On the Optimal Labor Income Share*

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Labor's income share has attracted interest reflecting its decline. But, from an efficiency standpoint, can we say what share would hold in the social optimum? We address this question using a microfounded endogenous growth model calibrated on U.S. data. In our baseline case the socially optimal labor share is 17 percent (11 percentage points) above the decentralized (historical) equilibrium. This wedge reflects the presence of externalities in R&D in the decentralized equilibrium, whose importance is conditioned by the degree of factor substitutability. We also study the dependence of both long-run growth equilibriums on different model parameterizations and relate our results to Piketty's "Laws of Capitalism."

JEL Codes: O33, O41.

1. Introduction

Although interest in labor's share of income has a long tradition in economics, current interest has crystallized around its apparent fall in recent decades across many countries.[1] Much of the public discussion suggests that the primary reason to be concerned with

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[1]The scope of economic interest in the labor share is extremely diverse—aside from the conventional political economy, inequality, and growth literatures, the labor share is an important consideration in inflation modeling (Galí and Gertler 1999; McAdam and Willman 2004) and the consequence of firm market power, de Loecker, Eeckhout, and Unger (2020).

a low labor income share is wealth inequality. That may be a valid concern. The current paper however provides another rationale for interest in the labor share. We demonstrate that even in a representative household model (in which wealth inequality is absent by definition), the level of the labor share determines whether the economy is allocating its resources efficiently. Thus, we can say that the labor share may be too low (or high)—even without distributional concerns.

Remarkably, there appears to be no investigation of this issue in the literature. This contrasts with equivalent discussions in the growth literature: since Ramsey (1928), the question of whether a decentralized economy saves "too little" is fundamental (e.g., de La Grandville 2012). Likewise, in terms of production, modern endogenous growth theory typically suggests that the presence of various distortions implies that the economy produces less output and less research and development (R&D) relative to the first-best allocation (the "social optimum") (e.g., Jones and Williams 2000; Alvarez-Pelaez and Groth 2005). These *distortions* include monopoly power and markups plus the existence of technological externalities. Their presence can mean that individual firms have weak incentives to work to their fullest capacity, or indeed to invest and innovate, if not all of the benefits accrue to them.

But what of the labor share of income? Does, for example, "too little" output in the decentralized economy translate into a labor share that is also somehow too low? Ex ante, it is by no means obvious. Given widespread interest in the labor share, this constitutes an important gap in our knowledge, which we seek to address.[2] In the context of a microfounded endogenous growth model calibrated on U.S. data, with labor- and capital-augmenting technical change in the aggregate production function, we find that in our baseline case the socially optimal labor share is markedly above the decentralized equilibrium. The decentralized labor share, in other words, is too low.

The key channels underlying that result are the following. First, the social planner saves more and thus has more physical capital in

[2]Note, we not only study the implications for the labor share but also the growth rate, employment in the research sector, consumption, capital accumulation, etc.

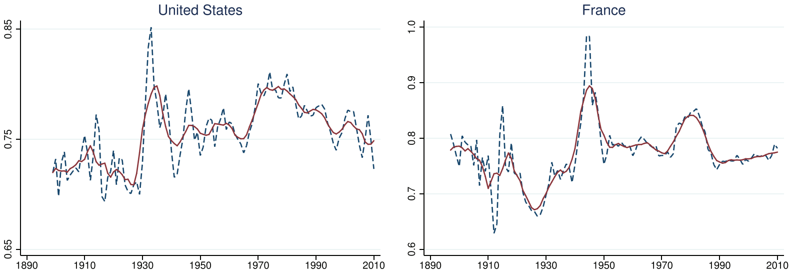
the long run. This is because the social planner takes into account the *social* rather than purely private return on production capital, and because the planner is able to internalize the positive returns to capital used in R&D. This abundance of capital makes labor relatively scarce. If, and only if, both factors have a substitution elasticity in production below one (i.e., are gross complements), this capital abundance pushes the labor share up.

Second, in comparison with the decentralized allocation, the social planner tends to allocate relatively more labor to the R&D sectors and less to the (final good) production sector. Because labor-augmenting R&D is the ultimate source of per capita growth, this increases the long-run growth rate. Yet, even though the social planner increases labor-augmenting R&D relative to the decentralized allocation, it increases capital-augmenting R&D even more. The ratio of capital- to labor-augmenting R&D is always higher in the social planner allocation. This effect increases unit productivity of capital in the steady state, augmenting both production and R&D sectors and feeding back once again to the steady-state growth rate and the labor share.

Another appealing aspect of our framework is that it can provide new convincing results on one important strand of the literature on the labor share, namely Piketty's "Laws of Capitalism" (Piketty 2014). These predict that the capital-output ratio and the capital income share should increase whenever the pace of economic growth declines. We say "convincing results" because in our setup all relevant variables (factor accumulation and factor intensity, growth, technological progress, marginal product of capital and thus the interest rate, etc.) are endogenous and modeled in a sufficiently flexible manner. Our analysis underscores that Piketty's laws should not be interpreted causally, but rather as correlations generated by changes in deeper characteristics of the economy. Crucially, though, in fact these correlations are not guaranteed to hold at all. For example, we find that increases in factor substitutability are able to raise the capital-output ratio, the capital income share, and the economic growth rate, thus simultaneously *violating* both of Piketty's laws.

Finally we demonstrate that, through the lens of our model, a fluctuating labor share would not in itself necessarily be a sign of inefficiency. Rather, the planner's solution shows that volatility in

Figure 1. Historical Labor Share: United States (1899–2010) and France (1897–2010)



Notes: The U.S. data are taken from Piketty and Zucman (2014) over the sample 1929–2010; prior to 1929 the labor share is extrapolated using the database by Groth and Madsen (2016), which provides compensation of employees and value-added data starting in 1898 based on historical source provided by Liesner (1989). The French data are also taken from Piketty and Zucman (2014). The dashed line is the level of the labor income share and the solid line is a simple moving-average process approximating its trend characteristics: $1/10 \sum_{j=-5}^{4} ls_{t-j}$ where ls_t is the labor share. See also Charpe, Bridji, and McAdam (2020) for a discussion and analysis of the properties of historical labor share measures.

this share is a natural outcome. Indeed, we know from historical sources such as Piketty and Zucman (2014) that labor income shares, even over long horizons (e.g., above 100 years), can fluctuate considerably (see figure 1 for the United States and France; see also Charpe, Bridji, and McAdam 2020).[3] By comparison, can we describe the decentralized labor share as being characterized by excessive volatility? The reasons an optimal allocation would produce oscillations, too, relate to the fact that there is an entrenched tension between capital- and labor-augmenting technologies. Labor-augmenting developments generate economic growth but also make capital relatively scarce, necessitating a reallocation of resources towards capital to overcome this scarcity. By the same token,

[3]Both aspects matter for any normative discussion on the labor share. For instance, if the labor share is falling yet still above its "optimal" level (or fluctuating around it), then, arguably, this might be interpreted passively, as a manifestation of recognized fluctuations in factor shares (e.g., Mućk, McAdam, and Growiec 2018). Indeed, given that long and persistent fluctuations in the labor share are observed in practice, we might also wonder whether such fluctuations are socially optimal.

capital-augmenting developments make labor relatively scarce and trigger the opposite reallocation.

The paper is organized as follows. Section 2 describes the model (also contained in Growiec, McAdam, and Mućk 2018). This is a non-scale model of endogenous growth with two R&D sectors, giving rise to capital- as well as labor-augmenting innovations, drawing from the seminal contributions of Romer (1990) and Acemoglu (2003).[4] The model economy uses the Dixit-Stiglitz monopolistic competition setup and the increasing variety framework of the R&D sectors. Two R&D sectors are included to enable an endogenous determination of factor shares. Both the social planner and decentralized allocations are solved for and compared. We see that the presence of markups arising from imperfect competition (and market power) and technological and R&D externalities, are the key reasons why the decentralized allocation produces relatively lower output growth and labor share.

Section 3 calibrates the model to U.S. data. We assume that a range of long-run averages from U.S. data (evaluated over 1929–2015) correspond to the decentralized balanced growth path (BGP) of the model. Around this central calibration, though, we extensively examine robustness of our result to alternative parameterizations.

Thereafter, in section 4 we solve the BGP of each allocation (i.e., decentralized and first best) and compare them. We list the channels and assumptions underlying the differences between both allocations. We find that—assuming that factors are gross complements in production—the decentralized labor share is indeed socially suboptimal. The difference, moreover, is large: about 17 percent (11 percentage points). We describe the mechanisms which underlie this wedge. For robustness, we also consider production characterized by Cobb-Douglas as well as gross substitutes. In the latter case, and almost only in that case, the socially optimal labor share falls below the decentralized one. However, already for $\sigma = 1.25$, which constitutes a mild degree of gross substitutability, its value is counterfactually

[4]The term "scale effect" states that an increase in an economy's labor endowment leads to a higher real growth rate. This relation arises from the (counterfactual) assumption that growth is proportional to the number of R&D workers (Jones 1995).

low (at around 0.5) and also associated with counterfactually high per capita growth rates.[5]

In section 5 we also study the dependence of both long-run growth equilibriums on model parameters and relate our results to Piketty's "Laws of Capitalism." We also consider the dynamic properties of the model around the balanced growth path (both in the decentralized and optimal allocation) in terms of oscillatory dynamics. Section 6 concludes. Additional material is found in the appendixes.

Finally, note that while making a first attempt at a new research question, we abstract from several issues. First, to repeat a remark made earlier, our concern is not about inequality among heterogeneous agents; there are many papers on this topic. Indeed, although the labor share and inequality are clearly related, they are by no means interchangeable (Atkinson, Piketty, and Saez 2011); an economy may well exhibit a socially optimal factor income division yet still be characterized by considerable inequality—as, for example, if there are different skill characteristics in the labor force (and thus appreciable wage dispersion), asymmetric corporate or union insider power, or if there is financial repression and rent seeking, etc. Indeed, one can draw an analogy with Ramsey (1928)—whose concern lay with the level of the socially optimal aggregate savings rate, not how savings behavior is distributed across economic agents (such as by wealth, age percentiles, etc.). Our concern therefore is somewhat more straightforward—namely, how would a social planner choose functional income shares. And, would that share be realistic[6] in terms of its central value (relative to the decentralized optimum) and its volatility (again compared with the decentralized optimum and historical averages). Second, we do not discuss policy designs able to alleviate the discrepancy between the decentralized allocation and the social optimum, nor the dynamics with which

[5]It may be, as Piketty and Zucman (2014) argue, that one might expect a higher elasticity of substitution in "high-tech" economies where there are lots of alternative uses and forms for capital.

[6]By way of realism, consider another "optimal" rule in growth theory: namely the golden rule savings ratio which in standard form equates the optimal savings rate with the capital income share (which is usually around 30 percent); see de la Grandville (2012). With the exception of some Asian economies and for some particular periods, such values are highly counterfactual.

they could be introduced.[7] Our results are obtained by comparing long-run equilibriums of two entirely separate model economies (decentralized and first-best allocation). Therefore we are silent on the possible evolution of the labor share along the transition path following the introduction of policy measures able to shift the decentralized allocation towards the first best. In consequence, we cannot say (i) if the labor share should rise or fall in the short to medium run, and (ii) how long the transition to the optimal labor share should take.

2. Model

The framework is a generalization of Acemoglu (2003) with capital- and labor-augmenting R&D, building on the earlier induced innovation literature from Kennedy (1964) onwards as well as general innovation in monopolistic competition and growth literatures (e.g., Dixit and Stiglitz 1977, Romer 1990, and Jones 1999).

By "generalization" we mean that we relax a number of features to make our conclusions more applicable to the studied question, as well as to correct for some counterfactual features (such as the aforementioned scale effects). Formally, (i) our model is non-scale: both R&D functions are specified in terms of percentages of population employed in either R&D sector; (ii) we also assume R&D workers are drawn from the same pool as production workers;[8] (iii) we assume more general R&D technologies which allow for mutual spillovers between both R&D sectors (cf. Li 2000) and for concavity in capital-augmenting technical change; (iv) in contrast to Acemoglu (2003), the BGP growth rate in our model depends on preferences via employment in production and R&D—the tradeoff is due to drawing researchers from the same employment pool as production

[7] Interestingly, Atkinson (2015) lists a number of proposals for reducing inequality trends, the first of which is that "the direction of technical change should be an explicit concern of policy-makers."

[8] Acemoglu (2003) assumes that labor supply in the production sector is inelastic and R&D is carried out by a separate group of "scientists" who cannot engage in production labor. Our assumption affects the tension between both R&D sectors by providing R&D workers with a third option, the production sector.

workers (a tradeoff not present in his model); and (v) we use *normalized* constant elasticity of substitution (CES) production functions[9] which, importantly, ensures valid comparative static comparisons in the elasticity of factor substitution. To start matters off, we consider the simpler case of the social planner allocation.[10]

2.1 The Social Planner's Problem

The social planner maximizes the representative household's utility from discounted consumption, c, given standard constant relative risk aversion (CRRA) preferences, (1).

$$\max \int_0^\infty \frac{c^{1-\gamma}-1}{1-\gamma} e^{-(\rho-n)t} dt, \tag{1}$$

where $\gamma > 0$ is the inverse of the intertemporal elasticity of substitution, $\rho > 0$ is the rate of time preference, and $n > 0$ is the (exogenous) growth rate of the labor supply.

