regard, we demonstrate that the piecewise linear model can well replicate the natural interest rate based on the non-linear model in the aftermath of the global financial crisis, although it can slightly under-estimate the natural rate due to ignoring uncertainty at the ZLB.

The remainder of the paper proceeds as follows. Section 2 describes the model used in our analysis and a strategy for estimating the natural interest rate. Section 3 presents our empirical results. Section 4 is the conclusion.

2. Model and Estimation Strategy

This section begins by describing the model used in our analysis. In the model economy, there are households, perfectly competitive final-good firms, monopolistically competitive intermediate-good firms, and a central bank that faces the ZLB constraint on the nominal interest rate. To ensure a better fit to the macroeconomic time series, the model features habit persistence in consumption preferences, price and wage stickiness, backward-looking components in price and wage settings, and monetary policy smoothing.[4] In the model, the natural rate of interest is defined as the real interest rate that would prevail if prices and wages were fully flexible without any markup shocks.[5]

[4]Regarding modeling choice, we consider nominal wage rigidities as in Erceg, Henderson, and Levin (2000) as well as price rigidities to incorporate additional non-linearities through a wage dispersion term and its variability. A medium-scale model with capital accumulation could take more non-linearities into account, but we leave an analysis using such a larger-scale model for future work.

[5]This definition follows from Barsky, Justiniano, and Melosi (2014) and is the most commonly used in the literature that estimates the natural rate based on DSGE models. Cúrdia et al. (2015) estimate the efficient interest rate, which is

To obtain the estimates of the natural interest rate, we implement a two-step approach. First, we estimate a piecewise linear version of the model, in which the ZLB constraint is imposed but all the equilibrium conditions are linearized using the OccBin toolbox and a inversion filter. Next, given the estimated parameters, we solve the model in a fully non-linear and stochastic setting with the ZLB using a projection method and apply a particle filter to extract the sequence of the natural rate.

2.1 The Model

2.1.1 Households

Each household $h \in [0,1]$ consumes final goods $C_{h,t}$, supplies labor $l_{h,t} = \int_0^1 l_{f,h,t} df$ to intermediate-good firms $f \in [0,1]$, and purchases one-period riskless bonds $B_{h,t}$ so as to maximize the following utility function:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\prod_{k=1}^{t} d_k \right)^{-1} \left[\log \left(C_{h,t} - \gamma C_{t-1} \right) - \frac{l_{h,t}^{1+\eta}}{1+\eta} \right],$$

subject to the budget constraint

$$P_t C_{h,t} + B_{h,t} = W_{h,t}^n l_{h,t} + R_{t-1}^n B_{h,t-1} + T_{h,t},$$

where $\beta \in (0,1)$ is the subjective discount factor, $\gamma \in [0,1]$ is the degree of external habit persistence in consumption preferences (C_{t-1} is the aggregate consumption in period $t-1$), $\eta \geq 0$ is the inverse of the labor supply elasticity, P_t is the price of final goods, $W_{h,t}^n$ is the nominal wage for household h, R_t^n is the gross nominal interest rate, and $T_{h,t}$ is the sum of a lump-sum public transfer and profits received from firms. Following Christiano, Eichenbaum, and Rebelo (2011), a shock to the discount factor d_t affects the weight of the utility in period $t+1$ relative to the one in period t. In the present model, this shock is broadly interpreted as a shock to

defined as the real interest rate under flexible prices and perfect competition with zero markups.

aggregate demand. The log of the discount factor shock follows an AR(1) process,

$$\log d_t = \rho_d \log d_{t-1} + \varepsilon_{d,t}, \tag{1}$$

where $\rho_d \in [0, 1)$ is an autoregressive coefficient and $\varepsilon_{d,t}$ is a normally distributed innovation with mean zero and standard deviation σ_d. The first-order conditions for optimal decisions on consumption and bond-holding are identical among households, and therefore become

$$\Lambda_t = \frac{1}{C_t - \gamma C_{t-1}}, \tag{2}$$

$$\Lambda_t = \frac{\beta}{d_t} R_t \mathbb{E}_t \frac{\Lambda_{t+1}}{\Pi_{t+1}}, \tag{3}$$

where Λ_t is the marginal utility of consumption, and $\Pi_t = P_t/P_{t-1}$ denotes gross inflation.

2.1.2 Wage Setting

A labor packer collects differentiated labor $\{l_{f,h,t}\}$ from each household h and resells a labor package augmented by a constant elasticity of substitution (CES) aggregator $l_{f,t} = \left[\int_0^1 l_{f,h,t}^{(\theta_w-1)/\theta_w} dh\right]^{\theta_w/(\theta_w-1)}$ to intermediate-good firms indexed by f, where $\theta_w > 1$ represents the elasticity of substitution among labor varieties. Given the nominal wage for each household $W_{h,t}^n$, cost minimization yields a set of labor demand schedules $l_{f,h,t} = \left(W_{h,t}^n/W_t^n\right)^{-\theta_w} l_{f,t}$ and the aggregate wage index $W_t^n = \left(\int_0^1 {W_{h,t}^n}^{1-\theta_w} dh\right)^{1/(1-\theta_w)}$.

Given the demand for labor by the labor packers, labor unions representing each household h set nominal wages on a staggered basis, as in Erceg, Henderson, and Levin (2000). In each period, a fraction $1 - \xi_w \in (0, 1)$ of labor unions reoptimizes their nominal wages, whereas the remaining fraction ξ_w indexes nominal wages to the economy's trend growth γ_a and a weighted average of past inflation Π_{t-1} and steady-state inflation $\bar{\Pi}$. The labor unions that reoptimize their nominal wages in the current period then maximize expected utility as follows:

$$\mathbb{E}_t \sum_{j=0}^{\infty} \xi_w^j \beta^j \left(\prod_{k=1}^{j} d_k \right)^{-1} \times \left[\frac{\gamma_a^j W_{h,t}^n}{P_{t+j}} \prod_{k=1}^{j} \left(\Pi_{t+k-1}^{\iota_w} \bar{\Pi}^{1-\iota_w} \right) \Lambda_{h,t+j} l_{h,t+j} - \frac{l_{h,t+j}^{1+\eta}}{1+\eta} \right],$$

subject to the labor demand

$$l_{f,h,t+j} = \left[\frac{\gamma_a^j W_{h,t}^n}{W_{t+j}^n} \prod_{k=1}^{j} \left(\Pi_{t+k-1}^{\iota_w} \bar{\Pi}^{1-\iota_w} \right) \right]^{-\theta_w} l_{f,t+j},$$

where $l_{h,t} = \int_0^1 l_{f,h,t} df$ is the amount of labor supplied by each household h, and $\iota_w \in [0, 1)$ is the weight of wage indexation to past inflation relative to steady-state inflation. The first-order condition for the reoptimized wage $W_t^{n,o}$ is given by

$$\left(\frac{W_t^{n,o}}{W_t^n} \right)^{1+\eta\theta_w} = \frac{\theta_w}{\theta_w - 1} \times \frac{\mathbb{E}_t \sum_{j=0}^{\infty} \xi_w^j \beta^j \left(\prod_{k=1}^{j} d_k \right)^{-1} \left[\left(\prod_{k=1}^{j} \Pi_{t+k-1}^{\iota_w} \bar{\Pi}^{1-\iota_w} \frac{\gamma_a^j W_t^n}{W_{t+j}^n} \right)^{-(1+\eta)\theta_w} l_{d,t+j}^{1+\eta} \right]}{\mathbb{E}_t \sum_{j=0}^{\infty} \xi_w^j \beta^j \left(\prod_{k=1}^{j} d_k \right)^{-1} \left[\left(\prod_{k=1}^{j} \Pi_{t+k-1}^{\iota_w} \bar{\Pi}^{1-\iota_w} \frac{\gamma_a^j W_t^n}{W_{t+j}^n} \right)^{1-\theta_w} \Lambda_{t+j} \frac{W_{t+j}^n}{P_{t+j}} l_{d,t+j} \right]}, \tag{4}$$

where $l_{d,t} = \int_0^1 l_{f,t} df$ is the total labor demand. The aggregate nominal wage index $W_t^n = \left(\int_0^1 {W_{h,t}^n}^{1-\theta_w} dh \right)^{1/(1-\theta_w)}$ can be written as

$$W_t^n = \left[(1-\xi_w) \left(W_t^{n,o} \right)^{1-\theta_w} + \xi_w \left(\Pi_{t-1}^{\iota_w} \bar{\Pi}^{1-\iota_w} \gamma_a W_{t-1}^n \right)^{1-\theta_w} \right]^{\frac{1}{1-\theta_w}}. \tag{5}$$

2.1.3 Firms

The representative final-good firm produces output Y_t under perfect competition by choosing a combination of intermediate inputs

$\{Y_{f,t}\}$ so as to maximize profit $P_tY_t - \int_0^1 P_{f,t}Y_{f,t}df$, subject to a CES production technology $Y_t = \left[\int_0^1 Y_{f,t}^{(\theta_p-1)/\theta_p} df\right]^{\theta_p/(\theta_p-1)}$, where $P_{f,t}$ is the price of intermediate good f and $\theta_p > 1$ denotes the elasticity of substitution among the variety of intermediate goods. The first-order condition for profit maximization yields the final-good firm's demand for each intermediate good $Y_{f,t} = (P_{f,t}/P_t)^{-\theta_p} Y_t$ and the aggregate price index $P_t = \left(\int_0^1 P_{f,t}^{1-\theta_p} df\right)^{1/(1-\theta_p)}$.

