What Can the Data Tell Us about the Equilibrium Real Interest Rate?*

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The equilibrium real interest rate ($r*$) is the short-term real interest rate that, in the long run, is consistent with aggregate production at potential and stable inflation. Estimation of $r*$ faces considerable econometric and empirical challenges, including the "pile-up" problem in which maximum-likelihood estimation may underestimate the time variation in $r*$. In light of these challenges, this analysis considers Bayesian methods and examines the degree to which a researcher's prior views, rather than the data, shapes inference on the $r*$ process. I find that the data provide relatively little information on the $r*$ data-generating process, as the posterior distribution of this process lies very close to its prior. This result contrasts sharply with those for the trend growth or natural rate of unemployment processes. Conditional on a prior view that the $r*$ process involves only small quarter-to-quarter changes in $r*$, estimates of $r*$ show a gradual decline to a value below 1 percent in recent years. Auxiliary properties of the model, such as its output gap, are in line with consensus views.

JEL Codes: E5, E3, E4.

1. Introduction

The equilibrium real interest rate ($r*$) is the short-term real interest rate that, in the long run, is consistent with aggregate production

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at potential and stable inflation. Recent monetary policy discussions focus on $r*$ because it provides a gauge of a "long-run neutral" stance of monetary policy and hence affects assessments of the current stance of monetary policy. As a result, policymaker communications have increasingly emphasized this concept—with estimates reported in the Federal Reserve Board's *Monetary Policy Report to the Congress* in July 2018 and the Bank of England's *Inflation Report* in August 2018 (including, in both cases, an estimate from the model of this paper, based on the working paper version from several years ago).[1] Moreover, fiscal and other policy discussions also depend, in part, on the equilibrium real interest rate, as it affects discounting of future payoffs and hence any assessment of intertemporal and/or budgetary tradeoffs.[2]

However, estimation of $r*$ faces considerable econometric and empirical challenges. In the analysis herein, I demonstrate that the data used in a typical approach to estimation of the equilibrium real interest rate are not very informative regarding the process governing $r*$. As a result, estimates of $r*$ are dependent on modeling choices to an important degree. Indeed, the analysis shows that the degree to which the data inform estimates of the $r*$ process is substantially smaller than the degree to which the data inform estimates of the processes governing other unobserved variables entering the model, such as the processes for trend economic growth or the natural rate of unemployment. In this sense, the data provide even less guidance than for estimates of the natural rate of unemployment, a concept that has long been judged very difficult to measure (e.g., Staiger, Stock, and Watson 1997).

These empirical challenges are not surprising, and they reflect the fact that there is relatively little trend movement in interest rates.

[1]Policymaker interest in the equilibrium real interest rate predates this recent focus. While some notion of "equilibrium" real interest rates has a long tradition—dating to at least Knut Wicksell—there are also a number of alternative notions of "equilibrium," some of which are more short term and some of which are more long term. Ferguson (2004) presents a policymaker's assessment of how these various definitions contribute to (or confuse) policy discussions. Section 2 below references related research. Yellen (2015) represents an early discussion that arguably initiated recent focus among policymakers.

[2]For a policy-focused discussion, see the summary of work at the Council of Economic Advisers in Obstfeld and Tesar (2015).

One manifestation of this challenge confronted in classical econometric inference is the "pile-up" problem. As emphasized in Stock and Watson (1996), maximum-likelihood estimation of models in which a time series contains a small permanent component and a sizable transitory component tends to drive the estimated role of the permanent component toward zero—that is, the estimated variance of this component "piles up" near zero. Because real interest rates contain (at most) only a small permanent component, the "pile-up" problem is severe in analyses of the equilibrium real interest rate (Laubach and Williams 2003).[3]

In order to both illustrate and ameliorate these challenges, the analysis herein takes a Bayesian approach. Bayesian methods aid inference in two ways. First, the focus on the posterior distribution of the variance of the permanent component provides a more comprehensive lens with which to examine the range of reasonable settings for this parameter than the focus on point estimates associated with classical inference. This ameliorates the pile-up problem by removing focus on the point estimate. (More technically, albeit less central to my discussion of the economic issues, Kim and Kim 2013 suggest that the "pile-up" problem is much less severe for the Bayesian approach. This result is closely related to the well-known result that Bayesian inference regarding unit roots requires no special assumptions, in contrast to the classical approach (e.g., Sims and Uhlig 1991).)

Perhaps more importantly for understanding the information in the data for the $r*$ process, the Bayesian approach allows the researcher to impose a prior that the variance of the permanent component is likely to be nonzero, but not large, and then to assess the information in the data relative to this prior through examination of the entire posterior distribution. That is, the Bayesian approach allows the researcher to see how much the data has shifted the assessment of the parameters of interest. My analysis goes further and considers how alternative priors influence the posterior distributions, thereby illustrating further how much results depend on

[3]Since the working paper version of this analysis was released, Laubach and Williams have updated their analysis, including with coauthors; the main points herein are invariant to these updates, as reported later.

data and how much results depend on the specification brought to the data.

My primary finding is that the data provide relatively little information on the $r*$ data-generating process; indeed, the posterior distribution of this process lies very close to the prior distribution. This result contrasts sharply with those for the trend growth or natural rate of unemployment processes, for which the data are very informative. With this result in hand, estimates of $r*$ are presented for a prior specification in which the variance of innovations to the equilibrium real interest rate is moderate. While these estimates of $r*$ are highly uncertain, they are more stable than other estimates, with $r*$ at the end of 2017 between 0 percent and 1 percent at the posterior mode of the parameters. Looking over past history, the distribution of $r*$ values consistent with the posterior distribution of model parameters is highly asymmetric over the business cycle. For example, the central tendency of this distribution does not move much over the business cycle, but a tail follows the path of the actual real federal funds rate. This occurs because the posterior distribution of the variance of the permanent component of the real interest rate places some (small) probability on this component being fairly large and hence accounting for a notable portion of movements in the real interest rate.

Finally, I also compare auxiliary implications of the models herein to those from the frequently cited study of Laubach and Williams (2003). The models herein yield estimates of the output gap very similar to those from the Congressional Budget Office, in contrast to that of Laubach and Williams (2003).

