Optimal Central Bank Forward Guidance*

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When the future state of the economy is uncertain, yet a central bank has more information about the possible scenarios, how should the central bank communicate its private information to the public? This paper analyzes the optimal *tone* of central bank Delphic forward guidance using the Bayesian persuasion model (Kamenica and Gentzkow 2011). Assuming that monetary policy is an exogenously given function over the states of the economy and that the central bank is precommitted to a forward-guidance policy function, under certain conditions, the optimal tone of communication about the uncertain future is overly pessimistic.

JEL Codes: D82, D83, E58.

1. Introduction

Suppose central bank governors are required to offer an impromptu prediction of the state of the economy in the next one or two years. In the near future, they know that the economy could either experience a mild recession or stay expansionary. They think that both scenarios are possible, and that they need further information to offer a specific prediction. Then, what should they say to the public? What kind of tone should they use? Being vague by describing all the scenarios, including the above two, is one option. Or should they be optimistic, focusing only on the leading indicators that provide

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them with an expansionary forecast? Should they be pessimistic, emphasizing the negative indicators that predict a looming economic recession? Otherwise, should they simply point to the mixed signs from the economic indicators and the fact that they need further information?

Inadequate communication can mislead private agents' beliefs and risk evaluation about the future, thereby encouraging them to make premature decisions that may lead to either overly borrow to make inappropriately aggressive equipment investments or miss opportunities to grow faster. Moreover, it is not helpful to be always optimistic or always pessimistic because such central bank's communication may be regarded as uninformative or even incredible. Then, what is the optimal way of communicating the uncertain future?

Motivated by the above thought experiment, this paper describes a simple framework based on the Bayesian persuasion model (Kamenica and Gentzkow 2011) for analyzing optimal communication of a central bank's economic forecasts. The model consists of spaces with two signals, two states of the economy, and two messages. The state of the economy is about supply shocks. The monetary policy is exogenously determined as a response to the state of the economy: in the weak economy with an adverse cost shock, inflation is allowed to be high, while in the strong economy with an advantageous cost shock, inflation is controlled to be low. I have three results conditional on the characteristic of the exogenously given monetary policy regime (relatively unemployment fighting versus relatively inflation fighting) and the economic fundamental (the prior of an economic forecast of a strong or weak economy). First, given a relatively unemployment-fighting monetary policy regime and an economic environment with an a priori robust fundamental, it is optimal for the central bank to be overly pessimistic in communication. Second, under a relatively unemployment-fighting monetary policy regime and an economic environment with an a priori weak fundamental, central bank communication should be truthful. Third, given a relatively inflation-fighting monetary policy regime, it is optimal for the central bank to be uninformative.

The most interesting result is that the optimal central bank communication should be overly pessimistic if monetary policy regime is relatively unemployment fighting and the economic fundamental is a priori robust. The overly pessimistic communication means that, first, when the central bank receives a signal forecasting the weak economy more likely, it should send a message associated with the weak economy ("pessimistic message") with probability 1. Second, when the central bank receives a signal forecasting the strong economy more likely, it should mix the pessimistic message and an optimistic message, which is associated with the strong economy. By the overly pessimistic forward guidance, the central bank can mitigate the welfare damage caused by a large deviation from inflation target under a relatively unemployment-fighting regime.

In the model, the central bank is able to manage private sector's expected inflation by communicating the uncertain future. The private sector sets expected inflation with anticipating central bank's monetary policy action, while the monetary policy action is assigned by an exogenously preprogrammed function of the state of the economy. The state of the economy is realized stochastically, and the monetary policy action is confirmed only after the state is realized. Before the state is realized, the central bank obtains a better probability distribution over the states of the economy as its private information. Hence, communication of the central bank's economic forecast ("forward guidance") helps the private sector set better inflation expectations.