Each intermediate-good firm f produces a differentiated good $Y_{f,t}$ under monopolistic competition by choosing a labor input $l_{f,t}$, given the real wage $W_t = W_t^n/P_t$, and subject to the production function

$$Y_{f,t} = A_t l_{f,t},$$

where A_t represents total factor productivity. The log of the productivity level follows a non-stationary stochastic process,

$$\log A_t = \log \gamma_a + \log A_{t-1} + a_t, \tag{6}$$

where $\log \gamma_a$ represents the steady-state growth rate of productivity and a_t is a shock to the productivity growth. The productivity shock follows an AR(1) process,

$$a_t = \rho_a a_{t-1} + \varepsilon_{a,t}, \tag{7}$$

where $\rho_a \in [0, 1)$ is an autoregressive coefficient and $\varepsilon_{a,t}$ is a normally distributed innovation with mean zero and standard deviation σ_a. Assuming the existence of a shock to real marginal cost z_t, which is interpreted as an inefficient cost-push shock, the first-order condition for cost minimization is given by[6]

$$MC_t = \frac{W_t}{A_t} z_t. \tag{8}$$

The log of the cost-push shock follows an AR(1) process,

$$\log z_t = \rho_z \log z_{t-1} + \varepsilon_{z,t}, \tag{9}$$

[6]The first-order condition also indicates that the real marginal cost MC_t is identical across the intermediate-good firms.

where $\rho_z \in [0,1)$ is an autoregressive coefficient and $\varepsilon_{z,t}$ is a normally distributed innovation with mean zero and standard deviation σ_z.

In the face of the final-good firm's demand and marginal cost, the intermediate-good firms set the prices of their products on a staggered basis, as in Calvo (1983). In each period, a fraction $1-\xi_p \in (0,1)$ of intermediate-good firms reoptimizes their prices, whereas the remaining fraction ξ_p indexes prices to a weighted average of past inflation Π_{t-1} and steady-state inflation $\bar{\Pi}$. The firms that reoptimize their prices in the current period then maximize expected profit as follows:

$$\mathbb{E}_t \sum_{j=0}^{\infty} \xi_p^j \beta^j \left(\prod_{k=1}^{j} d_k \right)^{-1} \frac{\Lambda_{t+j}}{\Lambda_t} \times \left[\frac{P_{f,t}}{P_{t+j}} \prod_{k=1}^{j} \left(\Pi_{t+k-1}^{\iota_p} \bar{\Pi}^{1-\iota_p} \right) - MC_{t+j} \right] Y_{f,t+j},$$

subject to the final-good firm's demand

$$Y_{f,t+j} = \left[\frac{P_{f,t}}{P_{t+j}} \prod_{k=1}^{j} \left(\Pi_{t+k-1}^{\iota_p} \bar{\Pi}^{1-\iota_p} \right) \right]^{-\theta_p} Y_{t+j},$$

where $\iota_p \in [0,1)$ denotes the weight of price indexation to past inflation relative to steady-state inflation. The first-order condition for the reoptimized price P_t^o is given by

$$\frac{P_t^o}{P_t} = \frac{\theta}{\theta-1} \times \frac{\mathbb{E}_t \sum_{j=0}^{\infty} \xi_p^j \beta^j \left(\prod_{k=1}^{j} d_k \right)^{-1} \frac{\Lambda_{t+j}}{\Lambda_t} \left[\left(\prod_{k=1}^{j} \left[\left(\frac{\Pi_{t+k-1}}{\bar{\Pi}} \right)^{\iota_p} \frac{\bar{\Pi}}{\Pi_{t+k}} \right] \right)^{-\theta_p} MC_{t+j} Y_{t+j} \right]}{\mathbb{E}_t \sum_{j=0}^{\infty} \xi_p^j \beta^j \left(\prod_{k=1}^{j} d_k \right)^{-1} \frac{\Lambda_{t+j}}{\Lambda_t} \left[\left(\prod_{k=1}^{j} \left[\left(\frac{\Pi_{t+k-1}}{\bar{\Pi}} \right)^{\iota_p} \frac{\bar{\Pi}}{\Pi_{t+k}} \right] \right)^{1-\theta_p} Y_{t+j} \right]}. \tag{10}$$

The final-good's price $P_t = \left(\int_0^1 P_{f,t}^{1-\theta_p} df \right)^{1/(1-\theta_p)}$ can be written as

$$P_t = \left[(1-\xi_p) \left(P_t^o \right)^{1-\theta_p} + \xi_p \left(\Pi_{t-1}^{\iota_p} \bar{\Pi}^{1-\iota_p} P_{t-1} \right)^{1-\theta_p} \right]^{\frac{1}{1-\theta_p}}. \tag{11}$$

2.1.4 Market Clearing Conditions

The final-good market clearing condition is

$$Y_t = C_t, \tag{12}$$

whereas the labor market clearing condition leads to

$$l_t = \frac{\Delta_{p,t}\Delta_{w,t}Y_t}{A_t}, \tag{13}$$

where $l_t = \int_0^1 \int_0^1 l_{f,h,t} df dh$ is the aggregate labor input, $\Delta_{p,t} = \int_0^1 (P_{f,t}/P_t)^{-\theta_p} df$ is price dispersion across the intermediate-good firms, and $\Delta_{w,t} = \int_0^1 \left(W^n_{h,t}/W^n_t\right)^{-\theta_w} dh$ is wage dispersion across the labor unions. Equation (13) can be rewritten in terms of $l_{d,t} = \int_0^1 l_{f,t} df$ as

$$l_{d,t} = \frac{\Delta_{p,t}Y_t}{A_t}. \tag{14}$$

In the present model, the price and wage dispersion evolve according to

$$\Delta_{p,t} = (1-\xi_p)\left(\frac{P^o_t}{P_t}\right)^{-\theta_p} + \xi_p\left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\theta_p}\left(\frac{\Pi_{t-1}}{\bar{\Pi}}\right)^{-\iota_p\theta_p}\Delta_{p,t-1}, \tag{15}$$

$$\begin{aligned}\Delta_{w,t} &= (1-\xi_w)\left(\frac{W^{n,o}_t}{W^n_t}\right)^{-\theta_w} \\ &\quad + \xi_w\left(\frac{\Pi_t W_t}{\bar{\Pi}\gamma_a W_{t-1}}\right)^{\theta_w}\left(\frac{\Pi_{t-1}}{\bar{\Pi}}\right)^{-\iota_w\theta_w}\Delta_{w,t-1}.\end{aligned} \tag{16}$$

2.1.5 Flexible Wage and Price Equilibrium

Natural output Y^*_t and the natural rate of interest R^*_t are defined as the levels that would prevail if both wages and prices were perfectly flexible with no cost-push shocks. Such a flexible wage and price equilibrium is obtained with $\xi_w = \xi_p = 0$, $W^n_{h,t} = W^n_t$, $P_{f,t} = P_t$, and $z_t = 1$ for all h, f, and t in the model above and is characterized by the following equations:

$$\left(Y_t^* - \gamma Y_{t-1}^*\right)\left(\frac{Y_t^*}{A_t}\right)^{\eta} = \mu A_t, \tag{17}$$

$$R_t^* = \frac{d_t}{\beta}\left(\mathbb{E}_t \frac{Y_t^* - \gamma Y_{t-1}^*}{Y_{t+1}^* - \gamma Y_t^*}\right)^{-1}, \tag{18}$$

where $\mu = \frac{\theta_w - 1}{\theta_w}\frac{\theta_p - 1}{\theta_p}$ is the product of price and wage markups. Thus, the law of motion for natural output Y_t^* is determined by (17), given the sequence of total factor productivity A_t. The natural rate of interest R_t^* is determined by (18), given the sequences of natural output Y_t^* and the discount factor shock d_t.

2.1.6 Central Bank

A monetary policy rule is specified as

$$R_t^n = \max[\widehat{R}_t^n, 1], \tag{19}$$

where

$$\widehat{R}_t^n = (\widehat{R}_{t-1}^n)^{\phi_r}\left[\bar{R}\bar{\Pi}\left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\phi_\pi}\left(\frac{Y_t}{A_t}\right)^{\phi_y}\left(\frac{Y_t}{\gamma_a Y_{t-1}}\right)^{\phi_{\Delta y}}\right]^{1-\phi_r}\exp(\varepsilon_{r,t}). \tag{20}$$

$\widehat{R}_t^n$ denotes the notional nominal interest rate that the central bank would set according to a Taylor (1993) type monetary policy rule in the absence of the ZLB constraint, where $\bar{R}$ is the steady-state gross real interest rate, $\phi_r \in [0, 1)$ is the policy-smoothing parameter, and $\phi_\pi \geq 0$, $\phi_y \geq 0$, and $\phi_{\Delta y} \geq 0$ are the degrees of the interest rate policy response to inflation, detrended output, and output growth, respectively. $\varepsilon_{r,t}$ is a monetary policy shock, which is normally distributed with mean zero and standard deviation σ_r. The max function in (19) constrains the nominal interest rate to be greater than or equal to zero. If $\widehat{R}_t^n > 1$, the ZLB constraint is not imposed, i.e., $R_t^n = \widehat{R}_t^n$. If $\widehat{R}_t^n \leq 1$, the ZLB is binding, i.e., $R_t^n = 1$.

2.1.7 Equilibrium

An equilibrium is given by the sequences $\{Y_t, C_t, \Lambda_t, W_t, W_t^n, W_t^{n,o}, l_t, l_{d,t}, MC_t, \Pi_t, P_t, P_t^o, \Delta_{p,t}, \Delta_{w,t}, Y_t^*, R_t^*, R_t^n, \widehat{R}_t^n, d_t, A_t, a_t, z_t\}_{t=0}^{\infty}$ satisfying the equilibrium conditions (1)–(20) and two definitional equations, $W_t = W_t^n/P_t$ and $\Pi_t = P_t/P_{t-1}$.