2. Previous Literature on the Equilibrium Real Interest Rate

I have defined the equilibrium real interest rate ($r*$) as the level of the real (short-term) interest rate that is consistent, in the long run, with output at potential, unemployment at its natural rate, and inflation at the monetary policymaker's long-run objective (Laubach and Williams 2003). This concept is a long-run notion and may vary over time because of changes in the rate of economic growth, the degree of international capital mobility, or other factors that affect the interest rate that balances savings and investment.[4] In the approach herein, $r*$ is estimated using a "trend/cycle" model. That is, the approach herein is statistical in nature and identifies $r*$ as the long-run random-walk component. (Formally, this is simply the Beveridge-Nelson trend for the real interest rate, estimated within a restricted econometric system of equations.)

Laubach and Williams (2003) introduce the approach taken herein—a relatively simple trend/cycle approach to estimate the equilibrium real interest rate. In their model, fluctuations in output and inflation move along an "IS curve" linking the output gap to the real interest rate and a Phillips curve linking inflation and the output gap. This system treats the level of potential output, the output gap, and the equilibrium real interest rate as unobservable state variables, and derives estimates of these concepts using output, inflation, and the real interest rate as observable variables via the Kalman filter. Estimates from their model for the U.S. equilibrium real interest rate are updated frequently on the website of the Federal Reserve Bank of New York, and their measure was about 3/4 percent in the fourth quarter of 2017 (as of August 2018).[5]

[4]The equilibrium rate of interest is distinct from the short-run Wicksellian natural rate of interest, which is the level of the real (short-term) interest rate that is consistent with price stability in the short-to-medium run (e.g., Woodford 2003; Edge, Kiley, and Laforte 2008; and Curdia et al. 2015). In particular, the natural rate of interest should be expected to fluctuate considerably over the business cycle—that is, it contains transitory components; in contrast, the equilibrium real interest rate consists solely of the permanent component and is likely to fluctuate less, if at all, over the business cycle and instead should evolve slowly over time. In the literature, the distinction between the equilibrium real interest rate and the natural rate is not always drawn as finely as herein; as a result, these definitions provide guidance that should help the reader understand the analysis, but care must be taken when comparing the discussion herein to that in the broader literature. Kiley (2013) notes the long record, emphasized by Paul Samuelson many decades ago, of economists to use the same term for different concepts, or different terms for the same concept, in discussions of unobserved economic states.

[5]The simple IS curve used in Laubach and Williams (2003) ignores factors other than the short-term real interest rate (relative to its equilibrium rate) that affect the output gap. Research has also highlighted the importance of a broad array of financial conditions in the business cycle: Gilchrist and Zakrajsek (2012) and Kiley (2014b) document important contributions of credit spreads to output movements (using, respectively, forecasting techniques and simple "IS-curve" analysis). Borio, Disyatat, and Juselius (2013) and Arseneau and Kiley (2014)

Clark and Kozicki (2005) highlight a number of challenges associated with the estimation of the equilibrium real interest rate using the trend/cycle approach. First, as emphasized in Laubach and Williams (2003), classical inference regarding the equilibrium interest rate is plagued by the "pile-up" problem associated with estimation of a small permanent component of a time series with substantial short-run variation (as discussed in Stock and Watson 1996). Second, the filtered estimates of $r*$ usually differ substantially between the "one-sided" and "two-sided" estimates—a result that holds even if the true population parameters of the model are known, because estimation of long-run trends is more precise after substantial data has been accumulated when time series have important short-run components. Finally, "real-time" challenges associated with measurement and data revisions are substantial. For each of these reasons, Clark and Kozicki (2005) suggest that $r*$ estimates may have limited use in policy discussions.[6]

The model developed herein draws on the approach of Laubach and Williams (2003). In particular, the model consists of an IS curve and Phillips curve. These equations are augmented with an equation linking cyclical fluctuations in unemployment to those in output (i.e., an "Okun's law" equation).[7] This addition nods in the direction of the literature that emphasizes the role of labor market indicators in assessments of resource-utilization gaps (e.g., Basistha and Startz 2008 and Fleischman and Roberts 2011).

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have found that credit measures may improve estimates of the output gap and natural rate of unemployment. Moreover, the fiscal stance can vary substantially, with effects on aggregate production and income (Follette and Lutz 2010). The working paper version of this research examined the effect of such factors on estimates of the $r*$ process. While such additional factors alter parameter estimates a bit, they do not affect my assessment of the degree to which the data inform estimates of the $r*$ process, and hence these considerations are not explored in this analysis.

[6]Orphanides and Williams (2007) analyze this idea in detail and argue that imprecision in estimates of $r*$ point to a limited role for such estimates in policy discussions. This finding is closely related to the challenges related to using output gap estimates in policy discussions (e.g., Orphanides and van Norden 2002).

[7]A large literature has focused on fluctuations in the unemployment gap and natural rate of unemployment, rather than the output gap; perhaps the most well-known reference is Staiger, Stock, and Watson (1997).

An alternative approach looks at long-term interest rates to examine movements in $r*$, with Bomfim (2001) representing an early example of this approach. Because long-term interest rates reflect expectations for the path of interest rates in the future, it is reasonable to expect such rates to provide information about market participants' expectations for $r*$. Bauer and Rudebusch (2017) is a recent example of research pursuing this line of reasoning. At the same time, this approach involves computational challenges, as the effective lower bound (ELB) on nominal interest rates affects expectations for future interest rates and this nonlinearity cannot be gauged through simple trend/cycle decompositions using the Kalman filter. Johannsen and Mertens (2018) explore the effect of the ELB on estimates of $r*$.

Finally, a number of researchers examine the equilibrium real interest rate outside of the unobserved-component/state-space modeling approach. One approach is to examine factors that have contributed to the average level of interest rates across decades and countries, with guidance from economic theory. For example, Pescatori and Furceri (2014) and Hamilton et al. (2015) consider the links between average real interest rates and factors such as the rate of potential growth in output (globally or within individual countries) or the global saving rate. Summers (2014) perceives an imbalance between global saving and investment, stemming in part from reduced growth prospects in advanced economies, as a motivation for a persistently low equilibrium real interest rate (and the risk of prolonged "secular stagnation"—that is, a state in which aggregate demand falls persistently short of aggregate supply). Gagnon, Johannsen, and López-Salido (2016) consider the influence that demographic factors may have had on the equilibrium real interest rate in a calibrated dynamic general equilibrium model, and Del Negro et al. (2017) examine the role of safety/liquidity premiums in an estimated dynamic stochastic general equilibrium model.