Before it observes the private information, the central bank designs forward-guidance policy function to maximize social welfare. The social welfare function reflects both inflation stability and output gap stability. Central bank communication changes the private sector's beliefs about the likelihood of future states and affects the expected inflation. In this way, inflation surprises can be managed by the central bank even though the realized inflation will be exogenously bounded by the preprogrammed monetary policy function. Once the optimal forward-guidance policy function is determined, the central bank is pre-committed to it, and the private sector is informed of the forward-guidance policy function. The model is one-shot. However, the result does not change under a dynamic model: as forward-guidance policy function is pre-committed, there is no room for the private sector's learning.

The novel feature of this study is that it analyzes the optimal *tone* for the central bank communication to convey information about the uncertain future. The Bayesian persuasion model allows to assign a probability distribution over the message space (see Table 3). For example, suppose the central bank privately forecasts a weak economy more likely, while there is a small but positive probability that a strong economy is realized. In this environment, the tone of the communication is formulated by a probability distribution over the message space. The tone is referred to as *truthful* if probability 1 is assigned to the pessimistic message in the above scenario, which means the truthful communication of the central bank's private information. However, the tone is referred to as *overly optimistic* if positive probabilities are assigned to both the optimistic message as well as the pessimistic message. By contrast, suppose the central bank privately forecasts the strong economy more likely, while there is a small but positive probability of the forecast proven wrong. The tone is referred to as *overly pessimistic* if a positive probability is also assigned to the pessimistic message along with the optimistic message.

In the sense that this paper deals with a central bank's communication about the future, it follows the line of literature on forward guidance. Gürkaynak, Sack, and Swanson (2005) empirically show the effect of verbal communication. Rudebusch and Williams (2008), Campbell et al. (2012), Del Negro, Giannoni, and Patterson (2012), and Andrade et al. (2019), among others, admit that empirically central bank forward guidance includes the communication of the future uncertainty, so-called Delphic forward guidance. Given reality, this paper normatively approaches how to optimally communicate the future uncertainty.

There has been numerous research on optimal central bank communication, especially using cheap talk by Crawford and Sobel (1982) or global game by Morris and Shin (2002). Morris and Shin (2018) describe optimal communication when the central bank has private noisy signal on the realized state, focusing on the reflection problem between the market and the central bank. Bassetto (2019) shows that when the central bank receives a noisy signal about the realized state, the central bank's cheap talk can increase social welfare.[1] This paper, however, provides a new framework to analyze the

[1]The research area of central bank communication about the future uncertainty is nascent, so most papers are at the working paper stage. Notably, Cieslak, Malamud, and Schrimpf (2019) suggest that the central bank's optimal communication of the state of the economy should be clustered. Fujiwara and Waki (2019)

optimal *tone* of central bank communication of the uncertain future: given a private signal forecasting either a strong economy more likely or a weak economy more likely, how optimistic or pessimistic the central bank communication should be.

The remainder of the chapter proceeds as follows. The simple two-signal, two-state, two-message model is introduced in Section 2. Sections 3 presents the threefold optimal central bank forward guidance. Section 4 discusses how the overly pessimistic communication can be better than the truthful communication in certain economic environments. In the appendix, I present that the threefold results of forward guidance hold in a generalized economic environment that incorporates possibly self-fulfilling effect of the private sector's expected inflation and allows for endogenous asymmetric inflation-targeting bandwidth.

2. The Model

2.1 Environment

In this section, I describe the primitives: the players, their payoff functions, their actions, and the private information of the central bank.

There are Nature, a central bank, and the private sector. It is a one-shot problem where the central bank pre-commits on a communication function as well as a monetary policy function. After the pre-commitment, the central bank executes actions according to the policy functions without reoptimization. The private sector determines the expected inflation after communication but before the monetary policy action.

The private sector's payoff function is given by a quadratic loss function from inflation surprises:

$$-\left(\pi^e-\pi\right)^2, \tag{1}$$

incorporate Bayesian persuasion into the New Keynesian model to suggest that central bank's Delphic communication may not increase social welfare. Related to communication *tone* in forecasting, Beaudry and Willems (2022) analyze International Monetary Fund (IMF) data to find that an overly optimistic tone in forecasting is correlated with economic contractions with time lags.

where $\pi^e \in \mathbb{R}$ is expected inflation and $\pi \in \mathbb{R}$ is realized inflation. The expected inflation is the private sector's action.