The maximization is subject to the budget constraint (2) (i.e., the equation of motion of the aggregate per capita capital stock k), the "normalized" production function (3), the two R&D technologies (4)–(5), and the labor market clearing condition (6):[11]

$$\dot{k} = y - c - (\delta + n)k - \zeta \dot{a}, \tag{2}$$

$$y = y_0 \left(\pi_0 \left(\lambda_b \frac{k}{k_0} \right)^\xi + (1-\pi_0) \left(\frac{\lambda_a}{\lambda_{a0}} \frac{\ell_Y}{\ell_{Y0}} \right)^\xi \right)^{1/\xi}, \tag{3}$$

[9]Normalization essentially implies representing the production relations in consistent index number form. Its parameters then have a direct economic interpretation. Otherwise, the parameters can be shown to be scale dependent (i.e., a circular function of σ itself, as well as a function of the implicit normalization points). Subscript 0's denote the specific normalization points: geometric (arithmetic) averages for non-stationary (stationary) variables. See de la Grandville (1989), Klump and de la Grandville (2000), and Klump and Preissler (2000) for the seminal theoretical contributions. In our case, normalization is essentially important, since comparative statics on production function parameters are a key concern.

[10]It is simpler because, solving under the social optimum, we can impose symmetry directly and deal in terms of aggregates; see Bénassy (1998).

[11]There are three control (c, ℓ_a, ℓ_b) and three state $(k, \lambda_a, \lambda_b)$ variables in this optimization problem.

$$\dot{\lambda}_a = A\left(\lambda_a \lambda_b^{\phi} x^{\eta_a} \ell_a^{\nu_a}\right), \tag{4}$$

$$\dot{\lambda}_b = B\left(\lambda_b^{1-\omega} x^{\eta_b} \ell_b^{\nu_b}\right) - d\lambda_b, \tag{5}$$

$$1 = \ell_a + \ell_b + \ell_Y. \tag{6}$$

In (2) and (3), $y = Y/L$ and $k = K/L$ (i.e., output and capital per capita), where L is total employment and ℓ_a and ℓ_b are the shares (or "research intensity") employed in labor- and capital-augmenting R&D, respectively (and, respectively, generating increases in λ_a and λ_b). The remaining fraction of population ℓ_Y is employed in production. We assume that capital augmentation is subject to gradual decay at rate $d > 0$, which mirrors susceptibility to obsolescence and embodied character of capital-augmenting technologies; Solow (1960). This assumption is critical for the asymptotic constancy of unit capital productivity λ_b in the model, and thus for the existence of a BGP with purely labor-augmenting technical change.

The term π denotes the capital income share, and $\xi = \frac{\sigma-1}{\sigma}$, where $\sigma \in [0, \infty)$ is the elasticity of substitution between capital and labor. This parameter, important in many contexts,[12] turns out also to be critical in our analysis with the distinction as to whether factors are gross complements, i.e., $\sigma < 1$, or gross substitutes, $\sigma > 1$, in production.

Factor-augmenting innovations are created endogenously by the respective R&D sectors (Acemoglu 2003), increasing the underlying parameters λ_a, λ_b, as in (4) and (5). Parameters A and B capture the unit productivity of the labor- and capital-augmenting R&D process, respectively; ϕ captures the spillover from capital- to labor-augmenting R&D;[13] and ω measures the degree of decreasing returns

[12] CES function (3) nests the linear, Cobb-Douglas, and Leontief forms, respectively, when $\xi = 1, 0, -\infty$. The value of the elasticity of factor substitution has been shown to be a key parameter in many economic fields: e.g., the gains from trade (Saam 2008); the strength of extensive growth (de La Grandville 2016); multiple growth equilibriums, development traps, and indeterminacy (Azariadis 1996; Klump 2002; Kaas and von Thadden 2003; Guo and Lansing 2009); the response of investment and labor demand to various policy changes and shocks (Rowthorn 1999); etc.

[13] We assume $\phi > 0$, indicating that more efficient use of physical capital also increases the productivity of labor-augmenting R&D. Observe, there are *mutual* spillovers between both R&D sectors, with no prior restriction on their strength: $\dot{\lambda}_a = A\lambda_a^{1-\eta_a}\lambda_b^{\phi+\eta_a}k^{\eta_a}\ell_a^{\nu_a}$ and $\dot{\lambda}_b = B\lambda_a^{-\eta_b}\lambda_b^{1-\omega+\eta_b}k^{\eta_b}\ell_b^{\nu_b} - d\lambda_b$.

to scale in capital-augmenting R&D. By assuming $\omega \in (0,1)$ we allow for the "standing on shoulders" effect in capital-augmenting R&D, albeit we limit its scope insofar as it is less than proportional to the existing technology stock (Jones 1995).

The term $x \equiv \frac{\lambda_b k}{\lambda_a}$ captures the technology-corrected degree of capital augmentation of the workplace. This term represents positive spillovers from capital intensity in the R&D sector and will be constant along the BGP. The long-term endogenous growth engine is located in the linear labor-augmenting R&D equation. To fulfill the requirement of the existence of a BGP along which the growth rates of λ_a and λ_b are constant, we assume that $\eta_b\phi + \eta_a\omega \neq 0$.[14] Note that the above parameterization of R&D equations, with six free parameters in equations (4)–(5), is the most general one possible under the requirement of existence of a BGP with purely labor-augmenting technical change (Uzawa 1961; Jones 1999; Acemoglu 2003; Growiec 2007).

The last term in (2) captures a negative externality that arises from implementing new labor-augmenting technologies, with $\zeta \geq 0$. Motivated by León-Ledesma and Satchi (2019), we allow for a non-negative cost of adopting new labor-augmenting technologies: since workers (as opposed to machines) need to develop skills compatible with each new technology, it is assumed that there is a capital cost of such technology shifts (potentially representing training costs, learning-by-doing, etc.). We posit that new capital investments are diminished by $\zeta\dot{a}$, where $\dot{a} = g\lambda_a\left(\frac{\pi}{\pi_0}\right)^{1/\alpha}$, g being the economic growth rate (Growiec, McAdam, and Mućk 2018). For analytical simplicity we consider these costs exogenous to the firms.

Finally, R&D activity may be subject to duplication externalities; the greater the number of researchers searching for new ideas, the more likely is duplication. Thus research effort may be characterized by diminishing returns; Kortum (1993). This is captured by parameters $\nu_a, \nu_b \in (0,1]$: the higher is ν, the lower the extent of duplication.[15]

[14] All our qualitative results also go through for the special case $\eta_a = \eta_b = 0$, which fully excludes capital spillovers in R&D. The current inequality condition is not required in such cases.

[15] Observe that switching off all externalities and spillovers in (4)–(5) by setting $d = \omega = \eta_a = \eta_b = 0$ and $\nu_a = \nu_b = 1$ retrieves the original specification of R&D

Variables with subscript 0 ($\pi_0, y_0, k_0, \lambda_{a0}, \ell_{Y0}$) are CES normalization constants.

2.2 Decentralized Allocation

The construction of the decentralized allocation draws from Romer (1990), Acemoglu (2003), and Jones (2005). It has been also presented in Growiec, McAdam, and Mućk (2018). We use the Dixit and Stiglitz (1977) monopolistic competition setup and the increasing variety framework of the R&D sector. The general equilibrium is obtained as an outcome of the interplay between households; final goods producers; aggregators of bundles of capital- and labor-intensive intermediate goods; monopolistically competitive producers of differentiated capital- and labor-intensive intermediate goods; and competitive capital- and labor-augmenting R&D firms.

2.2.1 Households

Analogous to the social planner's allocation, we again assume that the representative household maximizes discounted CRRA utility:

$$\max \int_0^\infty \frac{c^{1-\gamma}-1}{1-\gamma} e^{-(\rho-n)t} dt$$

subject to the budget constraint:

$$\dot{v} = (r - \delta - n)v + w - c, \tag{7}$$

where $v = V/L$ is the household's per capita holding of assets, $V = K + p_a\lambda_a + p_b\lambda_b$. The representative household is the owner of all capital and also holds the shares of monopolistic producers of differentiated capital- and labor-intensive intermediate goods (priced p_a and p_b, respectively). Capital is rented at a net market rental rate

in Acemoglu (2003). Moreover, compared with models which use Cobb-Douglas production, equation (5) is akin to Jones's (1995) formulation of the R&D sector, generalized by adding obsolescence and positive spillovers from capital intensity. Thus, setting $d = \eta_b = 0$ retrieves Jones's original specification. And (4) is the same as in Romer (1990) but scale free (it features ℓ_b instead of $\ell_b \cdot L$), with a positive spillover from capital intensity and a direct spillover from λ_b; setting $\phi = \eta_a = 0$ retrieves the scale-free version of Romer (1990), cf. Jones (1999).

equal to the gross rental rate after depreciation: $r - \delta$. In turn, w is the market wage rate. Solving the household's optimization problem yields the familiar Euler equation:

$$\hat{c} = \frac{r - \delta - \rho}{\gamma}, \tag{8}$$

where $\hat{c} = \dot{c}/c$ is the per capita growth rate ("hats" denote growth rates).

2.2.2 Final Goods Producers

The role of final goods producers is to generate the output of final goods (which are then either consumed by the representative household or saved and invested, leading to physical capital accumulation), taking bundles of capital- and labor-intensive intermediate goods (Y_K, Y_L) as inputs. They operate in a perfectly competitive environment, where both bundles are remunerated at market rates p_K and p_L, respectively.

The final goods producers operate a normalized CES technology:

$$Y = Y_0 \left(\pi_0 \left(\frac{Y_K}{Y_{K0}} \right)^{\xi} + (1 - \pi_0) \left(\frac{Y_L}{Y_{L0}} \right)^{\xi} \right)^{\frac{1}{\xi}}. \tag{9}$$

The optimality condition implies that final goods producers' demand for capital- and labor-intensive intermediate goods bundles satisfies

$$\frac{p_K}{p_L} = \frac{\pi}{1 - \pi} \frac{Y_L}{Y_K}, \tag{10}$$

where $\pi = \pi_0 \left(\frac{Y_K}{Y_{K0}} \frac{Y_0}{Y} \right)^{\xi}$ is the elasticity of final output with respect to Y_K (in equilibrium it will be equal to the labor share).

2.2.3 Aggregators of Capital- and Labor-Intensive Intermediate Goods

There are two symmetric sectors whose role is to aggregate the differentiated (capital- or labor-intensive) goods into the bundles Y_K and Y_L demanded by final goods producers. It is assumed that the differentiated goods are imperfectly substitutable (albeit gross substitutes). The degree of substitutability is captured by parameter $\varepsilon \in (0,1)$:

$$Y_K = \left(\int_0^{N_K} X_{Ki}^{\varepsilon} di \right)^{\frac{1}{\varepsilon}}. \tag{11}$$

Aggregators operate in a perfectly competitive environment and decide upon their demand for intermediate goods, the price of which will be set by the respective monopolistic producers (discussed in the following subsection).

For capital-intensive bundles, the aggregators maximize

$$\max_{X_{Ki}} \left\{ p_K \left(\int_0^{N_K} X_{Ki}^{\varepsilon} di \right)^{\frac{1}{\varepsilon}} - \int_0^{N_K} p_{Ki} X_{Ki} di \right\}. \tag{12}$$

There is a continuum of measure N_K of capital-intensive intermediate goods producers. Optimization implies the following demand curve:

$$X_{Ki} = x_K(p_{Ki}) = \left(\frac{p_{Ki}}{p_K} \right)^{\frac{1}{\varepsilon - 1}} Y_K^{\frac{1}{\varepsilon}}. \tag{13}$$

Equivalent terms follow for labor-intensive intermediate goods producers.

2.2.4 Producers of Differentiated Intermediate Goods

It is assumed that each of the differentiated capital- or labor-intensive intermediate goods producers, indexed by $i \in [0, N_K]$ or $i \in [0, N_L]$, respectively, has monopoly over its specific variety. It is therefore free to choose its preferred price p_{Ki} or p_{Li}. These firms operate a simple linear technology, employing either only capital or only labor.

For the case of *capital-intensive* intermediate goods producers, the production function is $X_{Ki} = K_i$. Capital is rented at the gross rental rate r. The optimization problem is

$$\max_{p_{Ki}} (p_{Ki} X_{Ki} - r K_i) = \max_{p_{Ki}} (p_{Ki} - r) x_K(p_{Ki}). \tag{14}$$

The optimal solution implies $p_{Ki} = r/\varepsilon$ for all $i \in [0, N_K]$. This implies symmetry across all differentiated goods: they are sold at equal prices, thus their supply is also identical, $X_{Ki} = \bar{X}_K$ for all i. Market clearing implies

$$K = \int_0^{N_K} K_i di = \int_0^{N_K} X_{Ki} di = N_K \bar{X}_K, \qquad Y_K = N_K^{\frac{1-\varepsilon}{\varepsilon}} K. \quad (15)$$

The demand curve implies that the price of intermediate goods is linked to the price of the capital-intensive bundle as in $p_K = p_{Ki} N_K^{\frac{\varepsilon-1}{\varepsilon}} = \frac{r}{\varepsilon} N_K^{\frac{\varepsilon-1}{\varepsilon}}$.

The *labor-intensive* sector follows symmetrically: $X_{Li} = L_{Yi}$, $L_Y = \ell_Y L = \int_0^{N_L} L_{Yi} di$, and $p_{Li} = w/\varepsilon$, $p_L = p_{Li} N_L^{\frac{\varepsilon-1}{\varepsilon}} = \frac{w}{\varepsilon} N_L^{\frac{\varepsilon-1}{\varepsilon}}$, where w is the market wage rate.