3. Implementation and Results

3.1 The Model

The model includes equations for the dynamics of output y, inflation Δp, and the unemployment rate u. Output and unemployment have permanent, or trend, and cyclical components. The set of equations governs the cyclical dynamics of inflation π (a Phillips curve), unemployment u (Okun's law), and the output gap y (the IS curve). The cyclical component of a variable is denoted by a superscript "c" and the trend component by a superscript "T" (e.g., $X = X^c + X^T$).

3.1.1 Cyclical Dynamics

Three equations describe the cyclical dynamics of the model. Because $r*$ captures the real interest rate necessary to maintain output at potential over the long run, the key equation is the link between interest rates and output; indeed, the simple model of Laubach and Williams (2003) relies on this equation for identification of both the business cycle (output gap) and $r*$. I specify the links between (the cyclical component of) output and real interest rates via an "IS curve" similar to that in related research:

$$y_t^c = \gamma_r \sum_{j=1}^{2} (r_{t-j} - r_{t-j}^*) + \rho_1 y_{t-1}^c + \rho_2 y_{t-2}^c + e_t^{yc}. \tag{1}$$

The parameter γ_r governs the interest sensitivity of output, while ρ_1 and ρ_2 determine the autocorrelation patterns.

Unemployment is determined by Okun's law:

$$u_t^c = -\beta \left(0.4 y_t^c + 0.3 y_{t-1}^c + 0.2 y_{t-2}^c + 0.1 y_{t-3}^c\right). \tag{2}$$

Note that the equation assumes that unemployment gap fluctuations are a (fixed) distributed lag of output gap fluctuations, capturing the well-known tendency for unemployment to lag movements in output. This relationship is drawn from Braun (1990).

Inflation dynamics are governed by

$$\pi_t = s_1 \pi_{t-1} + s_2 \frac{\sum_{j=2}^{4} \pi_{t-j}}{3} + (1 - s_1 - s_2) E_{t-1} \pi^{LR} + \alpha_u u_t^c + e_t^{\pi}. \tag{3}$$

In the Phillips curve, inflation is influenced by its own lags, the unemployment gap, and the level of long-run inflation expectations. The role for long-run inflation expectations in the Phillips curve has considerable empirical support (e.g., Kiley 2014a, 2014b), and survey measures of long-run inflation expectations will be used in the empirical analysis.

3.1.2 Trends

Output, unemployment, and the real interest rate include trend components.

The trend for output y^T includes I(1) and I(2) components. The I(1) component is a shift to the trend level:

$$y_t^T = y_{t-1}^T + g_t + e_t^{yt}. \tag{4}$$

The I(2) component is a shift to the trend growth rate g:

$$g_t = g_{t-1} + e_t^{yg}. \tag{5}$$

The trend for the unemployment rate is a random walk (as in Staiger, Stock, and Watson 1997 and related literature):

$$u_t^T = u_{t-1}^T + e_t^{ut}. \tag{6}$$

Finally, the equilibrium real interest rate is a random walk:

$$r*_t = r*_{t-1} + e_t^{rt}. \tag{7}$$

The specification follows Laubach and Williams (2003). Note that the model herein does not link movements in the equilibrium real interest rate to the trend growth rate. Clark and Kozicki (2005) and Hamilton et al. (2015) question the strength of this relationship, and I take the approach of including as few parameters as possible to assess movements in the equilibrium real interest rate.

Before turning to the observation equations, two aspects of the model structure are noteworthy. First, the model assumes that the trend growth rate, the trend unemployment rate, and the long-run/trend interest rate $r*$ contain unit roots—despite the fact that economic growth, interest rates, and the unemployment rate in the United States have fluctuated within a fairly narrow range. This specification is adopted because it is common in the literature and allows the model to capture the low-frequency movements in economic growth per capita, interest rates, and unemployment. Second, the basic structure of the model and the intuition for how key parameters determine the evolution of $r*$ through the Kalman filter follow directly from the standard signal-extraction problem developed in Muth (1960). Recall that Muth (1960) shows that the optimal estimate of the long-run (trend) value of a variable ($\hat{x}_t^T$) that consists of the sum of normally distributed random-walk and iid noise components ($x_t = x_t^T + x_t^N$) is given by

$$\hat{x}_t^T = (1-\lambda)\sum_{i=0}^{\infty}\lambda^i x_{t-i} \tag{8}$$

$$\lambda = 1 + \frac{1}{2}\frac{\sigma_T^2}{\sigma_N^2} - \frac{\sigma_T}{\sigma_N}\sqrt{1 + \frac{1}{4}\frac{\sigma_T^2}{\sigma_N^2}},$$

where σ_T and σ_N are the standard deviations of the trend and noise components of the observed variable. If variation in the trend component is small relative to that in the noise, estimates of the trend are insensitive to recent data (i.e., $lim_{\frac{\sigma_T}{\sigma_N}\to 0} = 1$; conversely, if variation in the trend component is large relative to that in the noise, estimates of the trend are very sensitive to recent data (i.e., $lim_{\frac{\sigma_T}{\sigma_N}\to\infty} = 0$). In the model herein, signal extraction is substantially more complicated. However, a special case fixes intuition. If the output gap and real interest rate are observed, all the parameters of the model are known, and $r*$ is not observed, then the observed error in equation (1) equals $-2\gamma r *_t + e_t^{yc}$. In this case, the relevant signal-to-noise ratio is the ratio of the standard deviations of the innovation to $r*$ (multiplied by the interest elasticity of output) to that of the error in the IS curve. The data shape this signal-to-noise ratio if the data inform estimates of the posteriors of these two parameters.

3.1.3 Observation Equations

Finally, a set of observation equations for output and unemployment link the model back to observable data. The inflation observation equation is the Phillips curve (as its dependent variable is observed inflation).

Output is the sum of its trend and cycle,

$$y_t = y_t^T + y_t^c, \tag{9}$$

as is the unemployment rate,

$$u_t = u_t^T + u_t^c. \tag{10}$$

Note that the other observable variables that enter the cyclical equations (the real interest rate and long-run inflation expectations) are treated as exogenous to the system.