The ex post payoff of the central bank is given by a linear combination of the quadratic losses from both the inflation gap and the output gap. The inflation gap is measured by the deviation of realized inflation from the target, and the output gap is measured by cyclical unemployment. Let u^{NAR} and π^T denote the natural unemployment rate and inflation target, respectively, where u and π stand for the unemployment and inflation rates. Then, the payoff function is given by

$$W(u, \pi) = -(u - u^{NAR})^2 - \alpha \cdot (\pi - \pi^T)^2, \tag{2}$$

where u^{NAR} and π^T are parameters. The unemployment u is determined by a classic Phillips curve:

$$u - u^{NAR} = \theta + \gamma \cdot (\pi^e - \pi), \tag{3}$$

where γ is a parameter of unemployment for inflation surprises and θ represents the state of the economy. In addition, the realized inflation π is determined by an exogenously given monetary policy function $\pi(\cdot)$ and the realized state of the economy θ.

$$\text{(monetary policy) } \pi : \ \Theta \ \rightarrow \mathbb{R} \tag{4}$$

Nature informs θ to the central bank only as the private information. Nature draws the state of the economy from an exogenously given probability distribution Φ. The timing when the state of the economy is realized is *after* the communication occurs, and this formulation makes the communication be about the future uncertainty. The disposition of the state of the economy is adverse or advantageous cost-push shocks.

There are two possible states of the economy: $\Theta = \{\theta_S, \theta_W\}$. Here, θ_S and θ_W are real numbers related to the unemployment rate where $\theta_W > \theta_S$. When the economy is hit by an adverse cost-push shock, the state is θ_W (the "weak economy"); then, cyclical unemployment soars, assuming all other factors remained the same. When the economy is hit by an advantageous cost shock, the state is θ_S (the "strong economy"); then, cyclical unemployment is relatively low.

Table 1. Prior Distribution for Signals

	s_S	s_W
Prob.	ψ_0	$(1-\psi_0)$

Table 2. Stochastic Evolution of the State of the Economy

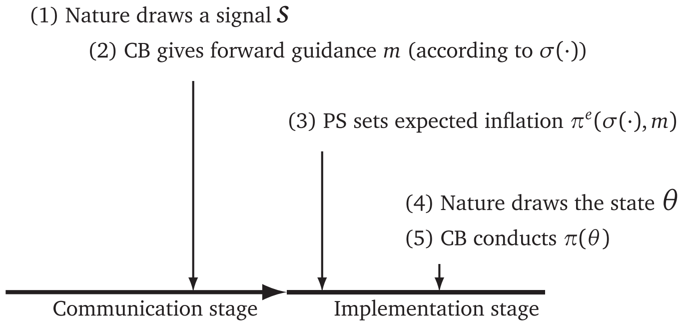
	θ_S	θ_W
s_S	ψ_S	$(1-\psi_S)$
s_W	$(1-\psi_W)$	ψ_W

In addition to the state of the economy, θ, Nature draws one more kind of private information for the central bank with a time lead: a private signal, s, which corresponds to the central bank's economic forecasts to the state of the economy. There are two possible private signals: $\{s_S,\ s_W\}$. Here, s_S denotes a signal to forecast the future *strong* economy with higher probability, whereas s_W denotes a signal to forecast the future *weak* economy with higher probability. The probability $Pr(s_S)$ of a signal to forecast future strong economy with higher probability is denoted by $\psi_0 \in [0,1]$. The private sector does not know the realized signal; however, they know the prior probability distribution over signals as summarized in Table 1.