Aggregating across all intermediate goods producers, we obtain that their total profits are equal to $\Pi_K N_K = rK\left(\frac{1-\varepsilon}{\varepsilon}\right)$ and $\Pi_L N_L = wL_Y\left(\frac{1-\varepsilon}{\varepsilon}\right)$ for capital- and labor-intensive goods, respectively. Streams of profits per person in the representative household are thus $\pi_K = \Pi_K/L$ and $\pi_L = \Pi_L/L$, respectively. Hence, the total remuneration channeled to the capital-intensive sector equals $p_K Y_K = \frac{r}{\varepsilon} K = rK + \Pi_K N_K$, whereas the total remuneration channeled to the labor-intensive sector equals $p_L Y_L = \frac{w}{\varepsilon} L_Y = wL_Y + \Pi_L N_L$.

In equilibrium, factor shares then amount to

$$\pi = \pi_0 \left(\frac{KY_0}{YK_0}\right)^{\xi} \left(\frac{N_K}{N_{K0}}\right)^{\xi\left(\frac{1-\varepsilon}{\varepsilon}\right)}, \quad (16)$$

$$1 - \pi = (1 - \pi_0) \left(\frac{Y_0 L_Y}{Y L_{Y0}}\right)^{\xi} \left(\frac{N_L}{N_{L0}}\right)^{\xi\left(\frac{1-\varepsilon}{\varepsilon}\right)}. \quad (17)$$

Incorporating all these choices into (9), and using the definitions $\lambda_b = N_K^{\frac{1-\varepsilon}{\varepsilon}}$ and $\lambda_a = N_L^{\frac{1-\varepsilon}{\varepsilon}}$ retrieves production function (3).

2.2.5 Capital- and Labor-Augmenting R&D Firms

The role of capital- and labor-augmenting R&D firms is to produce innovations which increase the variety of available differentiated intermediate goods, either N_K or N_L, and thus indirectly also

λ_b and λ_a. Patents never expire, and patent protection is perfect. R&D firms sell these patents to the representative household, which sets up a monopoly for each new variety. Patent price, p_b or p_a, which reflects the discounted stream of future monopoly profits, is set at the competitive market. There is free entry to R&D.

R&D firms employ labor only: $L_a = \ell_a L$ and $L_b = \ell_b L$ workers are employed in the labor- and capital-augmenting R&D sector, respectively. There is also an externality from the physical capital stock per worker, working through the capital spillover term in the R&D production function. Furthermore, the R&D firms perceive their production technology as linear in labor, while in fact it is concave due to duplication externalities (Jones 1995).

Incorporating these assumptions and using the notion $x \equiv \lambda_b k/\lambda_a$, capital-augmenting R&D firms maximize

$$\max_{\ell_b} \left(p_b \dot{\lambda}_b - w\ell_b\right) = \max_{\ell_b} \left((p_b Q_K - w)\ell_b\right), \tag{18}$$

where $Q_K = B\left(\lambda_b^{1-\omega} x^{\eta_b} \ell_b^{\nu_b - 1}\right)$ is treated by firms as an exogenously given constant (Romer 1990; Jones 2005). Analogously, labor-augmenting R&D firms maximize

$$\max_{\ell_a} \left(p_a \dot{\lambda}_a - w\ell_a\right) = \max_{\ell_a} \left((p_a Q_L - w)\ell_a\right), \tag{19}$$

where $Q_L = A\left(\lambda_a \lambda_b^{\phi} x^{\eta_a} \ell_a^{\nu_a - 1}\right)$ is treated as exogenous.

Free entry into both R&D sectors implies $w = p_b Q_K = p_a Q_L$. Purchase of a patent entitles the holders to a per capita stream of profits equal to π_K and π_L, respectively. While the production of any labor-augmenting varieties lasts forever, there is a constant rate d at which production of capital-intensive varieties becomes *obsolete*. This effect is external to patent holders and thus is not strategically taken into account when accumulating the patent stock.[16]

2.2.6 Equilibrium

We define the *decentralized equilibrium* as the collection of time paths of all the respective quantities: $c, \ell_a, \ell_b, k, \lambda_b, \lambda_a, Y_K, Y_L$,

[16] In other words, by solving a static optimization problem, capital-augmenting R&D firms do not take the dynamic (external) obsolescence effect into account.

$\{X_{Ki}\}, \{X_{Li}\}$ and prices r, w, p_K, p_L, $\{p_{Ki}\}$, $\{p_{Li}\}, p_a, p_b$ such that (i) households maximize discounted utility subject to their budget constraint; (ii) profit maximization is followed by final goods producers, aggregators and producers of capital- and labor-intensive intermediate goods, and capital- and labor-augmenting R&D firms; (iii) the labor market clears: $L_a+L_b+L_Y = (\ell_a+\ell_b+\ell_Y)L = L$; (iv) the asset market clears: $V = vL = K + p_a\lambda_a + p_b\lambda_b$, where assets have equal returns: $r - \delta = \frac{\pi_L}{p_a} + \frac{\dot{p}_a}{p_a} = \frac{\pi_K}{p_b} + \frac{\dot{p}_b}{p_b} - d$; and, finally, (v), such that the aggregate capital stock satisfies $\dot{K} = Y - C - \delta K - \zeta \dot{a} L$.

2.3 Solving for the Social Planner Allocation

In this section, we first solve analytically for the BGP of the social planner (SP) allocation of our endogenous growth model and then linearize the implied dynamical system around the BGP.

2.3.1 Balanced Growth Path

Any neoclassical growth model can exhibit balanced growth only if technical change is purely labor augmenting or if production is Cobb-Douglas; Uzawa (1961). That condition holds here too. Hence, once we presume a CES production function, the analysis of dynamic consequences of any technical change which is not purely labor augmenting must be done outside the BGP.

Along the BGP, we obtain the following growth rate of key model variables:

$$g = \hat{\lambda}_a = \hat{k} = \hat{c} = \hat{y} = A(\lambda_b^*)^{\phi} (x^*)^{\eta_a} (\ell_a^*)^{\nu_a}, \tag{20}$$

where stars denote steady-state values. Hence, ultimately long-run growth is driven by labor-augmenting R&D. This can be explained by the fact that labor is the only non-accumulable factor in the model, it is complementary to capital along the aggregate production function, and the labor-augmenting R&D equation is linear with respect to λ_a. The following variables are constant along the BGP: $y/k, c/k, \ell_a, \ell_b$, and λ_b (i.e., asymptotically there is no capital-augmenting technical change).

2.3.2 Euler Equations

Having set up the Hamiltonian (with co-state variables μ_k, μ_a, μ_b),

$$\begin{aligned}\mathcal{H}(c, \ell_a, \ell_b, k, \lambda_a, \lambda_b; \mu_k, \mu_a, \mu_b) = {} & \frac{c^{1-\gamma}-1}{1-\gamma} e^{-(\rho-n)t} \\ & + \mu_k(y - c - (\delta+n)k - \zeta \dot{a}) \\ & + \mu_a A \left(\lambda_a \lambda_b^{\phi} x^{\eta_a} \ell_a^{\nu_a}\right) \\ & + \mu_b \left(B\left(\lambda_b^{1-\omega} x^{\eta_b} \ell_b^{\nu_b}\right) - d\lambda_b\right),\end{aligned} \tag{21}$$

where

$$y = y_0 \left(\pi_0 \left(\lambda_b \frac{k}{k_0}\right)^{\xi} + (1-\pi_0)\left(\frac{\lambda_a}{\lambda_{a0}} \frac{1-\ell_a-\ell_b}{\ell_{Y0}}\right)^{\xi}\right)^{1/\xi}, \tag{22}$$

computed its derivatives, and eliminated the co-state variables, after tedious algebra the following Euler equations are obtained for the SP:[17]

$$\hat{c} = \frac{1}{\gamma}\left(\frac{y}{k}\left(\pi + \frac{1-\pi}{\ell_Y}\left(\frac{\eta_a \ell_a}{\nu_a} + \frac{\eta_b \ell_b}{\nu_b}\right)\right) - \delta - \rho\right), \tag{23}$$

$$\varphi_1 \hat{\ell}_a + \varphi_2 \hat{\ell}_b = Q_1, \tag{24}$$

$$\varphi_3 \hat{\ell}_a + \varphi_4 \hat{\ell}_b = Q_2, \tag{25}$$

where

$$\varphi_1 = \nu_a - 1 - (1-\xi)\pi \frac{\ell_a}{\ell_Y}, \tag{26}$$

$$\varphi_2 = -(1-\xi)\pi \frac{\ell_b}{\ell_Y}, \tag{27}$$

[17] A sufficient condition for all transversality conditions to be satisfied in the social optimum (as well as in the decentralized equilibrium) is that $(1-\gamma)g + n < \rho$.

$$\varphi_3 = -(1-\xi)\pi\frac{\ell_a}{\ell_Y}, \tag{28}$$

$$\varphi_4 = \nu_b - 1 - (1-\xi)\pi\frac{\ell_b}{\ell_Y}, \tag{29}$$

and

$$\begin{aligned} Q_1 = {} & -\gamma\hat{c} - \rho + n + \hat{\lambda}_a\left(\frac{\ell_Y\nu_a}{\ell_a} + 1 - \eta_a - \eta_b\frac{\ell_b\nu_a}{\ell_a\nu_b}\right) \\ & - \phi\hat{\lambda}_b + ((1-\xi)\pi - \eta_a)\hat{x}, \end{aligned} \tag{30}$$

$$\begin{aligned} Q_2 = {} & -\gamma\hat{c} - \rho + n + \hat{\lambda}_a + \hat{\lambda}_b\left(\frac{\pi}{1-\pi}\frac{\ell_Y\nu_b}{\ell_b} + (\phi+\eta_a)\frac{\nu_b\ell_a}{\nu_a\ell_b} + \eta_b\right) \\ & + ((1-\xi)\pi - \eta_b)\hat{x} + d\left(\frac{\pi}{1-\pi}\frac{\ell_Y\nu_b}{\ell_b} + (\phi+\eta_a)\frac{\nu_b\ell_a}{\nu_a\ell_b} - \omega + \eta_b\right). \end{aligned} \tag{31}$$

2.3.3 Steady State and Linearization of the Transformed System

The above Euler equations and dynamics of state variables are then rewritten in terms of *stationary* variables which are constant along the BGP, i.e., in coordinates: $u = (c/k), \ell_a, \ell_b, x, \lambda_b$, and with auxiliary variables $z = (y/k), \pi, g$. The full steady state of the transformed system is listed in appendix A.1. This nonlinear system of equations is solved numerically, yielding a steady state of the detrended system and thus a BGP of the model in original variables.[18] All further analysis of the social planner allocation is based on the (numerical) linearization of the five-dimensional dynamical system of equations (23)–(25), (2), and (5), taking the BGP equality (20) as given.

2.4 Solving for the Decentralized Allocation

When solving for the decentralized allocation (DA), we broadly follow the steps carried out in the case of the social planner (SP)

[18] We do not have a formal proof of BGP uniqueness, but the large number of numerical checks we have performed (e.g., varying initial conditions of the numerical algorithm, modifying values of model parameters), is suggestive that the BGP is indeed unique and depends smoothly on model parameters.

allocation. We first solve analytically for the BGP of our endogenous growth model and then linearize the implied dynamical system around the BGP.

2.4.1 Balanced Growth Path

Along the BGP, we obtain the following growth rate of the key model variables:

$$g = \hat{k} = \hat{c} = \hat{y} = \hat{w} = \hat{p}_b = \hat{p}_{Li} = \hat{\lambda}_a = A(\lambda_b^*)^{\phi}\,(x^*)^{\eta_a}\,(\ell_a^*)^{\nu_a}. \quad (32)$$

The following quantities are constant along the BGP: $y/k, c/k, \ell_a, \ell_b$, $Y_K/Y, Y_L/Y$, and λ_b (again, note, asymptotically, the absence of capital-augmenting technical change). The following prices are also constant along the BGP: $r, p_a, p_K, p_L, \{p_{Ki}\}$.

2.4.2 Euler Equations

The decentralized equilibrium is associated with the following Euler equations describing the first-order conditions:

$$\hat{c} = \frac{\varepsilon\pi\frac{y}{k} - \delta - \rho}{\gamma}, \quad (23')$$

$$\varphi_1\hat{\ell}_a + \varphi_2\hat{\ell}_b = \tilde{Q}_1, \quad (24')$$

$$\varphi_3\hat{\ell}_a + \varphi_4\hat{\ell}_b = \tilde{Q}_2, \quad (25')$$

where

$$\tilde{Q}_1 = -\varepsilon\pi\frac{y}{k} + \delta + \hat{\lambda}_a\frac{\ell_Y}{\ell_a} - \phi\hat{\lambda}_b + ((1-\xi)\pi - \eta_a)\hat{x} \quad (30')$$

$$\tilde{Q}_2 = -\varepsilon\pi\frac{y}{k} + \delta + \hat{\lambda}_a + (\hat{\lambda}_b + d)\left(\frac{\pi}{1-\pi}\frac{\ell_Y}{\ell_b}\right) - \hat{\lambda}_b(1-\omega) - d + ((1-\xi)\pi - \eta_b)\hat{x} \quad (31')$$

and φ_1 through φ_4 are defined as in (26)–(29). The full steady state of the transformed system is listed in appendix A.2. All further analysis of the decentralized allocation is based on the (numerical) linearization of the five-dimensional dynamical system of equations (23′)–(25′), (2), and (5), taking the BGP equality (32) as given.