Because total factor productivity A_t is non-stationary, as specified by (6), we rewrite the equilibrium conditions in terms of stationary variables detrended by A_t, as follows: $y_t = Y_t/A_t$, $c_t = C_t/A_t$, $\lambda_t = \Lambda_t A_t$, $w_t = W_t/A_t$, $w_t^n = W_t^n/A_t$, $w_t^{n,o} = W_t^{n,o}/A_t$, $mc_t = MC_t/A_t$, and $y_t^* = Y_t^*/A_t$, so that we can derive a non-stochastic steady state for the detrended variables.

2.2 Estimation of Parameters

To parameterize the model, we estimate a piecewise linear version of the model, in which the ZLB constraint is imposed but all the equilibrium conditions are linearized. More specifically, such a piecewise linear model is solved using the OccBin toolbox developed by Guerrieri and Iacoviello (2015), and the likelihood function is evaluated with the inversion filter following Guerrieri and Iacoviello (2017). While it is computationally very intensive to estimate a fully non-linear model using a projection method and a particle filter, the piecewise linear approach considerably reduces the computational burden. Regarding estimation accuracy, Atkinson, Richter, and Throckmorton (2020) demonstrate that the piecewise linear and fully non-linear approaches result in similar parameter estimates.

The model is fitted to four U.S. quarterly time series: the per capita real GDP (gross domestic product) growth rate ($100\Delta \log GDP_t$), the inflation rate of the GDP implicit price deflator ($100\Delta \log PGDP_t$), the federal funds rate (FF_t), and the log of hours worked ($100 \log H_t$).[7] Following Wolters (2018), the data on hours worked are adjusted for low-frequency movements due to sectoral and demographic changes so that the data are consistent with the model. The sample period is from 1987:Q3 to 2019:Q4. The beginning of the sample period is set at the time when Alan Greenspan became the Chairman of the Board of Governors of the

[7]The series of hours worked is demeaned.

Federal Reserve System, because thereafter, the style of the Fed's policy conduct seems consistent and stable. The end of the sample is determined to exclude the COVID-19 pandemic period. The linearized equilibrium conditions and observation equations are presented in Appendix A.

The parameters are estimated using Bayesian methods. The prior distributions of parameters are presented in the second to fourth columns of Table 1. For most of the parameters, each prior mean is set at the corresponding prior mean used in the literature including Smets and Wouters (2007). The prior mean of the policy-smoothing parameter ϕ_r is set at 0.5, which is lower than that in Smets and Wouters (2007) because a higher value of the estimated ϕ_r would lead to a non-convergence problem in the next step for solving our non-linear model.[8] As for the steady-state values of output growth, inflation, and real interest rates and hours worked ($\bar{a}$, $\bar{\pi}$, $\bar{r}$, $\bar{l}$), the priors are centered at the sample mean. The prior mean of the AR(1) coefficient for the discount factor shock ρ_d is 0.75, whereas those for the productivity and cost-push shocks (ρ_a, ρ_z) are 0.5. For the standard deviations of the shocks ($100\sigma_d$, $100\sigma_a$, $100\sigma_z$, $100\sigma_r$), we assign inverse-gamma distributions with a mean of 0.5 and a standard deviation of 2.0.

Following Guerrieri and Iacoviello (2017), 50,000 posterior draws are generated using the random-walk Metropolis-Hastings algorithm, and the first 10,000 draws are discarded. The posterior mean and 90 percent credible interval for each parameter are reported in the middle columns of Table 1, labeled "Piecewise Linear." For the sake of comparison, we estimate a linear version of the model that omits the ZLB constraint, and the results are shown in the last two columns of Table 1. The parameter estimates are very similar to each other, in contrast to the finding of Hirose and Inoue (2016), who conduct a Monte Carlo analysis and demonstrate that omitting the ZLB in estimation causes biased estimates of parameters.

In the subsequent analysis, the parameters are fixed at the posterior mean estimates of the piecewise linear model.

[8]For the same reason, relatively tight priors are used for the parameters that determine the persistency of endogenous variables in the model.

Table 1. Prior and Posterior Distributions of Parameters

	Prior			Posterior			
				Piecewise Linear		Linear (No ZLB)	
Parameter	**Distribution**	**Mean**	**S.D.**	**Mean**	**90% Interval**	**Mean**	**90% Interval**
γ	Beta	0.500	0.050	0.664	[0.609, 0.716]	0.654	[0.597, 0.708]
η	Gamma	2.000	0.250	1.763	[1.372, 2.212]	1.824	[1.438, 2.249]
ξ_w	Beta	0.500	0.050	0.681	[0.561, 0.756]	0.687	[0.598, 0.752]
ι_w	Beta	0.500	0.050	0.488	[0.407, 0.568]	0.492	[0.409, 0.576]
ξ_p	Beta	0.500	0.050	0.868	[0.817, 0.913]	0.841	[0.792, 0.887]
ι_p	Beta	0.500	0.050	0.451	[0.360, 0.545]	0.490	[0.404, 0.579]
ϕ_π	Gamma	1.500	0.250	1.575	[1.353, 1.825]	1.608	[1.361, 1.876]
ϕ_y	Gamma	0.500	0.100	0.168	[0.143, 0.194]	0.141	[0.120, 0.164]
$\phi_{\Delta y}$	Gamma	0.125	0.050	0.238	[0.134, 0.350]	0.265	[0.155, 0.381]
ϕ_r	Beta	0.500	0.050	0.750	[0.706, 0.789]	0.803	[0.770, 0.836]
$\bar{a}$	Normal	0.356	0.100	0.348	[0.250, 0.452]	0.361	[0.267, 0.453]
$\bar{\pi}$	Normal	0.529	0.100	0.576	[0.489, 0.670]	0.536	[0.443, 0.631]
$\bar{r}$	Gamma	0.276	0.100	0.215	[0.146, 0.293]	0.240	[0.162, 0.322]
$\bar{l}$	Normal	0.000	0.100	–0.058	[–0.146, 0.037]	0.016	[–0.018, 0.058]
ρ_d	Beta	0.750	0.050	0.831	[0.794, 0.862]	0.851	[0.819, 0.881]
ρ_a	Beta	0.500	0.050	0.418	[0.352, 0.487]	0.412	[0.345, 0.477]
ρ_z	Beta	0.500	0.050	0.573	[0.481, 0.663]	0.610	[0.525, 0.697]
$100\sigma_d$	Inv. Gamma	0.500	2.000	0.404	[0.311, 0.523]	0.359	[0.274, 0.466]
$100\sigma_a$	Inv. Gamma	0.500	2.000	0.558	[0.502, 0.618]	0.559	[0.503, 0.622]
$100\sigma_z$	Inv. Gamma	0.500	2.000	7.612	[3.613, 15.18]	5.256	[2.870, 9.135]
$100\sigma_r$	Inv. Gamma	0.500	2.000	0.119	[0.104, 0.136]	0.107	[0.096, 0.121]

Note: Each posterior mean and 90 percent credible interval are calculated from 50,000 draws (the first 10,000 draws are discarded) generated using the Metropolis-Hastings algorithm.

2.3 Non-linear Solution and Filtering

Given the posterior mean estimates for the piecewise linear model obtained above, the model is solved in a fully non-linear and stochastic setting with the ZLB constraint using a projection method. The model has seven endogenous state variables (output y_{t-1}, inflation Π_{t-1}, the real wage w_{t-1}, the notional nominal interest rate $\widehat{R}^n_{t-1}$, price dispersion $\Delta_{p,t-1}$, wage dispersion $\Delta_{w,t-1}$, and natural output y^*_{t-1}) and four exogenous shocks (the discount factor shock d_t, the productivity shock a_t, the cost-push shock z_t, and the monetary policy shock $\varepsilon_{r,t}$). The policy functions satisfying the detrended equilibrium conditions can be written as

$$\mathbb{S}_t = g(\mathbb{S}_{t-1}, \tau_t),$$

where $\mathbb{S}_{t-1} = [y_{t-1}, \Pi_{t-1}, w_{t-1}, \widehat{R}^n_{t-1}, \Delta_{p,t-1}, \Delta_{w,t-1}, y^*_{t-1}]'$ and $\tau_t = [d_t, a_t, z_t, \varepsilon_{r,t}]'$.

To compute the policy functions, we employ a projection method with an index function approach as in Aruoba, Cuba-Borda, and Schorfheide (2018), Gust et al. (2017), and Nakata (2017). For interpolating the policy functions within a time iteration algorithm, we adapt a standard linear interpolation to reduce approximation errors that tend to be large particularly when the ZLB binds due to large negative shocks to the economy.[9] The details of the solution method are described in Appendix B.

According to an artificial sample of 40,000 periods simulated from the non-linear solution of the model, the economy is at the ZLB for 3.8 percent of quarters, and the average duration of ZLB spells is 3.4 quarters. These statistics are in line with the simulation results in the previous studies that employ non-linear New Keynesian models. Fernández-Villaverde et al. (2015) simulate a small-scale model calibrated for the U.S. economy and show that the economy spends 5.5 percent of quarters at the ZLB and that the average duration at the ZLB is 2.1 quarters. Gust et al. (2017) estimate a medium-scale model in a non-linear setting using U.S. data from 1983:Q1 to 2014:Q1, and the simulation of their estimated model demonstrates

[9] Because of the high dimensionality of state variables, we use a supercomputer system to utilize parallel computing.

that the economy is at the ZLB for about 4 percent of quarters on average and that the average duration of the ZLB spells is just over 3.5 quarters.

We apply a particle filter as in Fernández-Villaverde and Rubio-Ramírez (2007) to extract the sequence of the state variables and then compute the estimates of the natural interest rate.[10] The data used for filtering are the same as those used for the parameter estimation in Section 2.2. To facilitate the use of the particle filter, measurement errors are added in the observation equations. The size of the measurement errors of output growth, inflation, the nominal interest rate, and hours worked are, respectively, set to be 1 percent of the standard deviations of the data so that we can reduce the effect of measurement errors on the filtered estimates of the natural interest rate and its related variables. We use 100,000 particles and confirmed that any further increase in the number of particles delivered almost the same results as those presented below.