The communication of the central bank concerns the private signal rather than the state of the economy. When its communication occurs, the central bank does not know which state of the economy will be realized but knows better about the stochastic evolution of the state of the economy than the private sector. Each private signal can be recast as a probability distribution over the state of the economy. The stochastic evolution of the state of the economy conditional on each signal is summarized in Table 2, and the Markov chain is common knowledge to the private sector.

Assume that $\psi_S \in (1/2, 1]$ as well as $\psi_W \in (1/2, 1]$. Hence, $\psi_S > 1 - \psi_W$. If the central bank observes the signal s_S, it is more likely that θ_S is realized; however, there is a small but positive probability that θ_W is realized. Similarly, if the central bank observes the signal s_W, it is more likely that θ_W is realized; nevertheless, there

Figure 1. Timeline



is a small but positive probability that θ_S is realized.[2] Owing to an empirical condition, I assume that $\{\theta_W \psi_W + \theta_S \cdot (1 - \psi_W)\} > 0$.[3] This condition sets a lower bound for θ_S, especially when $\theta_S < 0$.

With two types of private information, the timeline (Figure 1) is divided into two stages: the communication stage and the monetary policy implementation stage. In the communication stage, the central bank and private sector play in turn. The stage begins with the central bank observing the private signal. Then, the central bank sends a message according to a pre-committed communication policy function. There are two possible messages: an optimistic message and a pessimistic message. Here, the message space is denoted by $M = \{m_{opt},\ m_{pes}\}$. The communication policy function prescribes the central bank a probability distribution over the message space conditional on the realized state of the economy.

Reflecting the fact that the central bank communicates the uncertain future, the central bank's communication policy function

[2]Intuitively, this formation between the economic forecast and realized future state of the economy reflects the following observation: even if the central bank privately forecasts the strong economy, it is still just a forecast. There is a chance that the economy deteriorates rapidly. Conversely, even if the central bank (privately) forecasts the weak economy, there is a chance that economic conditions improve.

[3]The empirical conditions stems from the Phillips curve: Cogley and Sargent (2005) have a lagged term in their equation (1). The lagged term plays a role as a positive bias for cyclical unemployment. The theoretical results of this paper hold without the empirical condition.

is referred to as a *forward-guidance* policy function.[4] It is the objective of policy design as a functional of which domain is the set of signals. Let $\sigma(\cdot)$ denote the policy function:

$$\text{(forward-guidance policy function) } \sigma : \; S \; \rightarrow \Delta(M), \tag{5}$$

where $\Delta(M)$ means the set of probability distributions over the message space M. Let Σ denote the set of all feasible communication policy functions.

The choice of optimal forward-guidance policy function $\sigma(\cdot)$ is of interest in this paper. Once the optimal forward-guidance policy function is imposed, the central bank is pre-committed to the policy function, and it sends messages according to the policy prescription. Forward guidance does not affect the stochastic evolution of the state of the economy.

Observe that in comparison with the Kamenica and Gentzkow (2011), the communication stage corresponds to the conventional Bayesian persuasion model. I refer to the "state of the world" in Kamenica and Gentzkow (2011) as a "private signal," the "signal" in Kamenica and Gentzkow (2011) as a "forward-guidance policy function," and the "signal realization" in Kamenica and Gentzkow (2011) as a "message." The term "state" in this paper is reserved for the basis of monetary policy in the implementation stage. Each private signal can be recast as a probability distribution over the state space.

The private sector knows that the central bank communicates the future uncertainty, and the central bank's message does not guarantee a certain state of the economy. Also, the private sector is aware that the central bank's pre-committed communication policy is designed strategically to maximize the social welfare. The private sector updates their posterior after the central bank's message is received. The private sector decides, first, whether to "listen to" (Bergemann and Morris 2016 refer to this action as "obey") the announcement of the central bank and, second, how to formulate expected inflation. Once the private sector listens to the message, then it updates its beliefs about the true state of the economy. Each

[4] More specifically, it is Delphic forward guidance according to Campbell et al. (2012).

posterior is determined by the realized message as well as the precommitted forward-guidance policy function from which the message is drawn.