3.2 Comparison to Laubach and Williams (2003)

As much of the related literature builds off Laubach and Williams (2003), it is useful to summarize the similarities and differences of the model herein relative to their specification. Both models include IS and Phillips curves. While the IS curve herein is similar to that in Laubach and Williams (2003), the Phillips curve herein includes a role for long-term inflation expectations, following the literature summarized in Kiley (2014a, 2014b), whereas Laubach and Williams (2003) use an acclerationist Phillips curve. The specifications of stochastic trends in output and the real interest rate are similar, except that Laubach and Williams (2003) impose a one-to-one pass-through of changes in trend growth to $r*$, whereas the specification herein assumes no such link, as suggested by Clark and Kozicki (2005). Finally, the approach herein includes the unemployment rate and Okun's law, as research suggests this aids estimation of the output gap in the United States (e.g., Basistha and Startz 2008).

3.3 Data

A look at the data helps explain some of my later results. Output is measured by real gross domestic product, and the empirical analysis considers real GDP divided by the civilian non-institutional population aged 16 and over to remove the effects on trend growth associated with underlying population trends. The unemployment rate is for the civilian non-institutional population aged 16 and over; inflation is measured by the (log) change in the price index for personal consumption expenditures (PCE) excluding food and energy expressed at an annual rate (e.g., core PCE inflation, as the primary interest herein is in the relationship between slack and inflation, and the effects of volatile food and energy prices are largely orthogonal to this interest). Finally, the real interest rate is measured by the nominal federal funds rate minus the change in core PCE prices from four quarters earlier. (This backward-looking measure of the real interest rate is commonly used in related studies.[8])

Focusing first on variables common to trend/cycle models and as reported in figure 1, the unemployment rate shows a clear cycle around recessions (as identified by the National Bureau of Economic Research); this cycle, in conjunction with Okun's law, is highly informative about the overall business cycle. Indeed, a casual look at the change in real GDP, which is volatile from quarter to quarter, suggests that it may be useful to inform an assessment of the state of overall resource utilization with labor market data. Inflation has a highly persistent component over the period since 1960 but has been fairly stable around 2 percent for the past two decades. Finally, the real interest rate appears to have both a clear cyclical component and a lower-frequency component—consistent with the general idea that there may be some trend component to the real interest rate.

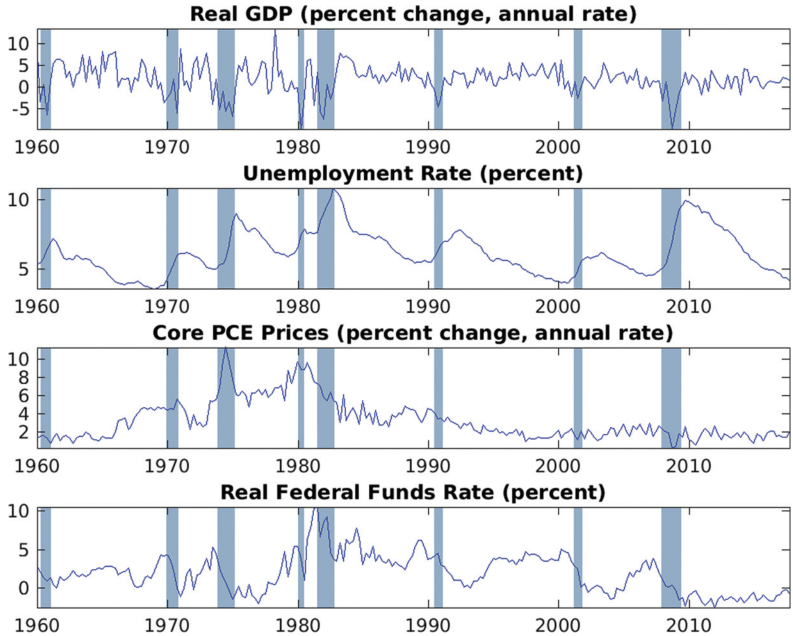
3.4 Estimation

3.4.1 Estimation Strategy

Output, inflation, unemployment, the real interest rate, and long-run inflation expectations are observed variables. Output, inflation, and unemployment are endogenous in the model, and other controls are treated as exogenous. This follows Laubach and Williams (2003) and Clark and Kozicki (2005). (An obvious extension is to treat other observables as "endogenous" within the model. Johannsen and Mertens (2018) pursue this idea by including a specification for the endogenous evolution of the short-term interest rate. I do

[8]Real GDP and the core PCE price index are from the Bureau of Economic Analysis; population and the unemployment rate are from the Bureau of Labor Statistics; and the nominal federal funds rate is from the Federal Reserve Board. All data are taken from the Federal Reserve's FRB/US model database, including any adjustments made by Federal Reserve staff. Brayton, Laubach, and Reifschneider (2014) describes the most recent FRB/US model and an easily accessible version of the data.

Figure 1. Key Data Series: Output, Unemployment, Inflation, and the Real Interest Rate



Sources: Real GDP and PCE prices, Bureau of Economic Analysis; unemployment rate, Bureau of Labor Statistics; federal funds rate, Federal Reserve Board. Inflation and real interest rate computed by author. Data accessed from Federal Reserve FRB/US database available at http://www.federalreserve.gov/econresdata/frbus/us-models-package.htm.
Note: Data accessed July 2, 2015. Shaded regions indicate recessions as identified by the National Bureau of Economic Research.

not adopt this approach, in part because treating the real interest rate as endogenous requires addressing the effects of the effective lower bound.)

The errors in all equations are assumed to follow a normal distribution, implying that the Kalman filter is the optimal method for parsing the data into trend and cycle. We construct the likelihood of the data via the Kalman filter. (Note that the zero lower bound, a nonlinearity that affects the course of the federal funds rate, does not imply that a nonlinear alternative to the Kalman filter is required, as the (real) federal funds rate is an exogenous variable in the system.)

A long literature has emphasized that estimation of "trend/cycle" decompositions via maximum likelihood is problematic because the variance of the trend processes for economic growth "pile up" near zero (Stock and Watson 1996). Bayesian methods do not face the same problems (Kim and Kim 2013). As earlier work on $r*$ has noted the empirical challenges associated with the pile-up problem, a Bayesian approach is attractive. In addition, a Bayesian approach allows imposition of prior views that capture common discussions in the literature, and a clear discussion of the information in the data through comparison of the prior and posterior distributions.