3. Results

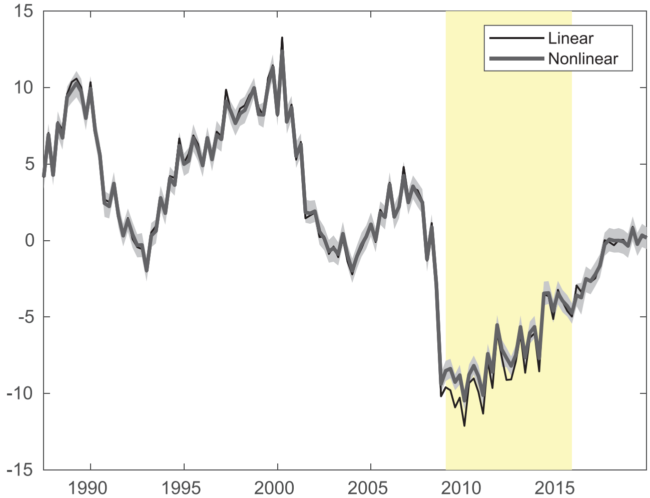
This section presents the estimate of the natural interest rate based on the non-linear model and compares it with that based on its linear counterpart. To understand the source of the difference between the two estimates, we investigate how the natural rate of interest is identified in each case. Moreover, we consider the piecewise linear model, which is used for estimating model parameters, to examine whether it can be a useful substitute for estimating the natural rate accurately.

3.1 Estimated Natural Rate of Interest

In Figure 1, the thick solid line shows the filtered mean estimate of the natural rate of interest on an annualized basis, based on the non-linear model. The gray shaded area is the 90 percent interval obtained from the distributions of particles in each period. The estimated natural rate measure peaked more than 10 percent at the end of 1980s and the beginning of 2000, then fell to about −10 percent in

[10] For a textbook treatment of a particle filter, see Fernández-Villaverde, Rubio-Ramírez, and Schorfheide (2016) or Herbst and Schorfheide (2015).

Figure 1. Natural Rate of Interest



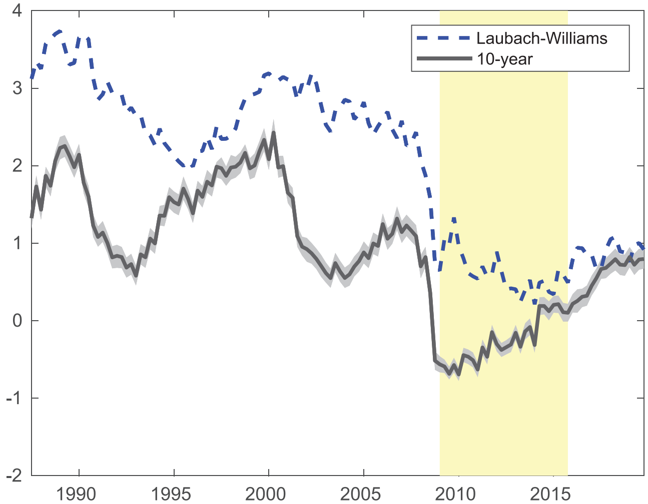
Note: The figure shows the filtered mean estimate of the natural interest rate, in annualized percentage terms, based on the non-linear model (thick solid line) with its 90 percent interval (shaded area) and that based on the linear model (thin solid line). The yellow area indicates the period when the federal funds rate was constrained by the ZLB. (For figures in color, see the online version of the paper at http://www.ijcb.org.)

the aftermath of the global financial crisis, and thereafter increased to slightly positive values toward the end of the sample period.[11] The overall cyclical movements and variability of the natural rate are similar to those estimated by Barsky, Justiniano, and Melosi (2014), who employ a medium-scale New Keynesian DSGE model with capital accumulation.

In the present framework, the natural interest rate is driven by structural shocks and hence exhibits short-run fluctuations. To make it comparable to the estimates of the long-run natural rate based on a semi-structural or reduced-form model as in Laubach and Williams

[11]In an earlier version of this paper (Hirose and Sunakawa 2017), the estimates of the natural rate were too volatile, ranging from −15 to 15 percent. This issue was resolved by employing the hours data constructed by Wolters (2018). As mentioned in Section 2.2, the data are adjusted for sectoral and demographic changes, and hence they are less volatile than the original data.

Figure 2. Long-Run Natural Rate of Interest



Note: The figure shows the filtered mean estimate of the 10-year natural interest rate, in annualized percentage terms, based on the non-linear model (thick solid line) with its 90 percent interval (shaded area) and the filtered estimate of the natural interest rate based on the Laubach-Williams (2003) model (dashed line). The yellow area indicates the period when the federal funds rate was constrained by the ZLB.

(2003), we calculate the 10-year natural rate R_t^{*10Y} according to the expectation hypothesis of the term structure of interest rates,

$$\log R_t^{*10Y} = \frac{1}{40} \sum_{i=0}^{39} \mathbb{E}_t \log R_{t+i}^*.$$

Figure 2 compares the filtered mean estimate of the 10-year natural rate based on the non-linear model (solid line) with the filtered estimate of the natural rate based on the Laubach-Williams (2003) model (dashed line, henceforth, the LW estimate).[12] Two notable differences emerge. First, cyclical patterns are somewhat different from each other. Second, our estimate of the 10-year natural rate is

[12]The latter estimate is available at the website of the Federal Reserve Bank of New York.

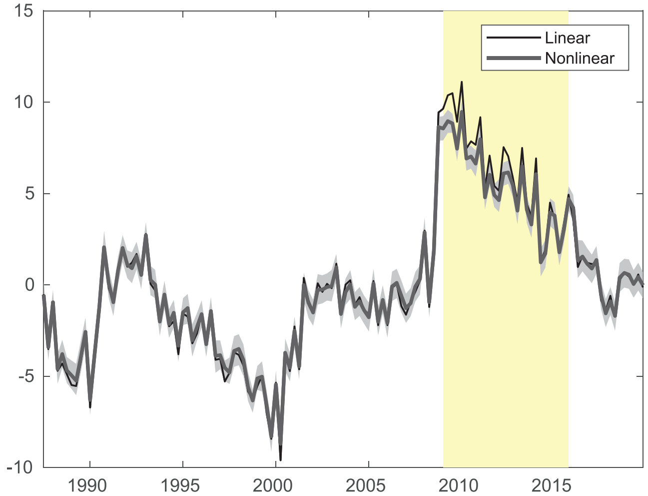
lower than the LW estimate throughout the sample period, whereas these two estimates are close to each other in the last few years. The second difference arises because of the difference in the estimation sample. The sample for the LW estimate begins in 1961:Q1, and hence the steady-state real interest rate in their model is estimated to be higher than that in our model.

The primary objective of this paper is to examine how and to what extent non-linearities, including the ZLB, affect the estimates of the natural interest rate. To this end, we estimate the natural rate using a linear counterpart of the model, as in the previous studies, and compare it with the one obtained above. The thin solid line in Figure 1 shows the filtered estimate of the natural interest rate based on the linearized version of the model with the same parameters and data set as used in the non-linear case.[13] The figure indicates that the natural rate based on the linear model is mostly very similar to that based on the non-linear model, except for the period after 2009, when the actual nominal interest rate was constrained by the ZLB. The difference amounts to 1.7 percent during 2009 and remains substantial thereafter.

This difference matters when we evaluate the Fed's monetary policy stance after the global financial crisis. In the present framework, the natural interest rate is a benchmark for whether monetary policy is too tight or too loose from a welfare perspective. Such a policy stance is measured by the interest rate gap—the difference between the actual (ex post) real interest rate and the natural rate. A larger value of the interest rate gap indicates a tighter policy stance. As shown in Figure 3, both measures of the interest rate gap were around zero from 2002 to 2008, but thereafter jumped to about 10 percent. More precisely, while the gap in the non-linear model increased to 9.5 percent in 2010:Q1, its linear counterpart increased to 11.1 percent. These jumps are due to the existence of the ZLB; that is, the Fed was not able to lower the policy rate any

[13] In the linear setting, the model is solved by the standard linear solution method and the filtered estimates of model variables are computed using the Kalman filter with zero measurement errors. As a preliminary analysis, we applied a particle filter to the linear model and obtained a very similar estimate of the natural rate. Thus, we confirmed that the difference in the filtering methods with and without measurement errors is not the major source of the difference in the estimated natural rates between the linear and non-linear models.

Figure 3. Interest Rate Gap



Note: The figure shows the real interest rate gap, in annualized percentage terms, based on the non-linear model (thick solid line) with its 90 percent interval (shaded area) and that based on the linear model (thin solid line). The yellow area indicates the period when the federal funds rate was constrained by the ZLB.

more in response to the severe economic downturn, and hence the policy stance was evaluated to be too tight. In this context, if we relied on the linear model, the estimated natural rate and the resulting interest rate gap would over-estimate the tightness of the policy stance by nearly 17 percent ($\fallingdotseq (11.1 - 9.5)/9.5$).