2.4.3 Departures from the Social Optimum

Departures of the decentralized allocation from the optimal one can be tracked back to specific assumptions regarding the information structure of the decentralized allocation. In the following we try to compare one-to-one all the differences in the equations related to the decentralized and to social planner outcome. As we shall show, those differences follow from the presence of wedges (such as markups from imperfect competition) and externalities (such as duplication externalities from R&D investment). These differences also show in different costs and returns that exist in the social planner and decentralized allocation.

Specifically, the points of comparison are as follows:

1. In the consumption Euler equation, comparing equations (23) with (23′), the term $\frac{y}{k}\left(\pi + \frac{1-\pi}{\ell_Y}\left(\frac{\eta_a \ell_a}{\nu_a} + \frac{\eta_b \ell_b}{\nu_b}\right)\right)$ is replaced by $\varepsilon\pi\frac{y}{k}$. This is due to two effects:
   (a) in contrast to the social planner, markets fail to account for the external effects of physical capital on R&D activity via the capital spillover terms (with respective elasticities η_b and η_a);
   (b) ε appears in the decentralized allocation due to imperfect competition in the labor- and capital-augmenting intermediate goods sectors.

   Both effects work in the same direction and buy the social planner much more capital in the steady state. The savings rate is much higher in the SP allocation, as two effects add up: (i) accounting for social instead of private returns on production capital, (ii) internalizing the positive returns to capital used in R&D. Therefore in the social planner's steady state, there is relatively lower consumption and output per unit of capital, and greater positive capital spillover in R&D, $x = \frac{\lambda_b k}{\lambda_a}$.

   This abundance of capital makes labor relatively scarce, which—if and only if both factors are gross complements

($\sigma < 1$)—increases the labor share in the social planner allocation relative to the decentralized equilibrium.

2. In the Euler equation for ℓ_a, the term $\left(\frac{\ell_Y \nu_a}{\ell_a} + 1 - \eta_a - \eta_b \frac{\ell_b \nu_a}{\ell_a \nu_b}\right)$, is replaced by $\frac{\ell_Y}{\ell_a}$. Analogously, in the Euler equation for ℓ_b, the term given by $\frac{\ell_Y \nu_b}{\ell_b} \frac{\pi}{1-\pi} + (1-\omega) + \eta_b + (\phi + \eta_a) \frac{\ell_a \nu_b}{\ell_b \nu_a}$ is replaced by $\frac{\ell_Y}{\ell_b} \frac{\pi}{1-\pi}$. This is due to two effects:
   (a) ν_a and ν_b are missing in the respective first components because markets fail to internalize the detrimental R&D duplication effects;
   b) the latter two components are missing because markets fail to account for the positive external effects of accumulating knowledge on future R&D productivity. These effects are included in the shadow prices of λ_a and λ_b in the social planner allocation but not in their respective market prices.

   The effect (a) reduces SP's investment in R&D, whereas the effect (b) increases it. In our baseline calibration and its robustness checks, on balance the latter effect robustly prevails and the social planner allocates less labor to production and more to R&D (greater ℓ_a and ℓ_b). Hence, this mechanism causes the social planner to accumulate more investment in both R&D sectors. This, given that labor-augmenting technological progress is the ultimate source of growth in the model, increases the long-run growth rate.

3. In the Euler equations for ℓ_a and ℓ_b (equations (24), (25), (24′), and (25′)) the shadow price of physical capital $\hat{c} - \rho + n$ is replaced by its market price $r - \delta = \varepsilon \pi \frac{y}{k} - \delta$, which is lower because it accounts for markups arising from imperfect competition.

   This mechanism causes the social planner to accumulate more capital-augmenting R&D ℓ_b relative to labor-augmenting R&D ℓ_a. Therefore the ratio of capital- to labor-augmenting R&D employment is always higher in the social optimum than in the decentralized allocation. Hence, unit capital productivity (λ_b) is higher in the SP steady state. This further adds to the capital spillover term x which, in turn, augments both R&D sectors, accelerates growth and—by making labor relatively scarce—increases the labor share.

3. Calibration of the Model

3.1 Empirical Calibration Components

The parameter calibration for the decentralized model based on magnitudes from historical data or empirical studies is listed in table 1. We assume that a range of long-run averages from U.S. data (1929–2015) correspond to the decentralized BGP of the model. Doing so allows us to calibrate the rates of economic and population growth, capital productivity and income share, and the consumption-to-capital ratio. Likewise, we assign CES normalization parameters to match U.S. long-run averages for factor income shares (we adjust the payroll share by proprietors' income, as in Mućk, McAdam, and Growiec 2018). This implies an average labor share of 0.67.[19]

Next, we turn to the elasticity of substitution between labor and capital (σ) which is the fundamental economic parameter in our analysis. We calibrate factors to be gross complements, i.e., $\sigma < 1$. This choice stems from a fact that the bulk of empirical studies for the U.S. aggregated production function document that the σ is systematically below unity (Klump, McAdam, and Willman 2012).[20] Most of the empirical evidence exploiting time-series variation for other countries also implies $\sigma < 1$ (McAdam and Willman 2013; Mućk 2017; Knoblach and Stöckl 2020).

However, the literature based predominantly on *cross-country* variation is rather inconclusive about the magnitude of σ. On one hand, several papers (Piketty and Zucman 2014; Karabarbounis and Neiman 2014) employ gross substitutes; however, the former paper

[19] Note that in the model, due to the inelastic labor supply and the firm profits being rebated to the household, markups do not directly affect factor shares.

[20] For instance, Arrow et al. (1961) found an aggregate elasticity over 1909–49 of 0.57 (similar to that of the more recent Antràs 2004). More recently, Klump, McAdam, and Willman (2007) reported $\hat{\sigma} \approx 0.7$. The tendency towards gross complementarity between factors is also confirmed at the industry level (Herrendorf, Herrington, and Valentinyi 2015; Laeven, McAdam, and Popov 2018) and firm level (Oberfield and Raval 2018). Importantly, the elasticity uncovered is found systematically below unity even if more flexible functional forms of aggregate production function are considered (Growiec and Mućk 2020).

Table 1. Calibrated Parameters (decentralized allocation)

Parameter	Symbol	Value	Target/Source
Income and Production			
GDP per capita Growth	g	0.0171	Geometric Average
Population Growth Rate	n	0.0153	Geometric Average
Capital Productivity	z_0, z^*	0.3442	Geometric Average
Consumption-to-Capital	u^*	0.2199	Geometric Average
Capital Income Share	π_0, π^*	0.3260	Arithmetic Average
Depreciation	δ	0.0600	Caselli (2005)
Factor Substitution Parameter	ξ	−0.4286	$\Rightarrow \sigma = 0.7$, Klump, McAdam, and Willman (2007)
Preferences			
Inverse Intertemporal Elasticity of Substitution	γ	1.7500	Barro and Sala-i-Martin (2003)
Time Preference	ρ	0.0200	Barro and Sala-i-Martin (2003)

Notes: This table shows the parameter values that are used in the central calibration of the model. The values are taken from historical averages (1929–2015) of the relevant U.S. time series.

calibrates the σ value and the latter estimates it in a cross-country panel context.[21] On the other hand, recent studies exploiting macro panels and allowing for factor augmentation in the supply-side system approach strongly conclude in favor of gross complementarity in production (Mućk 2017). Given this, we consider $\sigma < 1$ as the benchmark, but we do examine the $\sigma > 1$ case in our robustness exercises.

Finally, the table includes preference parameters (intertemporal elasticity of substitution, time preference) which are difficult to retrieve from historical data. For that reason, in our central calibration we rely on values typically found in the literature.

3.2 Model-Consistent Calibration Components

Next, conditional on the values in table 1, four identities included in the system (see appendix equations (A.9)–(A.17)) drive the calibration of other parameters in a model-consistent manner: r^*, λ_b^*, x^*, and ε. Employment in final production ℓ_Y^* is also set in a model-consistent manner (table 2).

In the absence of any other information, we agnostically assume that the share of population $1 - \ell_Y^*$ is split equally between employment in both (i.e., capital and labor) R&D sectors in the decentralized allocation—although notice these employment shares are endogenous in the social planner solution. For the model-consistent value of ℓ_Y^*, the relevant formula leads to values close to those typically considered for the non-routine cognitive occupational group (e.g., Jaimovich and Siu 2020, using Bureau of Labor Statistics data, show this ratio to be between 29 percent and 38 percent, over 1982–2012).

For the duplication externalities, we assume $\nu_a = \nu_b = 0.75$ following the (albeit single R&D sector) value in Jones and Williams (2000) (although, note again, we conduct extensive robustness checks on these values). The steady-state level of unit capital productivity λ_b^* is normalized to unity, and so are CES normalization parameters λ_{a0} and λ_{b0}.

[21] The latter paper moreover was estimated on a single-equation non-normalized basis which is known to have poor estimation properties in this context (León-Ledesma, McAdam, and Willman 2010; Klump, McAdam, and Willman 2012).

Table 2. Parameter and Steady-State Variable Calibration Conditional on Historical Average Calibration (decentralized allocation)

Parameter	Symbol	Value	Target/Source
Income and Production			
Net Real Rate of Return	$r^* - \delta$	0.0499	$r^* - \delta = \gamma g + \rho$
Substitutability between Intermediate Goods	ε	0.9793	$\varepsilon = \frac{r^*}{\pi^* z^*}$
R&D Sectors			
R&D Duplication Parameters	$\upsilon_a = \upsilon_b$	0.7500	Jones and Williams (2000)
Technology Augmenting Terms	$\lambda_{a0}, \lambda_{b0}$	1	Normalized to Unity
Unit Capital Productivity	λ_b^*	1	$\lambda_b^* = \lambda_{b0} \frac{z^*}{z_0} \left(\frac{\pi^*}{\pi_0}\right)^{\frac{1}{\xi}}$
Employment Share in R&D Sectors	ℓ_a^*, ℓ_b^*	0.2033	$\ell_a^* = \ell_b^*$ for $\ell_a^* + \ell_b^* = 1 - \ell_Y^*$
Capital–Labor Ratio in Efficiency Units†	x_0, x^*	61.7900	$x^* = x_0 \frac{\ell_Y^*}{\ell_{Y0}} \left(\frac{1}{1-\pi_0} \left(\frac{z^*}{z_0} \frac{\lambda_{b0}}{\lambda_b^*}\right)^{\xi} - \frac{\pi_0}{1-\pi_0} \right)^{-1/\xi}$

Notes: This table shows the parameter and steady-state variable values that are used in the central calibration of the model. The values are taken from values consistent with the structure of the model or from other relevant studies in the literature.
† $x_0 = \frac{\lambda_{b0} k_0}{\lambda_{a0}} = 61.79$.

The final step is to assign values to the remaining parameters, in particular the technological parameters of the R&D equations. We do this by solving the four remaining equations in system (A.9)–(A.17) with respect to the remaining parameters; see table B.1. Given this benchmark calibration, the steady state is a saddle point.

4. Is the Decentralized Labor Share Socially Optimal?

Given the model setup and its benchmark calibration, we can now come to our central question: is the decentralized labor share socially optimal? In table 3, columns 1 and 2 show the decentralized allocation (DA) and social planner (SP) outcomes for our benchmark calibration; columns 3 and 4, considered later, alternatively impose Cobb-Douglas and gross substitutes.[22]

The BGP of the DA solution features less physical capital, lower growth, and lower R&D activity, but a higher consumption rate (u is higher) than the SP. Moreover, with less capital and lower growth, the net real rate of return of capital is higher, and capital productivity is accordingly higher. Under gross complementarity of capital and labor, the relative scarcity of capital implies that also the labor share is lower. The theoretical underpinnings of these discrepancies have been discussed in section 2.4.3. But the magnitude of the labor share difference is perhaps less obvious. In fact, we see the striking result that the labor share in the social optimum is around 17 percent (11 percentage points) above the decentralized allocation. This means that looking at efficiency considerations only, the labor share not just is empirically too low today, but probably was too low even in the 1980s, before it embarked on a secular downward trend.

[22] We made a large number of numerical checks for existence, uniqueness, and stability of the steady state (e.g., varying initial conditions of the numerical algorithm, performing an eigenvalue analysis of the detrended system around the steady state, and modifying values of model parameters). Our results confirm that in the baseline calibration as well as across a large parameter space around it, the steady state of the model is unique, saddle-path stable, and depends smoothly on model parameters. Results of this analysis, beyond the ones reported in figures in our appendixes, are available on request.

Table 3. BGP Comparison under the Baseline Calibration

		(1)	(2)	(3)	(4)
		DA	**SP**		
Variable	**Symbol**		**Baseline Gross Comp.** $\boldsymbol{\sigma = 0.7}$ $\boldsymbol{\xi = -0.43}$	**CD** $\boldsymbol{\sigma = 1}$ $\boldsymbol{\xi = 0}$	**Piketty Gross Sub.** $\boldsymbol{\sigma = 1.25}$ $\boldsymbol{\xi = 0.20}$
Output Growth Rate	g	0.0171	0.0339	0.0425	0.0581
Consumption-to-Capital Ratio	u^*	0.2199	0.1628	0.1180	0.0856
Capital Productivity	z^*	0.3442	0.3071	0.2832	0.2743
Employment in Production	ℓ_Y^*	0.5934	0.4385	0.4160	0.3854
Employment in Labor-Augmenting R&D	ℓ_a^*	0.2033	0.2575	0.2447	0.2240
Employment in Capital-Augmenting R&D	ℓ_b^*	0.2033	0.3040	0.3393	0.3906
Relative Share	ℓ_a^*/ℓ_b^*	1	0.8470	0.7212	0.5735
Labor Income Share	$1-\pi^*$	0.6739	0.7854	0.6739	0.5243
Relative to DA (%)	$\frac{1-\pi^{*\mathrm{DA}}}{1-\pi^*} - 1$	0	0.1655	0	−0.2220
Capital Income Share	π^*	0.3261	0.2146	0.3261	0.4757
Net Real Rate of Return	$r^* - \delta$	0.0499	0.0059	0.0323	0.0704
Capital-Augmenting Technology	λ_b^*	1.0000	2.3696	3.3162	5.2600
Capital–Labor Ratio in Efficiency Units	x^*	61.7900	173.3363	342.7082	928.9625

Notes: This table shows the outcomes for the endogenous variables for the decentralized and social planner outcomes for different values of the aggregate elasticity of substitution.