For these reasons, I take a Bayesian approach. The objective is to estimate the parameter vector θ. Under the Bayesian approach, a prior distribution, represented by the density $p(\theta|\mathcal{M})$, is combined with the likelihood function $p(\mathcal{X}^o|\theta, \mathcal{M})$ for the observed data $\mathcal{X}^o(= \{X_t\}_{t=1}^T)$, to obtain, via Bayes's rule, the posterior:

$$p(\theta|\mathcal{X}^o, \mathcal{M}) \propto p(\mathcal{X}^o|\theta, \mathcal{M})p(\theta|\mathcal{M}).$$

To facilitate estimation, I must access the posterior $p(\theta|\mathcal{X}^o, \mathcal{M})$. Unfortunately, the posteriors are analytically intractable, owing to the complex ways θ enters the likelihood function. To produce draws from the posteriors, I resort to Markov-chain Monte Carlo.[9]

I employ a normal(0,2) prior for all "cyclical" parameters in the equations. Given the scaling of variables in the model, this prior is fairly uninformative.

The priors for the standard errors of the cyclical and trend equations assume an inverted gamma distribution. In the cyclical equations, I assume that the prior of the standard error of the shocks has a mean and standard error of 2. In the trend equations, I assume that the prior of the standard error of the shocks has a mean and standard error of 0.25. These assumptions imply the prior view that fluctuations in trend growth g, $r*$, and U^T are modest relative to cyclical variation. Our subsequent analysis will compare these prior distributions with the posterior distributions to highlight the information in the data for $r*$, and consider the robustness of the results to alternative choices.

[9] All my estimation procedures use Dynare (Adjemian et al. 2011).

Table 1. Moments of Posterior Distributions of Parameters

	Mean	Std. Dev.		Mean	Std. Dev.
IS Curve			**Standard Errors of Trends**		
ρ_1	1.30	0.11	e^{rt}	0.27	0.20
ρ_2	−0.34	0.10	e^{yt}	0.36	0.08
γ_r	0.09	0.04	e^{ut}	0.16	0.01
	—	—	e^{yg}	0.08	0.01
Okun's Law					
β	−0.56	0.05			
Phillips Curve			**Standard Errors of Dynamic Equations**		
$s1$	0.63	0.07	e^{π}	0.80	0.04
$s2$	0.15	0.08	e^{yc}	0.61	0.06
α_u	−0.10	0.05			

3.4.2 Results

My estimation sample spans from 1960:Q1 to 2017:Q4. The mean and standard deviations of the posterior distributions for each parameter, including shock standard errors, are reported in table 1.

The system has relatively typical properties: The AR(2) process for output shows a coefficient on the first lag (ρ_1) greater than 1 and a negative coefficient on the second lag (ρ_2); the coefficient on the output gap in the Okun's law equation (β) is near (but slightly greater than) 1/2 (similar to Fleischman and Roberts 2011), and the estimated mean and standard error of the posterior distribution for the Phillips-curve relationship between inflation and unemployment suggests that unemployment is not strongly associated with downward pressure on inflation ($\alpha_u > 0$, but small and estimated somewhat imprecisely).

The standard error estimates in table 1 are particularly informative about the degree to which properties of the system are informed by the data. In particular, the posterior distributions suggest that the standard error for the trend growth process and the process for the natural rate of unemployment are fairly tight around their estimated posterior means—with each posterior mean notably different than the prior mean (assumed to equal 0.25). In contrast, the mean of the posterior distribution for the $r*$ process lies close to the prior mean of 0.25 and has a large standard error. Moreover, the estimated standard error of the IS curve is precisely estimated at a value far from its prior mean, indicating that the data shape this posterior importantly. As highlighted when discussing the intuition behind the signal-extraction problem for $r*$ from equation (8), this pattern suggests that prior information is importantly shaping the estimated path for r*, rather than the data. The next section examines more closely the information in the data for the processes of the (unobserved) $r*$, g, and U^T processes and compares the model herein to that of Laubach and Williams (2003).

4. Implications of the Estimation Results

4.1 The Information in the Data

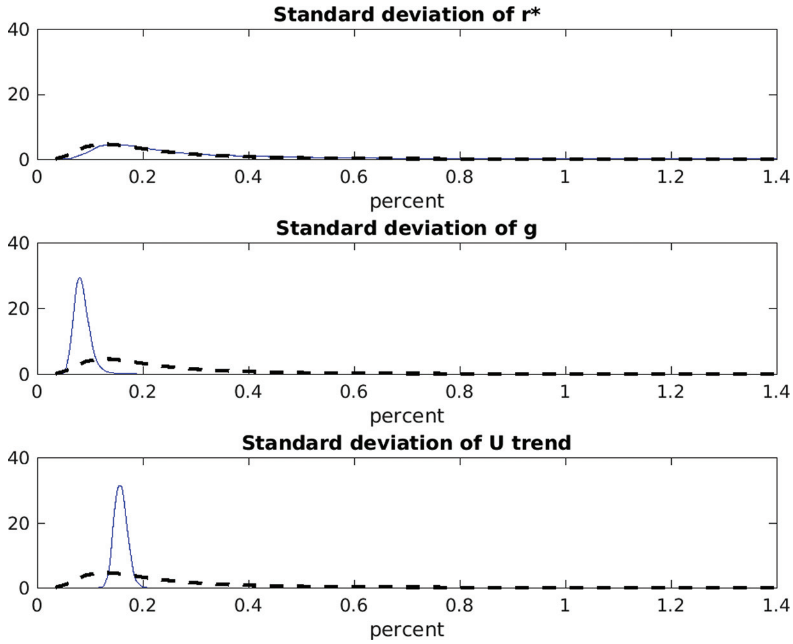
To illustrate the degree of precision, relative to the prior, in the posterior distribution of the $r*$ process, figure 2 plots the prior and posterior distributions for the standard error of the low-frequency components of $r*, g$, and U^T.

The posterior distribution for $r*$ is about as diffuse and centered on similar values as the prior distribution, indicating that the data contain relatively little information about the $r*$ process. This result contrasts sharply with those for the $g*$ and U^T processes, where the posterior distribution is much tighter than the prior distribution.

These results suggest that researchers with alternative prior views about the degree to which $r*$ has shifted over time are unlikely to be swayed by the information in the relationship between output and short-term interest rates—there is simply too little information in such co-movement over the past 50 years to provide much guidance.