3.2 What Accounts for the Difference?

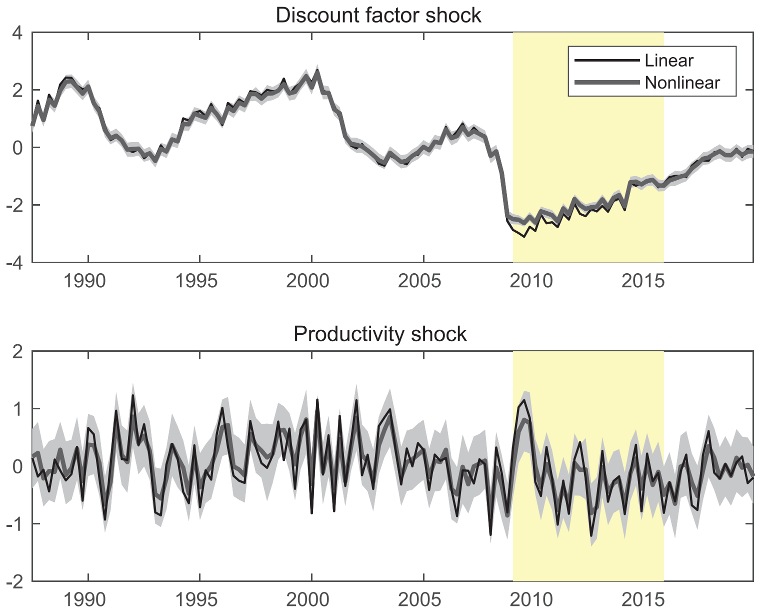
To understand what causes the difference between the two estimates of the natural interest rate, we consider how the natural rate is identified in each case. As addressed in Section 2.1, Equation (17), i.e., $\left(Y_t^* - \gamma Y_{t-1}^*\right)(Y_t^*/A_t)^\eta = \mu A_t$, determines natural output Y_t^*, given the sequence of total factor productivity A_t (or, equivalently, the productivity shock a_t). The natural rate R_t^* can be traced out from Equation (18), i.e., $R_t^* = d_t/\beta\left[\mathbb{E}_t(Y_t^* - \gamma Y_{t-1}^*)/(Y_{t+1}^* - \gamma Y_t^*)\right]^{-1}$, given the sequences of natural output Y_t^* and the discount factor shock d_t. Thus, the natural rate is pinned down by identifying the two shocks, a_t and d_t.

In the linear model, the productivity shock a_t is explicitly identified by the data on output and hours worked because detrending and log-linearizing the labor market clearing condition (13) yields $\tilde{y}_t = \tilde{l}_t$ and because the associated observation equations are $100\Delta \log GDP_t = \bar{a} + \tilde{y}_t - \tilde{y}_{t-1} + a_t$ and $100 \log H_t = \bar{l} + \tilde{l}_t$, where $\bar{a}$ and $\bar{l}$ are the steady-state growth rate and hours worked, respectively, and the variables with ˜ represent percentage deviations from their steady-state values. In the non-linear model, however, Equation (13) contains the price and wage dispersion, $\Delta_{p,t}$ and $\Delta_{w,t}$, and can be written as $y_t = l_t/(\Delta_{p,t}\Delta_{w,t})$ in detrended terms. These dispersion terms fluctuate so that $\Delta_{p,t} \geq 1$ and $\Delta_{w,t} \geq 1$, as the price and wage deviate from the steady state. Thus, y_t becomes lower than in the linear case where the dispersion terms are suppressed. Consequently, to satisfy the observation equation for output growth, a_t is identified to be larger in the non-linear case. From Equations (17) and (18), higher productivity raises natural output and results in the higher estimate of the natural rate.

Identification of the discount factor shock d_t is more complicated and influenced by the whole structure of the model. However, taking into account the finding that the two estimates of the natural interest rate differ from each other during the period when the nominal interest rate was bounded at zero, the existence of the ZLB, from which the linear model abstracts, possibly affects the identification of d_t in the non-linear model. The literature has established that the ZLB has a contractionary effect on the economy not only when the nominal interest is already binding at zero, but also when uncertainty exists about whether the ZLB will bind in the future.[14] Although such a contractionary effect lowers expected output and inflation, the filtering procedure pegs contemporaneous output and inflation to the corresponding observables, which are the same in the linear and non-linear cases. Then, in the non-linear case, the discount factor shock d_t must increase to satisfy the consumption Euler equation that equalize the marginal utility of consumption

[14]Hills, Nakata, and Schmidt (2016) quantify such an uncertainty effect on inflation in the face of the interest rate lower bound.

Figure 4. Estimated Shocks

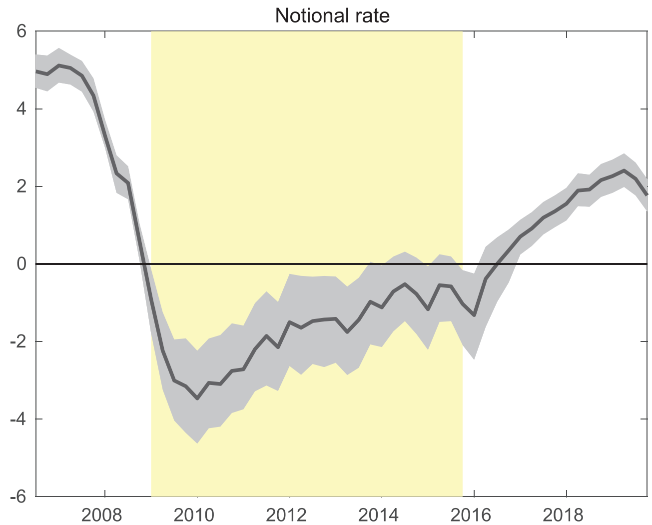


Note: The figure shows the filtered mean estimates of the discount factor shocks d_t and the productivity shocks a_t, in percentage terms, based on the non-linear model (thick solid lines) with their 90 percent intervals (shaded areas) and those based on the linear model (thin solid lines). The yellow areas indicate the period when the federal funds rate was constrained by the ZLB.

today with the expected discounted one in the future. As a result, the estimated natural rate becomes higher.

To quantify the differences in the sequences of identified shocks, Figure 4 shows the filtered mean estimates of the discount factor shocks d_t and the productivity shocks a_t, in percentage terms, based on the non-linear model (thick solid lines) and its linear counterpart (thin solid lines). The gray shaded areas are the 90 percent intervals for the filtered estimates in the non-linear model. The sequence of d_t identified in the non-linear model is remarkably different from that in the linear model after the global financial crisis. In particular, the difference is pronounced in 2009–10, when the notional nominal interest rate $\widehat{R}_t^n$ sharply fell below zero, as shown in Figure 5. Thus, whether the ZLB constraint is imposed or not substantially affects the identification of d_t.

Figure 5. Notional Rate



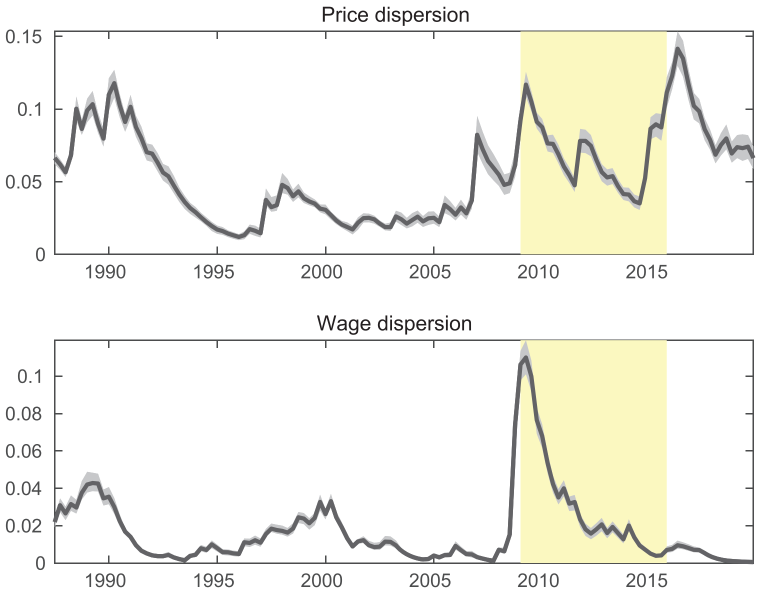
Note: The figure shows the filtered mean estimate of the notional nominal interest rate and its 90 percent interval (shaded area), in annualized percentage terms. The yellow area indicates the period when the federal funds rate was constrained by the ZLB.

On the other hand, the movements of the productivity shocks a_t are very similar between the two estimates, although temporary deviations are found occasionally. The finding of the small difference in a_t implies that the price and wage dispersion terms, $\Delta_{p,t}$ and $\Delta_{w,t}$, play a minor role in the non-linear model. Indeed, as shown in Figure 6, the estimates of $\Delta_{p,t}$ and $\Delta_{w,t}$ based on the non-linear model fluctuate little, i.e., 0.14 percent at most, even though they exhibit cyclical movements over the sample period.

3.3 The Natural Rate of Interest Based on the Piecewise Linear Model

The analysis thus far suggests that the existence of the ZLB constraint plays a crucial role in identifying the natural interest rate in a non-linear setting, but that the role of price and wage dispersion is negligible. These findings tempt researchers to exploit a piecewise linear model, in which the ZLB constraint is imposed but

Figure 6. Price and Wage Dispersion



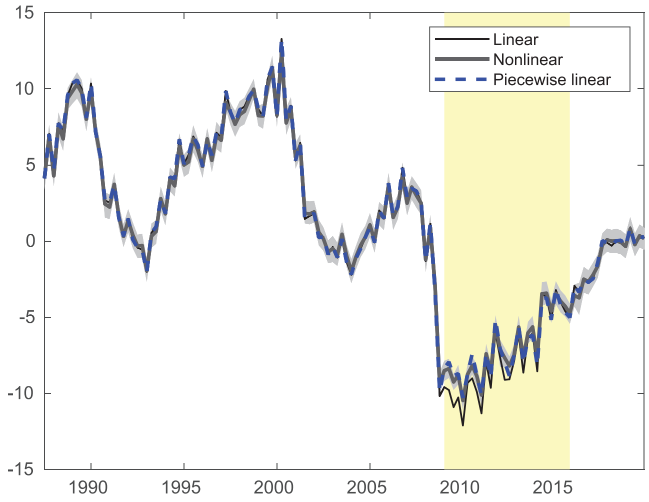
Note: The figure shows the filtered mean estimates of the price and wage dispersion and their 90 percent intervals (shaded areas) in terms of percentage deviation from the steady state. The yellow areas indicate the period when the federal funds rate was constrained by the ZLB.

all the equilibrium conditions are linearized, for estimating the natural rate measures, because such a model is easier to solve than a fully non-linear model. While the literature argues that the solution for this sort of piecewise linear models can give rise to an inaccurate assessment of the ZLB,[15] Atkinson, Richter, and Throckmorton (2020) demonstrate that the piecewise linear and fully non-linear approaches lead to similar results with regard to parameter estimates. In what follows, we examine whether the piecewise linear model can be a useful substitute for estimating the natural rate.