To further understand why this discrepancy is so high, let us decompose the capital income share, π, in the following two ways (recalling that $1-\pi$ is the labor income share):

$$\frac{\pi}{\pi_0} = \left(\frac{\lambda_b k}{k_0}\right)^{\xi}\left(\frac{y}{y_0}\right)^{-\xi} \Rightarrow \hat{\pi} = \xi(\hat{\lambda}_b + \hat{k} - \hat{y}), \tag{33}$$

$$\frac{\pi}{1-\pi} = \frac{\pi_0}{1-\pi_0}\left(\frac{x}{x_0}\frac{\ell_{Y0}}{\ell_Y}\right)^{\xi} \Rightarrow \hat{\pi} = \xi(1-\pi)(\hat{x} - \hat{\ell}_Y). \tag{34}$$

Equation (33) shows that under gross complementarity ($\sigma < 1$, or equivalently $\xi < 0$), the capital share increases with capital productivity and decreases with capital augmentation (i.e., the capital-augmenting technology improvements are "labor biased").

Equation (34), in turn, follows from the definition of the aggregate production function and the effective capital–labor ratio x. Given $\hat{\ell}_Y \equiv -\left(\frac{\ell_a}{\ell_Y}\hat{\ell}_a + \frac{\ell_b}{\ell_Y}\hat{\ell}_b\right)$, the dynamics of employment in the goods sector are equal to the inverse of the dynamics of total R&D employment. It then follows that dynamics of the labor share are *uniquely determined* by the sum of the dynamics of the capital spillover term x and R&D employment. As before, the sign of this relationship depends upon the substitution elasticity: if $\xi < 0$, then increases in R&D intensity reduce π, and thus increase the labor share, and vice versa.

Comparing the decentralized and the social planner's allocation through the lens of (33), we observe that the large difference in factor shares at the BGP is driven almost exclusively by the difference in the level of capital augmentation λ_b^*. This result suggests that technical change is quantitatively more important for explaining labor share developments than shifts in the capital-output ratio.

Equivalently, by (34), this large difference in the degree of capital augmentation shows up in the capital spillover term x^*. It is also strengthened by the discrepancy in employment in final production ℓ_Y^*, which is higher in the decentralized allocation because the planner devotes more resources to (both types of) R&D. Thanks to this, coupled with relatively more saving, the social planner achieves faster growth at the BGP but with a lower consumption-to-capital ratio and a lower rate of return to capital. All of these make for a higher labor share in the optimal allocation.

4.1 Impact of Parameter Variation on the Equilibrium Labor Share

The results just discussed hold for the benchmark calibration. Accordingly, we now consider sensitivity to deviations from that calibration. Figure 2 presents the impact of varying selected model parameters, holding others constant, on the BGP level of the labor share.

Essentially, all panels can be interpreted through the lens of equations (33) and (34). As agents become less patient (higher ρ), R&D intensity falls, as does the labor share. Similar reasoning pertains to the inverse intertemporal elasticity of substitution γ. That $\frac{\partial(1-\pi)}{\partial\eta_b} > 0$ arises from the usual property that, under our gross complements benchmark, improvements in capital-augmenting technical change are labor biased; analogously, $\frac{\partial(1-\pi)}{\partial\eta_a} < 0$. Likewise, we have under gross complements $\frac{\partial(1-\pi)}{\partial\nu_a} > 0, \frac{\partial(1-\pi)}{\partial\nu_b} < 0$. If capital depreciates faster, the capital (labor) share rises (falls).

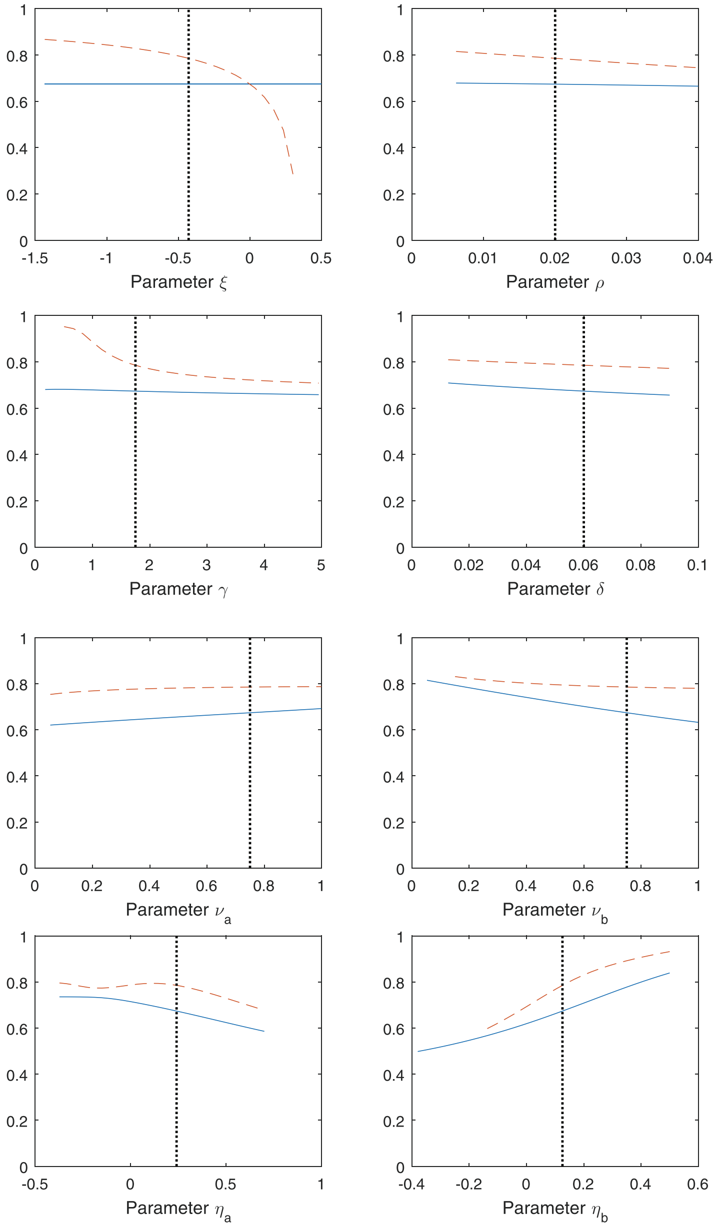
Finally, we see that under gross substitutes, $\sigma > 1$ $(\xi > 0)$, the DA labor share *exceeds* that of the SP. We discuss this case further below, but it is straightforward to motivate, since the previously discussed mechanisms go into reverse; capital-augmenting technical improvements tend to be capital biased, as output is now directed towards the relatively abundant, not the scarce factor of production.[23]

A more extensive study of the dependence of both BGPs on key model parameters $(\rho, \gamma, \nu_b, \eta_b)$ is included in figures D.1–D.3; the equivalent figure for the gross substitutes case is given in figure D.4. They are essentially a mirror image of our benchmark gross complements case.

Moreover, appendix C shows the impact of parameter variations on the equilibrium growth rate.

[23]Note that the lack of dependence of the BGP on ξ in the decentralized allocation follows from CES normalization (Klump and de La Grandville 2000), coupled with the fact that we have calibrated the normalization constants to the BGP of the decentralized allocation.

Figure 2. Dependence of Equilibrium Labor Share on Model Parameters



Notes: $1 - \pi$ on vertical axis; corresponding parameter support on the horizontal axis. Social planner allocation (dashed lines), decentralized equilibrium (solid lines). The vertical dotted line in each graph represents the baseline calibrated parameter value.

4.2 Impact of Elasticity of Substitution Variation on the BGP

Although we regard the gross complements case to be the more empirically relevant (at least for the aggregate economy), we also investigate the Cobb-Douglas and gross substitutes case. Accordingly, the SP is solved anew and presented in columns 3 and 4 of table 3, respectively.[24]

Both alternative parameterizations are markedly more growth friendly. Per capita output grows at the counterfactual rate of around 4–6 percent, exceeding both the previous SP and DA by a large margin, with an inflection point at $\xi \approx 0.25$ ($\sigma \approx 1.33$), after which it shoots through the roof. The fact that steady-state per capita growth is an increasing function of the substitution elasticity, though, is to be expected. Intuitively, easier factor substitution—by staving off diminishing returns—can prolong extensive growth (i.e., scarce factors can be substituted by abundant ones). The formal proof of this can be related through the properties of the normalized CES function as a general mean function.[25]

The consequences for labor's share of income, though, are dire. With gross factor substitutability,[26] the arguments of the previous section shift into reverse. Capital improvements are capital biased, and the incentives for capital accumulation are accordingly far higher in this regime. Hence the labor share declines with σ (or equivalently ξ).

It should also be emphasized that a balanced growth path does not exist in our model under sufficiently strong factor substitutability. Gross substitutability, as such, implies that Inada conditions at infinity are violated: the marginal product of per capita capital (MPK) remains bounded *above* zero as the capital stock goes to infinity. But then there is still the question whether the lower bound of MPK, multiplied by the savings rate, is high enough to exceed the capital depreciation rate. If so, and this happens only when

[24] The effect of a continuous variation in the substitution elasticity is graphed in figure D.5.

[25] See the discussion in Pitchford (1960) and the subsequent discussions in de La Grandville (1989); Klump and de La Grandville (2000); Klump and Preissler (2000), and Palivos and Karagiannis (2010).

[26] In the Cobb-Douglas case of $\xi = 0$, factor shares are constant and at their predetermined sample average. Thus $\pi|_{\xi=0} = \pi_0$.

σ exceeds a certain threshold $\bar{\sigma} > 1$, endogenous growth driven by capital accumulation appears (Jones and Manuelli 1990; Palivos and Karagiannis 2010). Combined with the existing growth engine of our model—labor-augmenting R&D—both sources of growth then lead to super-exponential, explosive growth. Then, even with diminishing returns to factors, capital intensity grows without bounds, labor becomes inessential in production, and hence the capital income share tends to unity. We rule such cases out of our analysis.

5. Additional Results

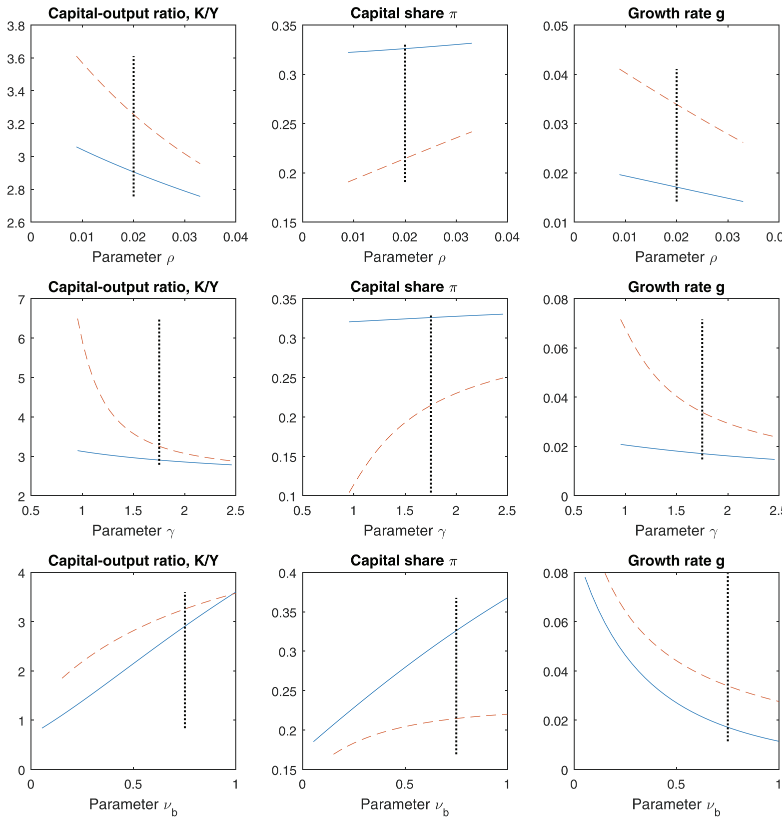
5.1 Comparing the Model with Piketty's Laws

As our model endogenizes both economic growth and factor shares, it constitutes an appropriate framework for studying the two "Fundamental Laws of Capitalism" formulated by Piketty (2014), i.e., (i) that the capital–output ratio K/Y rises whenever the economic growth rate g falls, and (ii) that the capital share π rises whenever the growth rate g falls. Our setup has the advantage over Piketty's that all three variables are endogenous, and hence one can legitimately observe whether changing some parameters implies co-movements that are or are not in line with Piketty's claims (i) and (ii). In addressing Piketty's laws with an R&D-based endogenous growth model, we follow the footsteps of Irmen and Tabakovic (2020). In contrast to their contribution, though, our setup departs from Cobb-Douglas technology.[27]

First, taking Piketty's claims (i) and (ii) together logically implies that K/Y and the capital share π are positively correlated, suggesting that capital and labor should be gross substitutes ($\sigma > 1$); see equation (33). This is a widely recognized issue with

[27] In Irmen and Tabakovic (2020), due to Cobb-Douglas technology, factor shares of capital, labor, and ideas in final output are always constant (their proposition 1). Factor shares in GDP, however, may vary because—foremost—GDP includes also new patented technological knowledge, and the proportion of final output to new technological knowledge within GDP is endogenous. By contrast, in our framework already factor shares in final output are variable. Therefore our setup is arguably better suited to identifying first-order effects of technical change on factor shares.