To illustrate the role of the prior in shaping parameter estimates, and following the suggestion in Mueller (2012), I shifted the prior mean of each of the standard deviations for the trends and examined the consequent change in the posterior mean; specifically, I

Figure 2. Prior and Posterior Distributions of Standard Errors of Trends



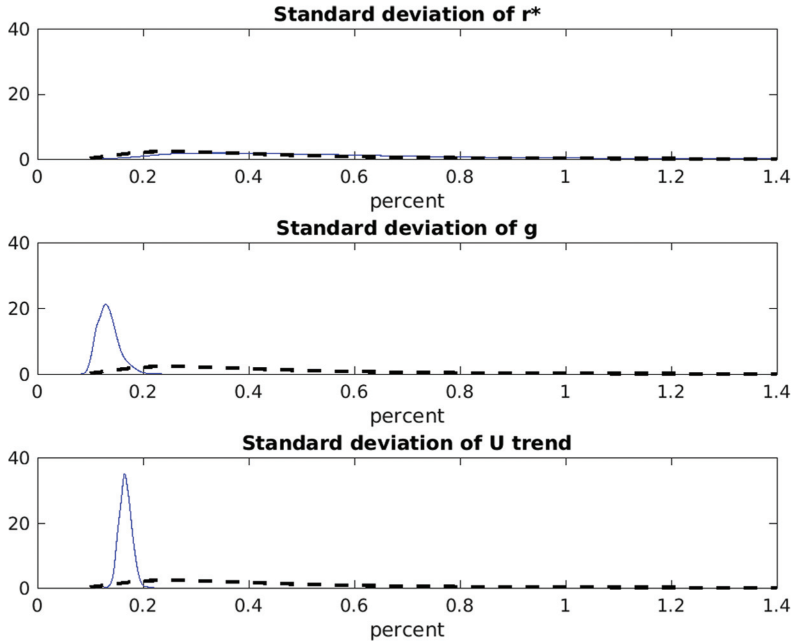
Source: Author's computations from model estimates.

Table 2. Ratio of Change in Posterior Mean of Trend Innovations' Standard Deviations to Change in Prior Mean

	e^{rt}	e^{yg}	e^{ut}
$\Delta E\sigma^{j,post}/\Delta E\sigma^{j,prior}$	1.00	0.24	0.04

doubled the prior mean from 0.25 to 0.5. Table 2 reports the ratio of the change in posterior mean to the change in prior mean for the standard deviations of $r*$, g, and U^T. This ratio is one—complete pass-through—for the $r*$ process, but much smaller for the g and U^T processes. This indicates that the data are not very informative for the $r*$ process—that is, the posterior moves with the prior.

Figure 3. Prior and Posterior Distributions for Standard Errors of Trends, Prior with Higher Mean for Standard Errors



Source: Author's computations from model estimates.

Figure 3 compares the estimates of the priors and posteriors for the standard deviations of $r*$, g, and U^T under the priors with the higher mean of 0.5. As before, the posterior for $r*$ resembles the prior, while those for the other trend processes are informed by the data.

All told, the exercises suggest that model choices, rather than the data, strongly affect the estimate for the $r*$ process—in contrast to results for the natural rate of unemployment or the trend growth rate of output. In the analysis herein, this dependence on prior views is explicitly outlined through analysis of the effects of prior views on posterior estimates. However, the results herein apply more generally, including to analyses that do not adopt a Bayesian perspective. It is common in the literature to impose parameter restrictions on estimation and make other modeling choices. Moreover, the $r*$ literature has noted that the pile-up problem makes identification of key parameters a challenge. The explicit Bayesian approach herein highlights the import of these assumptions and challenges.

4.2 Estimates of $r*$

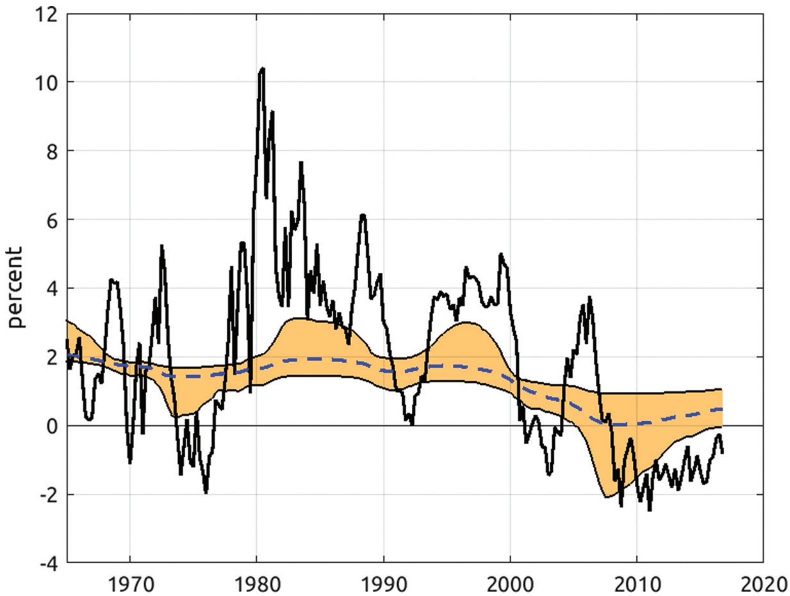
While the process for $r*$ is closely related to the assumed prior distribution, this prior captures the common view (as captured in the literature developing and using $r*$) that there may be some low-frequency component to the equilibrium short-term real interest rate. As a result, examination of the model's estimate of $r*$ and the range of estimates consistent with the posterior distribution of the model's parameters carries information about both where the equilibrium rate may currently stand and the degree to which alternative views are "reasonable" in light of the model. Figure 4 presents the smoothed (or two-sided) estimate of $r*$ along with the 68 percent confidence set based on the posterior distribution of model parameters.[10]

As illustrated by the comparison of the black line and the blue-dashed line (as well as the associated confidence set), the overwhelming share of movements in the real interest rate are viewed by the model as movements around $r*$, with only modest movements in $r*$. (For color versions of figures, see http://www.ijcb.org.) From the early 1960s through the 1980s, $r*$ is estimated to lie near 2 percent, close to the equilibrium rate assumed in the now-classic analysis of monetary policy rules in Taylor (1993). In recent years, the point estimate of $r*$ reached values between 0 percent and 1 percent.