As in the estimation of parameters described in Section 2.2, the piecewise linear version of the model is solved using the OccBin toolbox, and the filtered estimates are computed using the inversion

[15]See Boneva, Braun, and Waki (2016), Fernández-Villaverde et al. (2015), Gavin et al. (2015), Gust et al. (2017), Nakata (2016, 2017), Ngo (2014), and Richter and Throckmorton (2016a).

Figure 7. Natural Rate of Interest: Comparison with Piecewise Linear Model



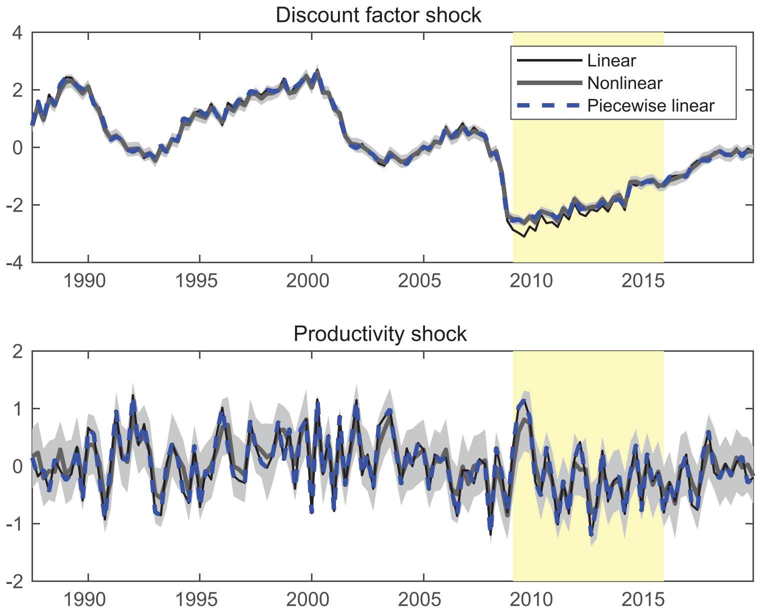
Note: The figure shows the filtered mean estimate of the natural interest rate, in annualized percentage terms, based on the non-linear model (thick solid line) with its 90 percent interval (shaded area) and those based on the linear model (thin solid line) and the piecewise linear model (dashed line). The yellow area indicates the period when the federal funds rate was constrained by the ZLB.

filter.[16] Figure 7 depicts the filtered estimate of the natural interest rate in the piecewise linear setting (dashed line) along with the estimates in the fully non-linear (thick solid line) and linear (thin solid line) settings. The estimate based on the piecewise linear model coincides with that based on the linear model before the global financial crisis, but in the aftermath, it is very close to that based on the fully non-linear model. Thus, the higher estimate of the natural rate in the non-linear model from 2009 to 2011 is well replicated by the piecewise linear model.

However, the natural rate in the piecewise linear model moves closer to that in the linear model after 2012, when the notional rate shown in Figure 5 approached toward zero. This is because the OccBin solution, by its construction, does not take into account uncertainty about the future. An uncertainty effect enhances the

[16]To run the inversion filter, the measurement errors are set to zero as in the Kalman filter for the linear model.

Figure 8. Estimated Shocks: Comparison with Piecewise Linear Model



Note: The figure shows the filtered mean estimates of the discount factor shocks d_t and the productivity shocks a_t, in percentage terms, based on the non-linear model (thick solid lines) with their 90 percent intervals (shaded areas) and those based on the linear model (thin solid lines) and the piecewise linear model (dashed lines). The yellow areas indicate the period when the federal funds rate was constrained by the ZLB.

contractionary effect of the ZLB because agents expect that monetary policy will be unable to accommodate negative shocks to the economy. As addressed in the previous subsection, the contractionary effect of the ZLB leads to the higher estimate of the natural rate in the non-linear model. This effect is reduced in the absence of uncertainty, and hence the piecewise linear model slightly under-estimates the natural rate.[17]

These mechanisms are confirmed by the upper panel in Figure 8. The estimated series of the discount factor shock d_t in the piecewise

[17]If the probability of binding at the ZLB were higher, the uncertainty effect would be larger, which would result in a larger difference in the natural rate between the non-linear and piecewise linear models.

linear model (dashed line) is substantially larger than that in the linear model (thin solid line) for a few years after the global financial crisis, whereas it is smaller to a lesser extent than that in the non-linear model (thick solid line) until the end of the sample period.

The lower panel in Figure 8 compares the estimates of the productivity shock a_t. The piecewise linear model abstracts from price and wage dispersion, as does the linear model, and hence the estimated productivity shocks in the two models exactly coincide with each other.

In summary, a piecewise linear model incorporating the ZLB can be a possible substitute for a fully non-linear model in estimating the natural rate measures, although it can slightly under-estimate the natural rate due to ignoring uncertainty at the ZLB.

4. Concluding Remarks

This paper has estimated the natural rate of interest in a non-linear New Keynesian model using U.S. macroeconomic data and compared it with the one estimated with the model's linear counterpart. We have found that the natural rate based on the non-linear model is substantially higher than that based on the linear model during the period when the nominal interest rate was bounded at zero. This difference is explained by a contractionary effect of the ZLB, which is omitted in the linear model. Although the existence of the price and wage dispersion terms potentially affects the estimate of the natural rate in the non-linear setting, their effects are negligible.

Whereas the present paper employs an empirically richer DSGE model than the prototypical New Keynesian model, existing studies, including Barsky, Justiniano, and Melosi (2014), Cúrdia et al. (2015), Del Negro et al. (2017), Edge, Kiley, and Laforte (2008), and Justiniano and Primiceri (2010), estimate the natural interest rate using medium-scale DSGE models with capital accumulation in a linear setting. Our analysis could be extended to exploit such a medium-scale model so that the estimated natural rate would be comparable to the rates obtained in these studies. We conjecture that our results regarding the higher estimate of the natural rate in a non-linear setting would still hold, even if we extended our model to a larger scale, because the main mechanism through which the ZLB can affect the identification of the natural rate remains unchanged.

Appendix A. Linearized Equilibrium Conditions and Observation Equations

Log-linearizing the detrended equilibrium conditions around the non-stochastic steady state, and rearranging the resulting equations, yields

$$\begin{aligned}
\tilde{y}_t &= \frac{\gamma_a}{\gamma_a+\gamma}(\mathbb{E}_t\tilde{y}_{t+1}+\mathbb{E}_t a_{t+1})+\frac{\gamma}{\gamma_a+\gamma}(\tilde{y}_{t-1}-a_t) \\
&\quad -\frac{\gamma_a-\gamma}{\gamma_a+\gamma}(\tilde{R}^n_t-\mathbb{E}_t\tilde{\Pi}_{t+1}-\tilde{d}_t), \\
\tilde{w}_t &= \tilde{w}_{t-1}-\tilde{\Pi}_t+\iota_w\tilde{\Pi}_{t-1}-a_t \\
&\quad +\beta(\mathbb{E}_t\tilde{w}_{t+1}-\tilde{w}_t+\mathbb{E}_t\tilde{\Pi}_{t+1}-\iota_w\tilde{\Pi}_t+\mathbb{E}_t a_{t+1}) \\
&\quad +\frac{(1-\xi_w)(1-\xi_w\beta)}{\xi_w(1+\eta\theta_w)}\left[\eta\tilde{l}_t+\frac{1}{\gamma_a+\gamma}(\gamma_a\tilde{y}_t-\gamma\tilde{y}_{t-1}+\gamma a_t)-\tilde{w}_t\right], \\
\tilde{y}_t &= \tilde{l}_t, \\
\tilde{\Pi}_t &= \frac{\beta}{1+\beta\iota_p}\mathbb{E}_t\tilde{\Pi}_{t+1}+\frac{\iota_p}{1+\beta\iota_p}\tilde{\Pi}_{t-1}+\frac{(1-\xi_p)(1-\xi_p\beta)}{\xi_p(1+\beta\iota_p)}(\tilde{w}_t+\tilde{z}_t), \\
\tilde{y}^*_t &= \frac{\gamma}{\gamma_a(1+\eta)-\gamma\eta}(\tilde{y}^*_{t-1}-a_t), \\
\tilde{R}^n_t &= \phi_r\tilde{R}^n_{t-1}+(1-\phi_r)\left[\phi_\pi\tilde{\Pi}_t+\phi_y\tilde{y}_t+\phi_{\Delta y}(\tilde{y}_t-\tilde{y}_{t-1}+a_t)\right]+\varepsilon_{r,t}, \\
\tilde{d}_t &= \rho_d\tilde{d}_{t-1}+\varepsilon_{d,t}, \\
a_t &= \rho_a a_{t-1}+\varepsilon_{a,t}, \\
\tilde{z}_t &= \rho_z\tilde{z}_{t-1}+\varepsilon_{z,t},
\end{aligned}$$

where the variables with a tilde ($\sim$) represent percentage deviations from their steady-state values.