Figure 3. Dependence of BGP on ρ, γ, and ν_b: DA vs. SP



Notes: Social planner allocation (dashed lines), decentralized equilibrium (solid lines). The vertical dotted line in each graph represents the baseline calibrated parameter value.

Piketty's claims (see, e.g., Oberfield and Raval 2018). In our baseline parameterization, we assume gross complements instead.

Second, inspection of figure 3 reveals that under the baseline calibration, both in the decentralized equilibrium and the social planner allocation:

- when households become more patient (ρ goes down) or more willing to substitute consumption intertemporally (γ goes down), only law (ii) holds: the growth rate g goes up, the K/Y ratio goes up, and the capital share π goes down;

- when the capital spillover exponent ν_b in capital-augmenting R&D goes up, both laws are verified: the growth rate g goes down, the K/Y ratio goes up, and the capital share π goes up.

Third, we find (figure D.5) that as the elasticity of substitution goes up, the optimal growth rate g goes up hand in hand with the capital share π and the K/Y ratio. In such case, *both* of Piketty's laws are violated.

5.2 Is the Decentralized Economy Characterized by Excessive Volatility?

In the data, we know that—irrespective of the concept utilized—labor shares are highly persistent and variable.[28] Although bounded within the unit interval and theoretically stationary, in the data labor income shares often appear to be characterized by marked volatility and long swings. In particular, around 80 percent of total labor share volatility in the United States (1929–2015) has been due to fluctuations in medium- to long-run frequencies (beyond the eight-year mark). As opposed to the short-run component of the labor share, its medium- to long-run component has also been procyclical (Growiec, McAdam, and Mućk 2018).

Other than undermining the case for aggregate Cobb-Douglas production, this also raises the question of whether our framework can generate and rationalize these long cycles. Growiec, McAdam, and Mućk (2018) have confirmed this conjecture for the decentralized allocation of the current model. The question is however equally interesting for the social planner case. Are cycles in factor income shares socially optimal? If so, (stabilization) policies to mitigate labor share or real volatility might be appraised differently.[29]

Table 4 makes the relevant comparisons across our maintained cases. It shows that the decentralized allocation features relatively shorter cycles but also faster convergence to the BGP. Hence, it

[28] For international evidence, see Jalava et al. (2006); Bengtsson (2014); and Mućk, McAdam, and Growiec (2018).

[29] By design, our analysis focuses only on endogenous long swings in factor shares. The deterministic character of the model precludes any conclusions regarding the magnitude and persistence of short-run fluctuations.

Table 4. Dynamics around the BGP

Allocation	**Baseline**		**CD**		**Piketty**	
	DA	**SP**	**DA**	**SP**	**DA**	**SP**
Pace of Convergence* (% per year)	6.3%	4.2%	5.8%	3.7%	5.2%	2.9%
Length of Full Cycle† (Years)	52.6	76.7	79.8	83.2	144.0	100.3
Labor Share Cyclicality	+	+	0	0	–	–
Amplitude of $1 - \pi$ Relative to y/k	62.0%	48.0%	NA	NA	28.0%	44.0%

Notes: *Computed as $1 - e^{rr}$ where $rr < 0$ is the real part of the largest stable root; †Computed as $2\pi/ir$ where $ir > 0$ is the imaginary part of two conjugate stable roots (if they exist). "NA" denotes not available/applicable. See table 3 for the definitions of "Baseline," "CD," and "Piketty."

cannot be claimed directly that the decentralized equilibrium has *excessive volatility* of the labor share. If both allocations were to start from the same initial point outside of the BGP, then the decentralized allocation would exhibit a greater frequency but smaller amplitude of cyclical variation.

Having scrutinized the robustness of this dynamic result by extensively altering the parameterization of the model, we conclude that while the decentralized equilibrium generally exhibits shorter cycles, the ordering of both allocations in terms of the pace of convergence can sometimes be reversed. This finding lends partial support to the claim that the decentralized equilibrium is perhaps likely to feature greater labor share volatility compared with the social optimum. However, it is worthwhile to point out that oscillations in the labor income share can still be socially optimal in this model.

Moreover, we also obtain quantitative predictions on the cyclical co-movement of the original model variables (including the economic growth rate g and the labor share $1-\pi$).[30] It turns out, both for the decentralized and optimal allocation, that all variables except for the consumption-capital ratio $u = c/k$ oscillate when converging to the steady state, with the same frequency of oscillations. The level of capital-augmenting technology λ_b, the capital spillover term x, and labor-augmenting R&D employment ℓ_a are always countercyclical, employment in production ℓ_Y is always procyclical, whereas the cyclicality of capital-augmenting R&D ℓ_b is ambiguous (in the baseline calibration, ℓ_b is procyclical in the decentralized allocation but countercyclical in the optimal one). Furthermore, as long as capital and labor are gross complements, the labor income share $1-\pi$ is unambiguously procyclical as well. These features of cyclical co-movement align well with the empirical evidence for the U.S. medium-term cycle. In particular, the U.S. labor share is indeed procyclical over the medium-to-long run—despite its countercyclicality along the business cycle (Growiec, McAdam, and Mućk 2018; Mućk, McAdam, and Growiec 2018).

[30]This is done by inspecting the eigenvector associated with the largest stable root of the Jacobian of the system at the steady state.

6. Conclusions

Modern endogenous growth theory tends to suggest that the socially optimal level of economic activity dominates (i.e., exceeds) the decentralized outcome. The decentralized outcome produces too little output because of monopoly behavior, markups, and externalities related to reaping the private returns to innovation. In this paper, we have confirmed this conclusion using a microfounded, calibrated two-sector R&D endogenous growth model. Due to externalities between the two R&D sectors, in our model the decentralized allocation produces also a socially suboptimal level of R&D and, particularly, too little capital-augmenting R&D. This, in addition to a suboptimal level of capital accumulation, translates into too low equilibrium growth.

But what of the labor share? Despite its importance, the conclusions for this variable have perhaps surprisingly not yet been drawn in the literature. Our objective was to bridge that knowledge gap. We found that if the elasticity of factor substitution σ is below unity (as the bulk of evidence suggests for the aggregate U.S. economy), then the decentralized labor share is indeed socially suboptimal. The difference, moreover, is large, around 17 percent in our baseline calibration.

Effectively, the only parameter which can reverse this ordering is the elasticity of substitution. However in the gross substitutes case ($\sigma > 1$) it tends to yield counterfactual outcomes. For example, an elasticity of $\sigma = 1.25$, only slightly above Cobb-Douglas, produces a decentralized labor share above the social planner one, but then the latter is as low as 0.52; as a simple point of comparison, according to the International Labor Organization (ILO) definition of the labor share (using annual data from 1960 to the present), no G-7 country has fallen below a labor share of 0.5. Moreover, such a mild perturbation away from Cobb-Douglas already produces equilibrium per capita growth rates of around 6 percent per annum.

In the future, our results should be contrasted with findings from a highly needed prospective study of optimal factor shares under inequality in factor ownership. Such a study could uncover the associated efficiency versus inequality tradeoff. We expect that the discrepancy between the optimal and decentralized labor share would then be even larger than 17 percent because the social planner might increase the labor share not just to improve efficiency of production under gross complementarity ($\sigma < 1$) but also to reduce income inequality (given that capital incomes tend to be relatively more concentrated).

Appendix A. Steady State of the Transformed System

A.1 Social Planner Allocation

The steady state of the transformed dynamical system implied by the social planner solution satisfies

$$g = \hat{\lambda}_a = \hat{k} = \hat{c} = \hat{y} = A(\lambda_b^*)^{\phi}\,(x^*)^{\eta_a}\,(\ell_a^*)^{\nu_a} \tag{A.1}$$

$$\gamma g + \delta + \rho = z\left(\pi + \frac{1-\pi}{\ell_Y}\left(\frac{\eta_a \ell_a}{\nu_a} + \frac{\eta_b \ell_b}{\nu_b}\right)\right) \tag{A.2}$$

$$g = z - \zeta\frac{\dot{a}}{k} - u - (\delta + n) \tag{A.3}$$

$$d = B\left(\lambda_b^{-\omega} x^{\eta_b} \ell_b^{\nu_b}\right) \tag{A.4}$$

$$(1-\gamma)g + n - \rho = d\left(\frac{\pi}{1-\pi}\frac{\ell_Y \nu_b}{\ell_b} + (\phi + \eta_a)\frac{\nu_b \ell_a}{\nu_a \ell_b} - \omega + \eta_b\right) \tag{A.5}$$

$$(1-\gamma)g + n - \rho = -g\left(\frac{\ell_Y \nu_a}{\ell_a} - \eta_a - \eta_b\frac{\ell_b \nu_a}{\ell_a \nu_b}\right) \tag{A.6}$$

$$\frac{\pi}{\pi_0} = \left(\frac{\lambda_b}{\lambda_{b0}}\right)^{\xi}\left(\frac{z}{z_0}\right)^{-\xi} \tag{A.7}$$

$$\frac{z}{z_0} = \frac{\lambda_b}{\lambda_{b0}}\left(\pi_0 + (1-\pi_0)\left(\frac{x_0}{x}\frac{\ell_Y}{\ell_{Y0}}\right)^{\xi}\right)^{1/\xi}. \tag{A.8}$$

This nonlinear system of equations is solved numerically, yielding the steady state of the detrended system, and thus the BGP of the model in original variables. All further analysis of the social planner allocation is based on the (numerical) linearization of the five-dimensional dynamical system of equations (23)–(25), (2), and (5), taking the BGP equality (20) as given.

A.2 The Decentralized Allocation

As in the case of the social planner, the Euler equations and dynamics of state variables are rewritten in terms of *stationary* variables. The steady state of the transformed system satisfies

$$g = \hat{\lambda}_a = \hat{k} = \hat{c} = \hat{y} = A(\lambda_b^*)^{\phi} (x^*)^{\eta_a} (\ell_a^*)^{\nu_a} \tag{A.9}$$

$$\gamma g + \rho = r - \delta \tag{A.10}$$

$$g = z - \zeta \frac{\dot{a}}{k} - u - (\delta + n) \tag{A.11}$$

$$d = B \left(\lambda_b^{-\omega} x^{\eta_b} \ell_b^{\nu_b} \right) \tag{A.12}$$

$$g \frac{\ell_Y}{\ell_a} = r - \delta \tag{A.13}$$

$$g = r - \delta + d \left(1 - \frac{\pi}{1 - \pi} \frac{\ell_Y}{\ell_b} \right) \tag{A.14}$$

$$r = \varepsilon \pi z \tag{A.15}$$

$$\frac{\pi}{\pi_0} = \left(\frac{\lambda_b}{\lambda_{b0}} \right)^{\xi} \left(\frac{z}{z_0} \right)^{-\xi} \tag{A.16}$$

$$\frac{z}{z_0} = \frac{\lambda_b}{\lambda_{b0}} \left(\pi_0 + (1 - \pi_0) \left(\frac{x_0}{x} \frac{\ell_Y}{\ell_{Y0}} \right)^{\xi} \right)^{1/\xi}. \tag{A.17}$$

This nonlinear system of equations is solved numerically, yielding the steady state of the detrended system, and thus the BGP of the model in original variables. All further analysis of the decentralized allocation is based on the (numerical) linearization of the five-dimensional dynamical system of equations (23′)–(25′), (2), and (5), taking the BGP equality (32) as given.