The lack of information in the data on the $r*$ process, as indicated by the proximity of the posterior distribution of the standard error of e^{rt} to its prior distribution shown in table 2 and figures 2 and 3, implies that the assumed prior for the $r*$ process has large effects on the estimate of $r*$ delivered by the model. This sensitivity can be seen in figure 5, which plots the estimate of $r*$ under the baseline prior (the blue-dashed line) and under the prior in which

[10]This confidence set describes the range of point estimates for $r*$ implied by the distribution of coefficients and also accounts for filtering uncertainty.

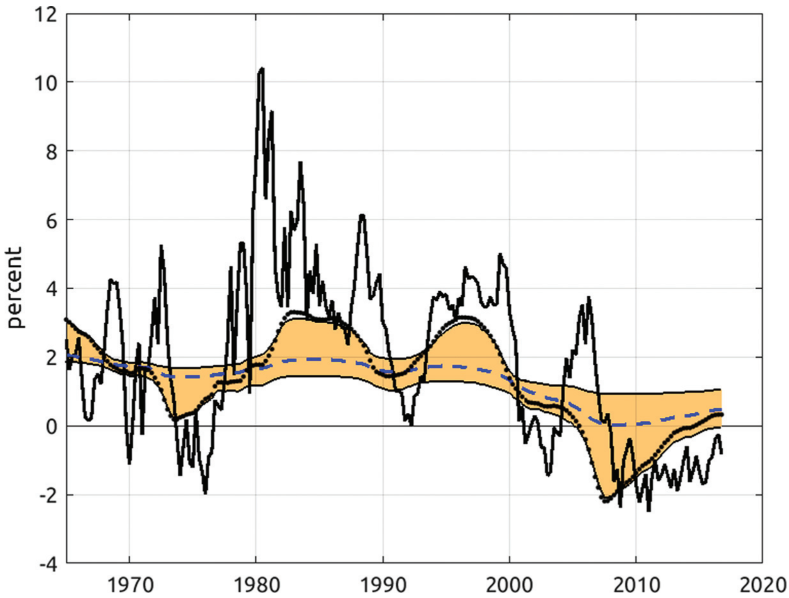
Figure 4. Estimate of $r*$ for Baseline Priors



Source: Author's computations from model estimates.
Notes: The black line is the observed (real) federal funds rate. The blue-dashed line is the median estimate of $r*$ implied by the posterior of the parameters. The shaded region is the 68 percent confidence set of $r*$ implied by the posterior of the parameters.

the mean of the standard error of e^{rt} is doubled (the black-dotted line). Because the data are relatively uninformative about the $r*$ process, the posterior under the alternative prior centers on a $r*$ process with greater variation. As a result, $r*$ shows more variation at a business cycle frequency—for example, falling to -2 percent in 2008 and recovering subsequently. This cyclical variation in $r*$ under the alternative prior may be unappealing; indeed, a prior distribution that avoids such variation is a key driver for the assumed baseline prior I have used. That said, the key analytical insight is that the data do not shift the posterior from the assumed prior, pointing to a need for researchers to consider carefully the degree to which their results on $r*$ are driven by a priori assumptions versus the information in the data.

Figure 5. Estimate of $r*$ for Alternative Priors



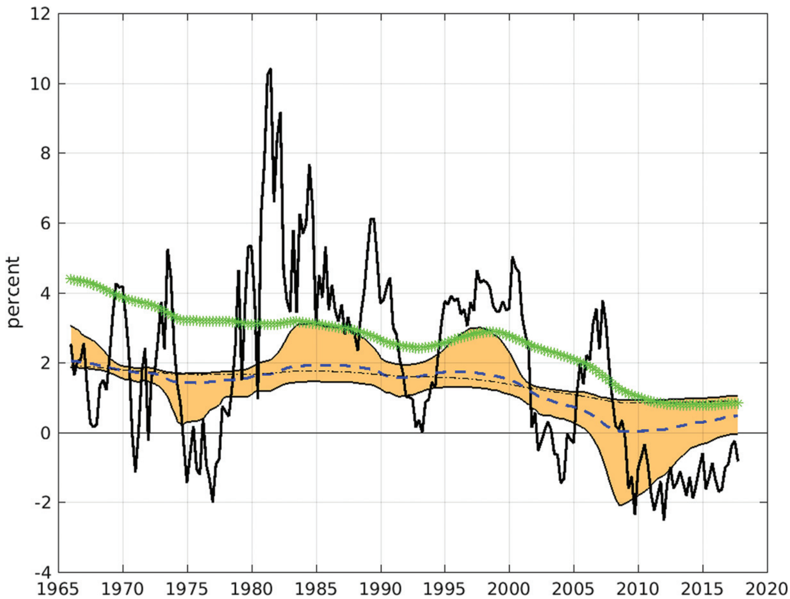
Source: Author's computations from model estimates.
Notes: The black line is the observed (real) federal funds rate. The blue-dashed line is the median estimate of $r*$ implied by the posterior of the parameters under the baseline prior. The shaded region is the 68 percent confidence set of $r*$ implied by the posterior of the parameters under the baseline prior. The black-dotted line is the median estimate of $r*$ implied by the posterior of the parameters under the alternative prior shown in figure 3.

4.3 Comparison to Laubach and Williams Results

Given the role their estimates have played in some policy discussions, it is useful to examine the estimates from the model herein to that produced by the Laubach and Williams (2003) model (figure 6). As in figure 4, the black and the blue-dashed lines present the data and $r*$ estimate, respectively, from the approach herein while the shaded region is the 68 percent confidence set. The green-starred line is the estimate from Laubach and Williams (2003).[11]

[11] This is the two-sided estimate as reported on the website of the Federal Reserve Bank of New York as of August 31, 2018.

Figure 6. Estimate of $r*$ from Laubach and Williams (2003)



Sources: Author's computations from model estimates and Federal Reserve Bank of New York (https://www.newyorkfed.org/research/policy/rstar).
Notes: The black line is the observed (real) federal funds rate. The blue-dashed line is the median estimate of $r*$ implied by the posterior of the parameters under the baseline prior. The shaded region is the 68 percent confidence set of $r*$ implied by the posterior of the parameters under the baseline prior. The green-starred line is the estimate from the model of Laubach and Williams (2003).