The observation equations are

$$\begin{bmatrix} 100\Delta\log GDP_t \\ 100\Delta\log PGDP_t \\ FF_t \\ 100\log H_t \end{bmatrix} = \begin{bmatrix} \bar{a} \\ \bar{\pi} \\ \bar{r}+\bar{\pi} \\ \bar{l} \end{bmatrix} + \begin{bmatrix} \tilde{y}_t-\tilde{y}_{t-1}+a_t \\ \tilde{\Pi}_t \\ \tilde{R}^n_t \\ \tilde{l}_t \end{bmatrix},$$

where $\bar{a} = 100 \log \gamma_a$, $\bar{\pi} = 100 \log \bar{\Pi}$, $\bar{r} = 100 \log \bar{R} (= 100 \log(\gamma_a/\beta))$, and $\bar{l}$ are, respectively, the steady-state growth rate, the inflation rate, the real interest rate, and hours worked.

Appendix B. Non-linear Solution Method

B.1 Recursive Forms of the Price- and Wage-Setting Equations

After detrending, the equilibrium conditions (10) and (4) can be written in the following recursive forms:

$$\begin{aligned}
\frac{P_t^o}{P_t} &= \frac{S_{p,t}}{F_{p,t}} \\
S_{p,t} &= \theta_p w_t z_t + \xi_p \beta d_t^{-1} \mathbb{E}_t \left[\left(\frac{\Pi_{t+1}}{\bar{\Pi}} \right) \left(\frac{\Pi_t}{\bar{\Pi}} \right)^{-\iota_p} \right]^{\theta_p} \\
&\quad \times \frac{y_{t+1}}{y_t} \frac{\lambda_{t+1}}{\lambda_t} S_{p,t+1}, \\
F_{p,t} &= (\theta_p - 1) + \xi_p \beta d_t^{-1} \mathbb{E}_t \left[\left(\frac{\Pi_{t+1}}{\bar{\Pi}} \right) \left(\frac{\Pi_t}{\bar{\Pi}} \right)^{-\iota_p} \right]^{\theta_p - 1} \\
&\quad \times \frac{y_{t+1}}{y_t} \frac{\lambda_{t+1}}{\lambda_t} F_{p,t+1}, \\
\left(\frac{W_t^{n,o}}{W_t^n} \right)^{1+\eta\theta_w} &= \frac{S_{w,t}}{F_{w,t}} \\
S_{w,t} &= \theta_w l_{d,t}^{\eta} \lambda_t^{-1} + \xi_w \beta d_t^{-1} \mathbb{E}_t \left[\left(\frac{\Pi_{w,t+1}}{\bar{\Pi}} \exp(a_{t+1}) \right) \right. \\
&\quad \times \left. \left(\frac{\Pi_t}{\bar{\Pi}} \right)^{-\iota_w} \right]^{(1+\eta)\theta_w} \frac{l_{d,t+1}}{l_{d,t}} \frac{\lambda_{t+1}}{\lambda_t} S_{w,t+1}, \\
F_{w,t} &= (\theta_w - 1) w_t + \xi_w \beta d_t^{-1} \mathbb{E}_t \left[\left(\frac{\Pi_{w,t+1}}{\bar{\Pi}} \exp(a_{t+1}) \right) \right. \\
&\quad \times \left. \left(\frac{\Pi_t}{\bar{\Pi}} \right)^{-\iota_w} \right]^{\theta_w - 1} \frac{l_{d,t+1}}{l_{d,t}} \frac{\lambda_{t+1}}{\lambda_t} F_{w,t+1},
\end{aligned}$$

where $\Pi_{w,t} = \Pi_t w_t / w_{t-1}$ and $l_{d,t} = \Delta_{p,t} y_t$.

B.2. Solution Algorithm

In what follows, we drop the time subscript and use -1 and $'$ for previous- and next-period variables, respectively. To solve for the policy functions on each grid point of the state space $(\mathbb{S}_{-1}, \tau)$, where $\mathbb{S}_{-1} = [y_{-1}, \Pi_{-1}, w_{-1}, \widehat{R}^n_{-1}, \Delta_{p,-1}, \Delta_{w,-1}, y^*_{-1}]'$ and $\tau = [d, a, z, \varepsilon_r]'$, we follow an index-function approach as in Aruoba, Cuba-Borda, and Schorfheide (2018), Gust et al. (2017), and Nakata (2017).[18] First, regime-specific expectation functions are defined as follows:

$$e_{\lambda,s}(\mathbb{S}_{-1}, \tau) \equiv \beta d^{-1} R^n \int_{\tau'} \left\{ \frac{1}{\gamma_a \exp(a')} \frac{\lambda'}{\Pi'} \right\} \Phi(\tau'|\tau) d\tau',$$

$$e_{sp,s}(\mathbb{S}_{-1}, \tau)$$

$$\equiv \theta_p w z + \xi_p \beta d^{-1} \int_{\tau'} \left\{ \left[\left(\frac{\Pi'}{\bar{\bar{\Pi}}} \right) \left(\frac{\Pi}{\bar{\bar{\Pi}}} \right)^{-\iota_p} \right]^{\theta_p} \frac{y' \lambda' S'_p}{y \lambda} \right\} \Phi(\tau'|\tau) d\tau',$$

$$e_{fp,s}(\mathbb{S}_{-1}, \tau)$$

$$\equiv \theta_p - 1 + \xi_p \beta d^{-1} \int_{\tau'} \left\{ \left[\left(\frac{\Pi'}{\bar{\bar{\Pi}}} \right) \left(\frac{\Pi}{\bar{\bar{\Pi}}} \right)^{-\iota_p} \right]^{\theta_p - 1} \frac{y' \lambda' F'_p}{y \lambda} \right\} \Phi(\tau'|\tau) d\tau',$$

$$e_{sw,s}(\mathbb{S}_{-1}, \tau) \equiv \theta_w l_d^{\eta} \lambda^{-1}$$

$$+ \xi_w \beta d^{-1} \int_{\tau'} \left\{ \left[\left(\frac{\Pi'_w \exp(a')}{\bar{\bar{\Pi}}} \right) \left(\frac{\Pi}{\bar{\bar{\Pi}}} \right)^{-\iota_w} \right]^{(1+\eta)\theta_w} \frac{l'_d \lambda' S'_w}{l_d \lambda} \right\} \Phi(\tau'|\tau) d\tau',$$

$$e_{fw,s}(\mathbb{S}_{-1}, \tau) \equiv (\theta_w - 1) w$$

$$+ \xi_w \beta d^{-1} \int_{\tau'} \left\{ \left[\left(\frac{\Pi'_w \exp(a')}{\bar{\bar{\Pi}}} \right) \left(\frac{\Pi}{\bar{\bar{\Pi}}} \right)^{-\iota_w} \right]^{\theta_w - 1} \frac{l'_d \lambda' F'_w}{l_d \lambda} \right\} \Phi(\tau'|\tau) d\tau',$$

where the index $s \in \{\text{NZLB}, \text{ZLB}\}$ is associated with the interest rate regime in which the notional nominal interest rate $\widehat{R}^n$ implied by its unconstrained policy function $g_{\widehat{R}^n,\text{NZLB}}(\mathbb{S}_{-1}, \tau)$ is either above

[18] See also Hirose and Sunakawa (2019) for details about the solution algorithm with an example of a prototypical New Keynesian model with the ZLB.

or below the lower bound. Then, the expectation functions are constructed as weighted averages of the regime-specific functions

$$e_x(\mathbb{S}_{-1},\tau) = e_{x,\mathrm{NZLB}}(\mathbb{S}_{-1},\tau)1_{\{\widehat{R}^n>1\}} + e_{x,\mathrm{ZLB}}(\mathbb{S}_{-1},\tau)1_{\{\widehat{R}^n\leq 1\}},$$

where $1_{\{D\}}$ is the indicator function that equals one if the condition D is true and zero otherwise.

We obtain the policy functions by a time iteration method, which takes the following steps.

(i) Make an initial guess for the expectation functions $e_s^{(0)} = \left(e_{\lambda,s}^{(0)}, e_{sp,s}^{(0)}, e_{fp,s}^{(0)}, e_{sw,s}^{(0)}, e_{fw,s}^{(0)}\right)$ for $s \in \{\mathrm{NZLB}, \mathrm{ZLB}\}$.

(ii) For $i = 1, 2, ...$ (i is an index for the number of iterations), taking as given the expectation functions previously obtained $e_s^{(i-1)}$, solve the relevant equations to obtain the policy functions $g_s^{(i)} = \left(g_{\Pi,s}^{(i)}, g_{\Delta_p,s}^{(i)}, g_{\Pi_w,s}^{(i)}, g_{\Delta_w,s}^{(i)}, g_{y,s}^{(i)}, g_{w,s}^{(i)}, g_{l_d,s}^{(i)}, g_{\widehat{R}^n,s}^{(i)}\right)$.

(iii) Update the expectation functions $e_s^{(i)}$ by interpolating the policy functions $g_s^{(i)}$.

(iv) Repeat Steps (ii)–(iii) until $\left\|e_s^{(i)} - e_s^{(i-1)}\right\|$ is small enough.