After receiving the message m, the private sector's posterior is $\mathcal{P}(\cdot \mid \sigma(\cdot), m)$. The private sector's maximizer is given by

$$\pi^e\Big(\sigma(\cdot), m\Big) = \int_\theta \pi(\theta)\ \mathcal{P}\left(d\theta \mid \sigma(\cdot), m\right). \tag{6}$$

Given the monetary policy function, the expected inflation is a function of the forward-guidance policy function and a realized message:[5]

$$\text{(the private sector's action) } \pi^e:\ \Sigma \times M \to \mathbb{R}. \tag{7}$$

I abuse the notation for π^e so that it represents both a realized value and a measurable function.

After the private sector's expected inflation is chosen, the monetary policy implementation stage begins. The central bank undertakes its monetary policy action $\pi(\theta)$ according to an exogenously given monetary policy function $\pi(\cdot)$ and the realized state of the economy θ. Then, the ex post payoff of the central bank can be rewritten as

$$\begin{aligned} & W(\pi^e\Big(\sigma(\cdot),\ m\Big), \pi(\theta), \theta) \qquad (8) \\ & \quad = -\left[\theta + \gamma \cdot \left\{\pi^e\left(\sigma(\cdot), m\right) - \pi(\theta)\right\}\right]^2 - \alpha \cdot (\pi(\theta) - \pi^T)^2, \end{aligned}$$

where $\gamma > 0$, and $\alpha > 0$. With this model, the goal of this study is to design the optimal *forward-guidance* policy function to maximize social welfare, which is ex ante expected payoff of the central bank. Social welfare is measured even before any signal is drawn by Nature.

2.2 Parameterization and Monetary Policy Regime

In this subsection, the forward-guidance policy function and monetary policy function are parameterized. Observe that the co-domain

[5]The choice of expected inflation π^e does not affect the stochastic evolution of the state of the economy. Hence, there are no self-fulfilling effects.

Table 3. Forward-Guidance Policy Function $\boldsymbol{\sigma : S \rightarrow \Delta(M)}$

$\boldsymbol{\sigma(s) \in \Delta(M)}$	$\boldsymbol{m_{opt}}$	$\boldsymbol{m_{pes}}$
s_S	ρ_{opt}	$(1 - \rho_{opt})$
s_W	$(1 - \rho_{pes})$	ρ_{pes}

of forward-guidance policy function σ is the set of probability distribution over the message space. The parameterization of policy function allows us to solve the optimization problem analytically. The abstract form of forward-guidance policy function is given by

$$\text{(forward-guidance policy function) } \sigma : \ S \ \rightarrow \Delta(M). \qquad (9)$$

Since there are two possible messages, a probability distribution over the message space can be parameterized by a probability mass on a specific message. That is, first, when the central bank observes the signal s_S, a probability distribution over message space can be represented by a probability mass $Pr(m_{opt} \mid s_S) \equiv \rho_{opt}$ on the optimistic message m_{opt}. Then, the probability mass on the pessimistic message m_{pes} is the complementary probability $(1 - \rho_{opt})$. Second, symmetrically, when the central bank observes the signal s_W, a probability distribution can be represented by a probability mass $Pr(m_{pes} \mid s_W) \equiv \rho_{pes}$ on the pessimistic message m_{pes}. Assume that the unconditional probabilities are not zero: $Pr(m_{opt}) = \psi_0 \cdot \rho_{opt} + (1 - \psi_0) \cdot (1 - \rho_{pes}) \neq 0$ and $Pr(m_{pes}) = \psi_0 \cdot (1 - \rho_{opt}) + (1 - \psi_0) \cdot \rho_{pes} \neq 0$. Table 3 summarizes the parameterization.

The optimal values of parameters ρ_{opt} and ρ_{pes} are of interest in this paper. Observe that the case in which $\rho_{opt} = (1 - \rho_{pes})$ means that the central bank's message is *uninformative*: given any message, the private sector's updated beliefs will be exactly the same to the prior.