Appendix B. Additional Parameters

We solve the four remaining equations in system (A.9)–(A.17) with respect to the remaining parameters; see table B.1. All these parameters are within admissible ranges. For instance, Pessoa (2005)

Table B.1. Baseline Calibration: Additional Parameters

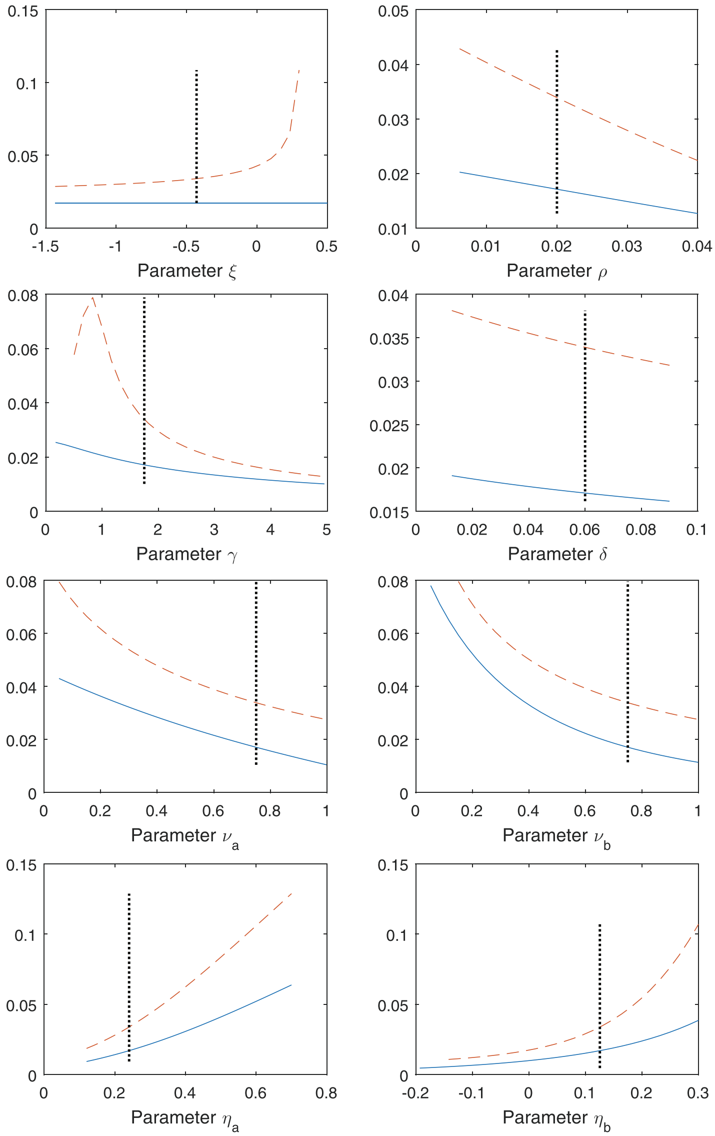
Parameter		**Value**
Labor-Augmenting R&D		
Unit Productivity	A	0.02
Capital Spillover Exponent	η_a	0.24
Capital-Augmenting R&D		
Unit Productivity	B	0.16
Capital Spillover Exponent	η_b	0.13
Degree of Decreasing Returns	ω	0.50
Obsolescence Rate	d	0.08
Spillover from Capital- to Labor-Augmenting Tech. Change	ϕ	0.30
Technology Choice Externality	ζ	115.28

estimates values for the obsolescence parameter d between 0 and 15 percent; our endogenously determined value is thus centered in that range. Comparing $\eta_a = 0.24$ with $\eta_b = 0.13$ signifies that, first of all, positive spillovers of capital intensity in R&D (effective capital augmentation of the R&D process) assuredly matter for R&D productivity, and second, that they are relatively more important for inventing new labor-augmenting technologies than capital-augmenting ones. Moreover, with $\phi = 0.3$, labor-augmenting R&D—the ultimate engine of long-run growth—is substantially reinforced by spillovers coming from the capital-augmenting R&D sector. On the other hand, $\omega = 0.5$ means that the scope for capital-augmenting R&D is quite strongly limited by decreasing returns. Given this benchmark calibration, as we said in the main text, the steady state is a saddle point.

Appendix C. Robustness Exercises: Impact of Parameter Variation on the Equilibrium Growth Rate

So far we have confirmed the received wisdom that the growth rate in the DA, g^{DA}, is socially suboptimal. This appears to be generally true in our model, regardless of its parameterization.

Figure C.1. Dependence of Equilibrium Growth on Model Parameters



Notes: The real economic growth rate g on vertical axis; corresponding parameter support on the horizontal axis. Social planner allocation (dashed lines), decentralized equilibrium (solid lines). The vertical dotted line in each graph represents the baseline calibrated parameter value.

Focusing on a reasonable parameter support, and assuming gross complements (see figure C.1), we can however identify a few credible cases where the difference between the two growth rates becomes small:

- A higher ρ (i.e., more impatience for current consumption), implies less capital and R&D accumulation and lower equilibrium growth than otherwise. If ρ is sufficiently large, then $g^{SP} \rightarrow_{+} g^{DA}$.[31]
- If the consumption smoothing motive is sufficiently weak (γ high), then $g^{SP} \rightarrow_{+} g^{DA}$.
- If the capital spillover exponents are weak, $\nu_a \rightarrow 0$ or $\nu_b \rightarrow 0$, then they attenuate the engine of long-run growth in the R&D equations and thus pull both g^{SP} and g^{DA} down.

Finally, note, departing from gross substitutes, we see the dramatic result that as σ (or equivalently ξ) increases, the gap $g^{SP} - g^{DA}$ hyperbolically widens; conversely, it narrows as substitution possibilities tend to zero (the Leontief case). We explore this in the next appendix.

[31] See also figure D.1.

Appendix D. Additional Figures

Figure D.1. Comparing Balanced Growth Paths, DA vs. SP: Dependence on the Time Preference

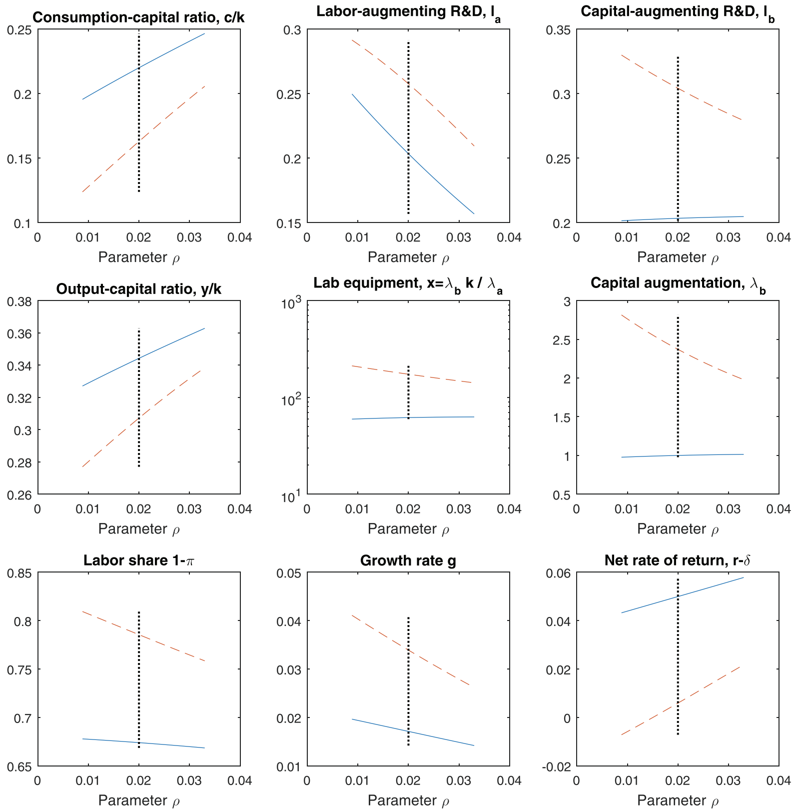


Figure D.2. Comparing Balanced Growth Paths, DA vs. SP: Dependence on the Intertemporal Elasticity of Substitution in Consumption

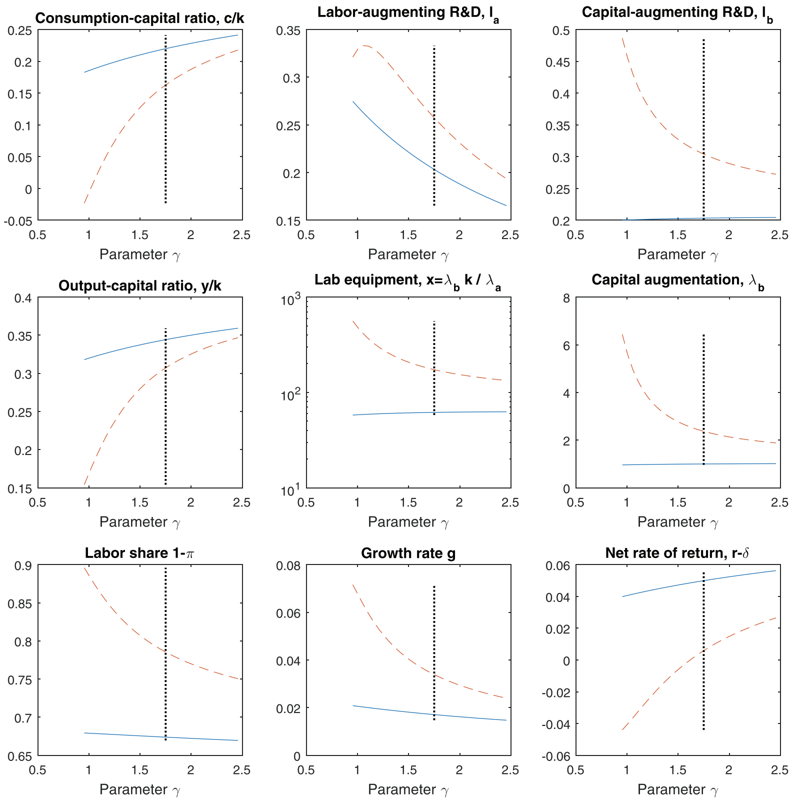


Figure D.3. Comparing Balanced Growth Paths, DA vs. SP: Dependence on the Capital Spillover Exponent in Capital-Augmenting R&D

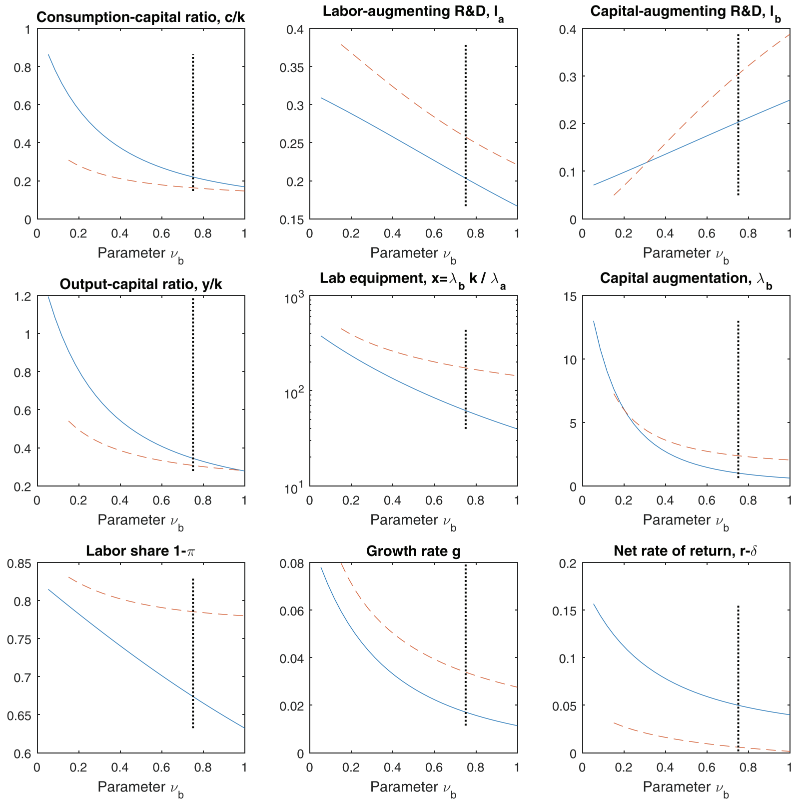
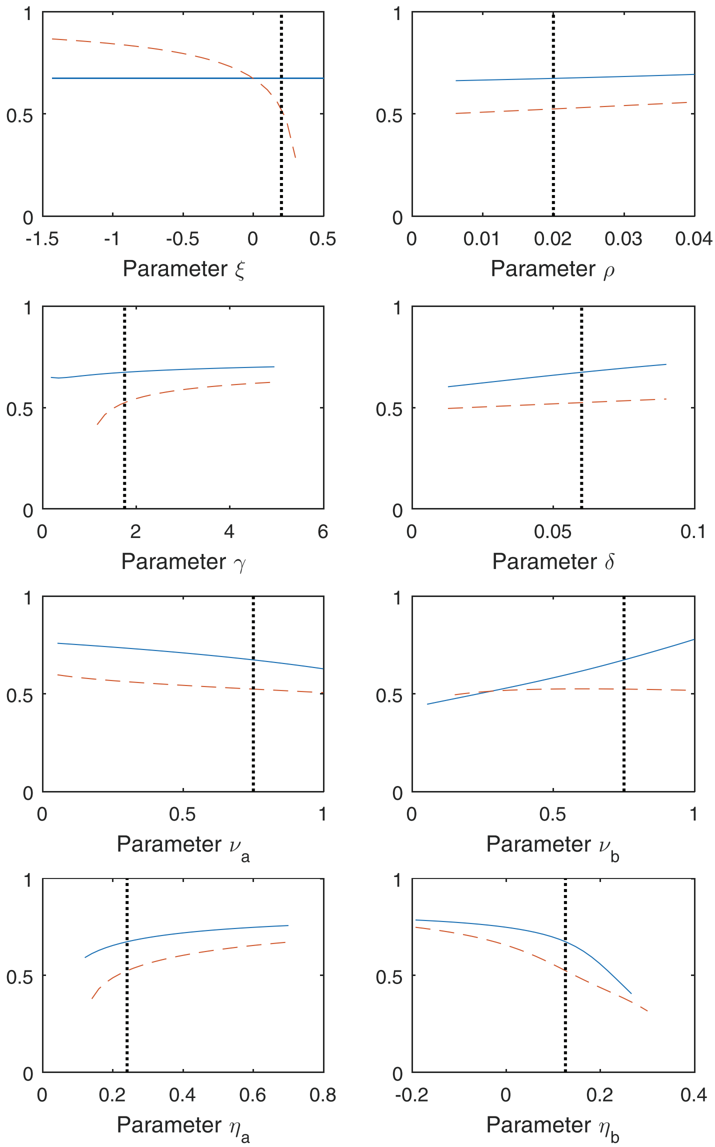
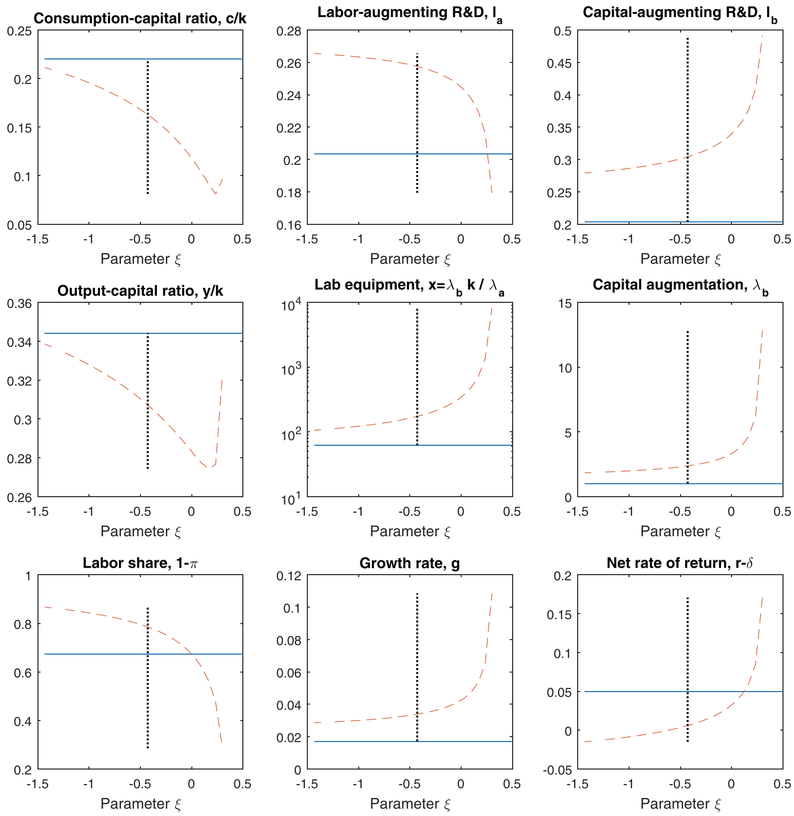