In Step (ii), taking as given the values of $e_{x,s}^{(i-1)}(\mathbb{S}_{-1,j},\tau_m)$ for $x \in \{\lambda, sp, fp, sw, fw\}$ at each grid point indexed by (j, m) and each regime $s \in \{\mathrm{NZLB}, \mathrm{ZLB}\}$, we have

$$\frac{\Pi_{jms}}{\bar{\Pi}} = \left(\xi_p^{-1} + (1-\xi_p^{-1})\left[\frac{e_{sp,s}^{(i-1)}(\mathbb{S}_{-1,j},\tau_m)}{e_{fp,s}^{(i-1)}(\mathbb{S}_{-1,j},\tau_m)}\right]^{1-\theta_p}\right)^{\frac{1}{\theta_p-1}} \left(\frac{\Pi_{-1,j}}{\bar{\Pi}}\right)^{\iota_p},$$

$$\Delta_{p,jms} = (1-\xi_p)\left[\frac{e_{sp,s}^{(i-1)}(\mathbb{S}_{-1,j},\tau_m)}{e_{fp,s}^{(i-1)}(\mathbb{S}_{-1,j},\tau_m)}\right]^{-\theta_p} + \xi_p\left(\frac{\Pi_{jms}}{\bar{\Pi}}\right)^{\theta_p}\left(\frac{\Pi_{-1,j}}{\bar{\Pi}}\right)^{-\iota_p}\Delta_{p,-1,j},$$

$$\frac{\Pi_{w,jms}\exp(a)}{\bar{\Pi}}$$
$$= \left(\xi_w^{-1} + (1-\xi_w^{-1})\left[\frac{e_{sw,s}^{(i-1)}(\mathbb{S}_{-1,j},\tau_m)}{e_{fw,s}^{(i-1)}(\mathbb{S}_{-1,j},\tau_m)}\right]^{\frac{1-\theta_w}{1+\eta\theta_w}}\right)^{\frac{1}{\theta_w-1}} \left(\frac{\Pi_{-1,j}}{\bar{\Pi}}\right)^{\iota_w},$$

$$\Delta_{w,jms} = (1-\xi_w)\left[\frac{e_{sw,s}^{(i-1)}(\mathbb{S}_{-1,j},\tau_m)}{e_{fw,s}^{(i-1)}(\mathbb{S}_{-1,j},\tau_m)}\right]^{\frac{-\theta_w}{1+\eta\theta_w}}$$
$$+ \xi_w\left(\frac{\Pi_{w,jms}\exp(a_m)}{\bar{\Pi}}\right)^{\theta_w}\left(\frac{\Pi_{-1,j}}{\bar{\Pi}}\right)^{-\iota_w}\Delta_{w,-1,j},$$

$$y_{jms} = e_{\lambda,s}^{(i-1)}(\mathbb{S}_{-1,j},\tau_m)^{-1} + \frac{\gamma y_{j,-1}}{\gamma_a\exp(a_m)}$$

$$w_{jms} = w_{-1}\Pi_{w,jms}/\Pi_{jms},$$

$$l_{d,jms} = \Delta_{p,jms}y_{jms},$$

$$\widehat{R}^n_{jms}$$
$$= (\widehat{R}^n_{-1,j})^{\phi_r}\left[\bar{R}\bar{\Pi}\left(\frac{\Pi_{jms}}{\bar{\Pi}}\right)^{\phi_\pi}\left(\frac{y_{jms}\exp(a_m)}{y_{-1}}\right)^{\phi_y}\right]^{1-\phi_r}\exp(\varepsilon_{r,m}).$$

Then, we can evaluate $\left(\Pi_{jms}, \Delta_{p,jms}, \Pi_{w,jms}, \Delta_{w,jms}, y_{jms}, w_{jms}, l_{d,jms}, \widehat{R}^n_{jms}\right)$ at each grid point (j,m) and each regime s and the policy functions $g_{x,s}^{(i)}(\mathbb{S}_{-1},\tau;\boldsymbol{\theta})$ for $x = \{\Pi, \Delta_p, \Pi_w, \Delta_w, y, w, l_d, \widehat{R}^n\}$ parameterized by a vector of polynomial coefficients $\boldsymbol{\theta}$ for computing the values off the grid points. Note that this procedure does not rely on any numerical optimization routines to solve the non-linear equations.

In Step (iii), the expectation functions are updated by

$$e_{\lambda,s}^{(i)}(\mathbb{S}_{j,-1},\tau_m)$$
$$= \beta d_m^{-1}R^n_{jms}\int_{\tau'}\left\{\frac{1}{\gamma_a\exp(a')}\frac{e_{\lambda}^{(i-1)}(\mathbb{S},\tau';\boldsymbol{\theta})}{g_{\Pi}^{(i)}(\mathbb{S},\tau';\boldsymbol{\theta})}\right\}\Phi(\tau'|\tau)d\tau',$$

$$
\begin{aligned}
e^{(i)}_{sp,s}(\mathbb{S}_{j,-1},\tau_m) &= \theta_p w_{jms} z_m \\
&\quad + \xi_p \beta d_m^{-1} \int_{\tau'} \left\{ \left[\left(\frac{g^{(i)}_{\Pi}(\mathbb{S},\tau';\boldsymbol{\theta})}{\bar{\Pi}} \right) \left(\frac{\Pi_{jms}}{\bar{\Pi}} \right)^{-\iota_p} \right]^{\theta_p} \right. \\
&\quad \left. \times \frac{g^{(i)}_{y}(\mathbb{S},\tau';\boldsymbol{\theta}) e^{(i-1)}_{\lambda}(\mathbb{S},\tau';\boldsymbol{\theta}) e^{(i-1)}_{sp}(\mathbb{S},\tau';\boldsymbol{\theta})}{y_{jms} e^{(i-1)}_{\lambda}(\mathbb{S}_{j,-1},\tau_m;\boldsymbol{\theta})} \right\} \Phi(\tau'|\tau) d\tau', \\
e^{(i)}_{fp,s}(\mathbb{S}_{j,-1},\tau_m) &= \theta_p - 1 \\
&\quad + \xi_p \beta d_m^{-1} \int_{\tau'} \left\{ \left[\left(\frac{g^{(i)}_{\Pi}(\mathbb{S},\tau';\boldsymbol{\theta})}{\bar{\Pi}} \right) \left(\frac{\Pi_{jms}}{\bar{\Pi}} \right)^{-\iota_p} \right]^{\theta_p - 1} \right. \\
&\quad \left. \times \frac{g^{(i)}_{y}(\mathbb{S},\tau';\boldsymbol{\theta}) e^{(i-1)}_{\lambda}(\mathbb{S},\tau';\boldsymbol{\theta}) e^{(i-1)}_{fp}(\mathbb{S},\tau';\boldsymbol{\theta})}{y_{jms} e^{(i-1)}_{\lambda}(\mathbb{S}_{j,-1},\tau_m;\boldsymbol{\theta})} \right\} \Phi(\tau'|\tau) d\tau', \\
e^{(i)}_{sw,s}(\mathbb{S}_{j,-1},\tau_m) &= \theta_w l^{\eta}_{d,jms} \lambda^{-1}_{jms} \\
&\quad + \xi_w \beta d_m^{-1} \int_{\tau'} \left\{ \left[\left(\frac{g^{(i)}_{\Pi_w}(\mathbb{S},\tau';\boldsymbol{\theta}) \exp(a')}{\bar{\Pi}} \right) \left(\frac{\Pi_{jms}}{\bar{\Pi}} \right)^{-\iota_p} \right]^{\theta_p} \right. \\
&\quad \left. \times \frac{g^{(i)}_{l_d}(\mathbb{S},\tau';\boldsymbol{\theta}) e^{(i-1)}_{\lambda}(\mathbb{S},\tau';\boldsymbol{\theta}) e^{(i-1)}_{sw}(\mathbb{S},\tau';\boldsymbol{\theta})}{l_{d,jms} e^{(i-1)}_{\lambda}(\mathbb{S}_{j,-1},\tau_m;\boldsymbol{\theta})} \right\} \Phi(\tau'|\tau) d\tau', \\
e^{(i)}_{fw,s}(\mathbb{S}_{j,-1},\tau_m) &= (\theta_w - 1) w_{jms} \\
&\quad + \xi_w \beta d_m^{-1} \int_{\tau'} \left\{ \left[\left(\frac{g^{(i)}_{\Pi_w}(\mathbb{S},\tau';\boldsymbol{\theta}) \exp(a')}{\bar{\Pi}} \right) \left(\frac{\Pi_{jms}}{\bar{\Pi}} \right)^{-\iota_p} \right]^{\theta_p} \right. \\
&\quad \left. \times \frac{g^{(i)}_{l_d}(\mathbb{S},\tau';\boldsymbol{\theta}) e^{(i-1)}_{\lambda}(\mathbb{S},\tau';\boldsymbol{\theta}) e^{(i-1)}_{fw}(\mathbb{S},\tau';\boldsymbol{\theta})}{l_{d,jms} e^{(i-1)}_{\lambda}(\mathbb{S}_{j,-1},\tau_m;\boldsymbol{\theta})} \right\} \Phi(\tau'|\tau) d\tau',
\end{aligned}
$$

where the values $(\Pi_{jms}, \Pi^w_{jms}, y_{jms}, w_{jms}, R^n_{jms})$ and the policy functions $g^{(i)}_x(\mathbb{S},\tau';\boldsymbol{\theta})$ evaluated at the next period's state $(\mathbb{S},\tau')$ are obtained in the previous step. Note that we interpolate $g^{(i)}_x(\mathbb{S},\tau';\boldsymbol{\theta})$ for $x \in \{\Pi, \Pi_w, y, l_d, \widehat{R}^n\}$ off the grid points (or equivalently

$e_x^{(i-1)}(\mathbb{S}, \tau'; \boldsymbol{\theta})$ for $x \in \{\lambda, sp, fp, sw, fw\}$) by piecewise linear interpolation. Numerical integrals are computed with regard to τ'.

For interpolating the policy functions at each regime, we set the number of grids as 5 for each endogenous state variable, 15 for the discount factor shock d_t, and 3 for each of the other shocks. Thus we have $5^6 \times 15 \times 3^3 = 6,328,125$ grid points in total.[19] The integrals over τ' are approximated by the Gauss–Hermite quadrature formula with three nodes for each shock. To deal with such a large number of grid points and quadrature nodes, we use a supercomputer system to run programming codes with parallel computing.

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[19]We can drop the natural output from the endogenous state variables in the time iteration, as it is relevant only for the natural rate.

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