In addition, for analytical simplicity, monetary policy function $\pi : \Theta \rightarrow \mathbb{R}$ takes a linear functional form as follows:

$$\pi(\theta_S) = \pi^T - v, \qquad \pi(\theta_W) = \pi^T + v, \qquad (10)$$

where $v \geq 0$. The monetary policy function reflects a simplified flexible inflation targeting under the Qvigstad rule (Walsh 2014).[6]

There are 62 countries that use inflation targeting with an explicitly stated inflation target (Foerster and Davig 2017). The majority of them (40 countries) use flexible inflation targeting with a range for target inflation rate as opposed to inflation targeting with pointwise target. For instance, the two pioneers of inflation targeting, New Zealand and Canada, adopted flexible inflation targeting while the United Kingdom, the third, adopted pointwise inflation targeting.

The optimal monetary policy under flexible inflation targeting is suggested by Qvigstad (2006), and coined as the "Qvigstad rule" by Walsh (2014). With flexible inflation targeting, the optimal monetary policy is characterized by the opposite sign of output gap and inflation deviation from the numerical target inflation point (also known as the inflation gap). It is an empirically supposed relationship based on the following evaluation: if the output gap is positive and inflation is above the target point, monetary policy is too accommodative. If the output gap is negative and inflation is below the target, monetary policy is too tight.

Equation (10) follows the Qvigstad rule. When the economy is hit by an adverse cost-push shock, monetary policy prescribes accommodative policy action so that inflation rate can rise above the target. When the economy is hit by advantageous cost shock, monetary policy prescribes tightening monetary policy action so that inflation rate decreases below the target. The degree of deviation from the inflation target v is a parameter. While the optimal values for ρ_{opt} and ρ_{pes} are to be solved, the value of v is assumed as exogenously given.

The monetary policy regime is defined by the magnitude of v. The threshold is $\left(\dfrac{\theta_W - \theta_S}{\gamma}\right)$. If monetary policy regime is "relatively unemployment fighting," i.e., $v > \left(\frac{\theta_W - \theta_S}{\gamma}\right)$, the central bank bears with sufficiently high inflation when unemployment is high under the adverse cost-push shock by the exogenously imposed

[6]More generalized formulations give the same results.

policy prescription. Otherwise, monetary policy is referred to as "relatively inflation fighting." Monetary policy regime is public information; the private sector knows whether it is relatively unemployment fighting or relatively inflation fighting before it chooses expected inflation.

3. Optimal Central Bank Forward Guidance

The main theorem of this paper is as follows:

THEOREM 1. *Optimal central bank forward-guidance policy function should be*

- *(i) overly pessimistic if monetary policy is relatively unemployment fighting and if it is unlikely to forecast a future weak economy;*
- *(ii) truthful if monetary policy is relatively unemployment fighting and if it is highly likely to forecast a future weak economy;*
- *(iii) uninformative if monetary policy is relatively inflation fighting.*

Sketch of Proof. The policy designer wants to maximize the ex ante expected payoff of the central bank, subject to the private sector's best response. To obtain the optimal policy, the idea of backward induction is applied: For any pre-committed forward-guidance policy function and any realized message, the private sector best responds to it. The private sector updates posterior and takes its action by setting the expected inflation.

The private sector updates its beliefs over the future state of the economy as follows: given a message m and a pre-committed forward-guidance policy function $\sigma : S \rightarrow \Delta(M)$,

$$Pr(\theta \mid \sigma(\cdot), m) = \sum_{s \in S} Pr(\theta \mid s) \cdot Pr(s \mid \sigma(\cdot), m). \tag{11}$$

Table 4. Posterior Using $\sigma : S \to \Delta(M)$ and $Pr(s_S) = \psi_0, Pr(s_W) = 1 - \psi_0$