The comparison illustrates that the model behaves quite differently from Laubach and Williams (2003)—both over history and in the recent sample. Indeed, the confidence set suggests that there are not likely coefficient combinations for the specification herein that would lead to estimates similar to those in Laubach and Williams (2003). In particular, the model of Laubach and Williams (2003) implies a very high value for the real interest rate—near 4 percent—early in the sample and a sizable decline to below 1 percent in recent years. Note that these movements do not match some view of the conventional wisdom, in which the long-run real interest rate was viewed as near 2 percent (Taylor 1993). In contrast, the model herein estimates that $r*$ was near 2 percent for most of the period prior to the most recent decade and a half.

The large decline in the Laubach and Williams (2003) estimate from near 4 percent to below 1 percent result is not surprising. Laubach and Williams (2003) estimate a high value for the standard error of $r*$ using median-unbiased estimation techniques, leading to a model that estimates a large decline in trend. Under the prior view that $r*$ is not highly variable, the estimated variance of the permanent component of interest rates in Laubach and Williams (2003) is judged unlikely by the approach herein. This result does not imply that the estimates in Laubach and Williams (2003) are "wrong" in any sense. Rather, it highlights that the data are not very informative about the $r*$ process, and a researcher with a prior view that $r*$ varies, but not too much, should not be swayed from that view by analyses such as Laubach and Williams (2003).

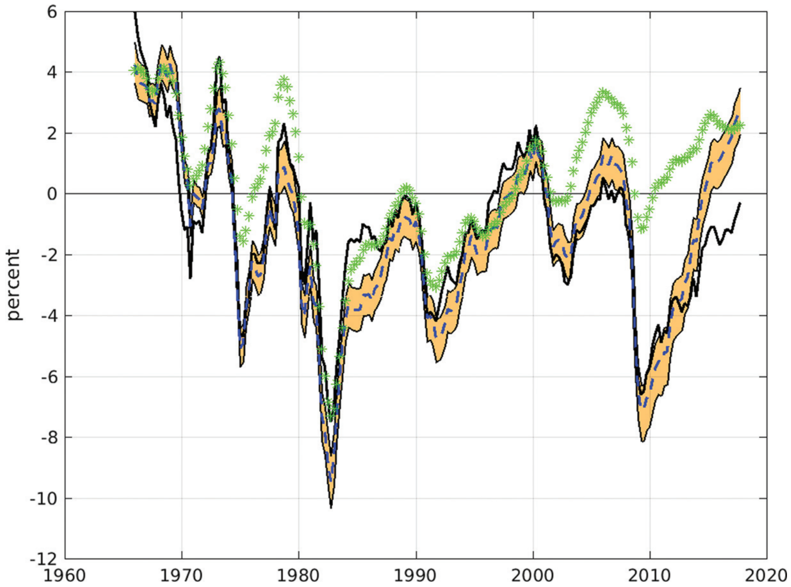
4.4 Auxiliary Implications of Models

The auxiliary implications of the models shed light on their properties and the corresponding estimates of $r*$. Because $r*$ is the real interest rate that closes the output gap in the long run, an especially interesting comparison of differences across frameworks can be seen in alternative assessments of the output gap.[12]

Figure 7 presents the estimate of the output gap from the Congressional Budget Office (CBO), the model herein (presented as the range consistent with 68 percent confidence set for the parameters, the shaded region, and the median value consistent with the posterior distribution of the parameters), and the Laubach and Williams (2003) model. (As above, the focus is on the properties of the model, not a full accounting of uncertainty regarding the estimates of the output gap, and consequently filter uncertainty is not considered in the shaded band—despite being an important consideration for overall uncertainty). The figure illustrates the similarity in assessments of resource utilization from the early 1960s through about the year 2000. After that period, the CBO and the model herein yield similar assessments of the output gap, but the Laubach

[12]This observation is intuitive and made elsewhere, e.g., Lubik and Matthes (2015).

Figure 7. Estimate of Output Gap from CBO, Model Herein, and Laubach and Williams (2003)



Sources: Author's computations from model estimates, Federal Reserve Bank of New York (Laubach and Williams 2003, https://www.newyorkfed.org/research/policy/rstar), and Congressional Budget Office (https://www.cbo.gov/system/files?file=2018-08/51137-2018-08-potentialgdp.xlsx).

and Williams (2003) model is quite different. For example, output is only slightly below potential in 2009 according to the Laubach and Williams (2003) model, but is very far below potential in the alternative assessments.[13]

Interpretation of these differences is difficult. The output gap is an unobserved variable, and there is no (statistical) objective sense in which the estimates from, for example, the CBO should be judged as closer to the truth than the estimate from, for example, Laubach and Williams (2003). Nonetheless, the failure of the Laubach and Williams (2003) measure to capture the severity of

[13]Holston, Laubach, and Williams (2017) report very similar estimates of the output gap to Laubach and Williams (2016). These estimates are available at https://www.newyorkfed.org/research/policy/rstar.

the Great Recession—a recession widely viewed as among the most severe in the United States since the Great Depression—suggests that alternative models of $r*$, and an acknowledgment of the degree to which prior views rather than data inform such estimates, are valuable.

5. Conclusion

My central conclusion is that the data provide relatively little information on the $r*$ data-generating process, as indicated by a posterior distribution for the $r*$ process that looks like the prior distribution I have assumed. This suggests some caution to researchers adopting either a Bayesian or classical approach to inference regarding $r*$: The data may not be informative, and hence specification choices, including priors (either explicit, as in a Bayesian approach, or implicit, as determined by a priori zero restrictions), may dominate results for $r*$. The lack of information in the data for the $r*$ process contrasts sharply with those for the trend growth or natural rate of unemployment processes, for which the data are very informative.

Second, conditional on the prior view that quarter-to-quarter movements in $r*$ are small, there has been a steady downward drift in the equilibrium real interest rate. The actors that may explain this trend decline have been analyzed in recent research, and include possible changes in the safety premium between a riskless rate and returns on risk assets (e.g., Del Negro et al. 2017) and changes in demographic factors that affect the balance between saving and investment (e.g., Gagnon, Johannsen, and López-Salido 2016). Further research in this area will help better pinpoint the role of these, and potentially other, forces in the years to come.

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