Figure D.4. Dependence of Equilibrium Labor Share on Model Parameters, for the Alternative Calibration of $\sigma = 1.25$ ($\xi = 0.2$)



Notes: $1 - \pi$ on vertical axis; corresponding parameter support on the horizontal axis. Social planner allocation (dashed lines), decentralized equilibrium (solid lines). The vertical dotted line in each graph represents the baseline calibrated parameter value.

Figure D.5. Dependence of BGP on Elasticity of Substitution, DA vs. SP



Notes: Social planner allocation (dashed lines), decentralized equilibrium (solid lines). The vertical dotted line in each graph represents the baseline calibrated parameter value.

References

Acemoglu, D. 2003. "Labor- and Capital-Augmenting Technical Change." *Journal of the European Economic Association* 1 (1): 1–37.

Alvarez-Pelaez, M. J., and C. Groth 2005. "Too Little or Too Much R&D?" *European Economic Review* 49 (2): 437–56.

Antràs, P. 2004. "Is the U.S. Aggregate Production Function Cobb-Douglas? New Estimates of the Elasticity of Substitution." *Contributions to Macroeconomics* 4: 1–34.

Arrow, K., H. B. Chenery, B. S. Minhas, and R. M. Solow. 1961. "Capital-Labor Substitution and Economic Efficiency." *Review of Economics and Statistics* 43: 225–50.

Atkinson, A. B. 2015. *Inequality: What Can Be Done?* Harvard University Press.

Atkinson, A. B., T. Piketty, and E. Saez. 2011. "Top Incomes in the Long Run of History." *Journal of Economic Literature* 49 (1): 3–71.

Azariadis, C. 1996. "The Economics of Poverty Traps — Part One: Complete Markets." *Journal of Economic Growth* 1 (4): 449–86.

Barro, R. J., and X. X. Sala-i-Martin. 2003. *Economic Growth*. MIT Press.

Bénassy, J.-P. 1998. "Is There Always Too Little Research in Endogenous Growth with Expanding Product Variety?" *European Economic Review* 42 (1): 61–69.

Bengtsson, E. 2014. "Labour's Share in Twentieth-Century Sweden: A Reinterpretation." *Scandinavian Economic History Review* 62 (3): 290–314.

Caselli, F. 2005. "Accounting for Cross-Country Income Differences." In *Handbook of Economic Growth*, Vol. 1A, ed. P. Aghion and S. N. Durlauf, 679–741 (chapter 9). North-Holland.

Charpe, M., S. Bridji, and P. McAdam. 2020. "Labor Share and Growth in the Long Run." *Macroeconomic Dynamics* 24 (7): 1720–57.

de La Grandville, O. 1989. "In Quest of the Slutsky Diamond." *American Economic Review* 79 (3): 468–81.

———. 2012. "How Much Should a Nation Save? A New Answer." *Studies in Nonlinear Dynamics and Econometrics* 16 (2): 1–36.

———. 2016. *Economic Growth: A Unified Approach*. Second edition. Cambridge University Press.

de Loecker, J., J. Eeckhout, and G. Unger. 2020. "The Rise of Market Power and the Macroeconomic Implications." *Quarterly Journal of Economics* 135 (2): 561–644.

Dixit, A. K., and J. E. Stiglitz. 1977. "Monopolistic Competition and Optimum Product Diversity." *American Economic Review* 67 (3): 297–308.

Galí, J., and M. Gertler. 1999. “Inflation Dynamics: A Structural Econometric Analysis.” *Journal of Monetary Economics* 44 (2): 195–222.

Groth, C., and J. B. Madsen. 2016. “Medium-term Fluctuations and the ‘Great Ratios’ of Economic Growth.” *Journal of Macroeconomics* 49 (September): 149–76.

Growiec, J. 2007. “Beyond the Linearity Critique: The Knife-Edge Assumption of Steady-State Growth.” *Economic Theory* 31: 489–99.

Growiec, J., P. McAdam, and J. Mućk. 2018. “Endogenous Labor Share Cycles: Theory and Evidence.” *Journal of Economic Dynamics and Control* 87 (February): 74–93.

Growiec, J., and J. Mućk. 2020. “Isoelastic Elasticity of Substitution Production Functions.” *Macroeconomic Dynamics* 24 (7): 1597–1634.

Guo, J. T., and K. J. Lansing. 2009. “Capital–Labor Substitution and Equilibrium Indeterminacy.” *Journal of Economic Dynamics and Control* 33 (12): 1991–2000.

Herrendorf, B., C. Herrington, and A. Valentinyi. 2015. “Sectoral Technology and Structural Transformation.” *American Economic Journal: Macroeconomics* 7 (4): 104–33.

Irmen, A., and A. Tabakovic. 2020. “Factor Income Distribution and Endogenous Economic Growth: Piketty Meets Romer.” *Economic Inquiry* 58 (3): 1342–61.

Jaimovich, N., and H. E. Siu. 2020. “Job Polarization and Jobless Recoveries.” *Review of Economics and Statistics* 102 (1): 129–47.

Jalava, J., M. Pohjola, A. Ripatti, and J. Vilmunen. 2006. “Biased Technical Change and Capital–Labor Substitution in Finland, 1902-2003.” *B.E. Journal of Macroeconomics* 6 (1): 1–20.

Jones, C. I. 1995. “R&D-Based Models of Economic Growth.” *Journal of Political Economy* 103 (4): 759–84.

———. 1999. “Growth: With or Without Scale Effects?” *American Economic Review* 89 (2): 139–44.

———. 2005. “Growth and Ideas.” In *Handbook of Economic Growth*, Vol. 1B, ed. P. Aghion and S. N. Durlauf, 1063–1111 (chapter 16). North-Holland.

Jones, C. I., and J. C. Williams. 2000. “Too Much of a Good Thing? The Economics of Investment in R&D.” *Journal of Economic Growth* 5 (1): 65–85.

Jones, L. E., and R. E. Manuelli. 1990. "A Convex Model of Equilibrium Growth: Theory and Policy Implications." *Journal of Political Economy* 98 (5, Part 1): 1008–38.

Kaas, L., and L. von Thadden. 2003. "Unemployment, Factor Substitution and Capital Formation." *German Economic Review* 4 (4): 475–95.

Karabarbounis, L., and B. Neiman. 2014. "The Global Decline of the Labor Share." *Quarterly Journal of Economics* 129 (1): 61–103.

Kennedy, C. 1964. "Induced Bias in Innovation and the Theory of Distribution." *Economic Journal* 74 (295): 541–47.

Klump, R. 2002. "Verteilung und Wirtschaftswachstum: Eine neoklassische Verallgemeinerung des Wachstumsmodell von Kaldor." In *Zur Theorie, Empirie und Politik der Einkommensverteilung*, ed. L. Menkhoff and F. Sell, 11–26. Berlin-Heidelberg-New York: Springer.

Klump, R., and O. de La Grandville. 2000. "Economic Growth and the Elasticity of Substitution: Two Theorems and Some Suggestions." *American Economic Review* 90 (1): 282–91.

Klump, R., P. McAdam, and A. Willman. 2007. "Factor Substitution and Factor-Augmenting Technical Progress in the US." *Review of Economics and Statistics* 89 (1): 183–92.

———. 2012. "The Normalized CES Production Functions: Theory and Empirics." *Journal of Economic Surveys* 26 (5): 769–99.

Klump, R., and H. Preissler. 2000. "CES Production Functions and Economic Growth." *Scandinavian Journal of Economics* 102 (1): 41–56.

Knoblach, M., and F. Stöckl. 2020. "What Determines the Elasticity of Substitution between Capital and Labor? A Literature Review." *Journal of Economic Surveys* 34 (4): 847–75.

Kortum, S. 1993. "Equilibrium R&D and the Patent-R&D Ratio: U.S. Evidence." *American Economic Review* 83 (2): 450–57.

Laeven, L., P. McAdam, and A. Popov. 2018. "Credit Shocks, Employment Protection, and Growth: Firm-level Evidence from Spain." ECB Working Paper No. 2166.

León-Ledesma, M. A., P. McAdam, and A. Willman. 2010. "Identifying the Elasticity of Substitution with Biased Technical Change." *American Economic Review* 100 (4): 1330–57.

León-Ledesma, M. A., and M. Satchi. 2019. "Appropriate Technology and Balanced Growth." *Review of Economic Studies* 86 (2): 807–35.

Li, C.-W. 2000. "Endogenous vs. Semi-endogenous Growth in a Two-R&D-Sector Model." *Economic Journal* 110 (462): C109–C122.

Liesner, T. 1989. *One Hundred Years of Economic Statistics*. Facts on File Publications.

McAdam, P., and A. Willman. 2004. "Supply, Factor Shares and Inflation Persistence: Re-Examining Euro-Area New-Keynesian Phillips Curves." *Oxford Bulletin of Economics and Statistics* 66 (s1): 637–70.

———. 2013. "Medium Run Redux." *Macroeconomic Dynamics* 17 (4): 695–727.

Mućk, J. 2017. "Elasticity of Substitution between Labor and Capital: Robust Evidence from Developed Economies." Working Paper No. 271, Narodowy Bank Polski, Economic Research Department.

Mućk, J., P. McAdam, and J. Growiec. 2018. "Will the 'True' Labor Share Stand Up? An Applied Survey on Labor Share Measures." *Journal of Economic Surveys* 32 (4): 961–84.

Oberfield, E., and D. Raval. 2018. "Micro Data and Macro Technology." *Econometrica* 89 (2): 703–32.

Palivos, T., and G. Karagiannis. 2010. "The Elasticity of Substitution as an Engine of Growth." *Macroeconomic Dynamics* 14 (5): 617–28.

Pessoa, A. 2005. "'Ideas' Driven Growth: The OECD Evidence." *Portuguese Economic Journal* 4 (1): 46–67.

Piketty, T. 2014. *Capital in the Twenty-First Century*. Harvard University Press. Translated from French by Arthur Goldhammer.

Piketty, T., and G. Zucman. 2014. "Capital is Back: Wealth–Income Ratios in Rich Countries, 1700–2010." *Quarterly Journal of Economics* 129 (3): 1255–1310.

Pitchford, J. 1960. "Growth and the Elasticity of Factor Substitution." *Economic Record* 36 (76): 491–504.

Ramsey, F. P. 1928. "A Mathematical Theory of Saving." *Economic Journal* 38 (152): 543–59.

Romer, P. M. 1990. "Endogenous Technological Change." *Journal of Political Economy* 98 (5, Part 2): S71–S102.

Rowthorn, R. 1999. "Unemployment, Wage Bargaining and Capital–Labour Substitution." *Cambridge Journal of Economics* 23 (4): 413–25.

Saam, M. 2008. "Openness to Trade as a Determinant of the Macroeconomic Elasticity of Substitution." *Journal of Macroeconomics* 30 (2): 691–702.

Solow, R. M. 1960. "Investment and Technical Progress." In *Mathematical Methods in the Social Sciences*, ed. K. Arrow, S. Karlin, and P. Suppes, 89–104. Stanford University Press.

Uzawa, H. 1961. "Neutral Inventions and the Stability of Growth Equilibrium." *Review of Economic Studies* 28 (1): 117–24.