$Pr(s\|\sigma(\cdot), m)$	s_S	s_W
m_{opt}	$\frac{\psi_0\rho_{opt}}{(1-\psi_0)(1-\rho_{pes})+\psi 0\rho_{opt}}$	$\frac{(1-\psi_0)(1-\rho_{pes})}{(1-\psi_0)(1-\rho_{pes})+\psi_0\rho_{opt}}$
m_{pes}	$\frac{\psi_0(1-\rho_{opt})}{(1-\psi_0)\rho_{pes}+\psi_0(1-\rho_{opt})}$	$\frac{(1-\psi_0)\rho_{pes}}{(1-\psi_0)\rho_{pes}+\psi_0(1-\rho_{opt})}$

For an exogenously given monetary policy function $\pi : \Theta \to \{\pi^T - v,\ \pi^T + v\}$, the expected inflation is set as

$$\pi^e(\sigma(\cdot),\ m) = \left(\pi^T - v\right) \cdot \left[\sum_{s\in S} Pr(\theta_S \mid s) \cdot Pr(s \mid \sigma(\cdot), m)\right] \quad (12)$$
$$+ \left(\pi^T + v\right) \cdot \left[\sum_{s\in S} Pr(\theta_W \mid s) \cdot Pr(s \mid \sigma(\cdot), m)\right].$$

For arbitrary ρ_{opt} and ρ_{pes}, for each $m \in M$, the posterior is as in Table 4. Then, the expected inflation conditional on each message is given by, respectively,

$$\pi^e(\rho_{opt},\ \rho_{pes},\ m_{opt}) \quad (13)$$
$$= \pi^T + v\left[\frac{(2\psi_W - 1)(1-\psi_0)(1-\rho_{pes}) - (2\psi_S - 1)\psi_0\rho_{opt}}{(1-\psi_0)(1-\rho_{pes}) + \psi_0\rho_{opt}}\right],$$

$$\pi^e(\rho_{opt},\ \rho_{pes},\ m_{pes}) \quad (14)$$
$$= \pi^T + v\left[\frac{(2\psi_W - 1)(1-\psi_0)\rho_{pes} - (2\psi_S - 1)\psi_0(1-\rho_{opt})}{(1-\psi_0)\rho_{pes} + \psi_0(1-\rho_{opt})}\right].$$

The efficacy of forward-guidance policy function relies on the private sector participating in the communication by updating their posterior. It is at issue to ensure that the private sector's expected payoff does not deteriorate by updating the posterior in order to give the private sector the incentive to adjust their belief after the central bank's forward guidance. In conventional Bayesian persuasion models, the incentive compatibility of the private sector participating in a communication becomes a restriction for the communication design (see the obedience condition in Bergemann and Morris 2016, p. 587). However, in this game structure of a central bank and the private sector, the incentive compatibility does not restrict the central bank's choice of forward-guidance policy function. It is due to the private sector's action space to be richer than the message space. In particular, for a given triple of $(\rho_{opt},\ \rho_{pes},\ m_{pes})$, the expected inflation can potentially be any number in between the upper bound and the lower bound of the inflation,

$$\pi^e(\rho_{opt},\ \rho_{pes},\ m_{pes}) \in \left[\pi^T - v,\ \ \pi^T + v,\right], \tag{15}$$

whereas the message space has only two elements, $M = \{m_{opt},\ m_{pes}\}$. Then, the private sector's expected payoff under the posterior is not less than the expected payoff under the prior. The policy designer can optimize the forward-guidance policy function relaxed from the private sector's incentive compatibility in communication.

Then, the policy designer solves the optimization problem for the forward guidance subject to the private sector's expected inflation only. The results are threefold conditional on the monetary policy regime, i.e., the value of $(\theta_W - \theta_S - \gamma v)$, and the economic fundamental, i.e., the probability of observing the signal that predicts the strong economy, $Pr(s_S) = \psi_0$. The mathematical derivation is elaborated in the online math appendix (Ko 2022).

Optimal Uninformative Forward-Guidance Policy Function. For the case where monetary policy regime is relatively inflation fighting, i.e., $\theta_W - \theta_S - \gamma v > 0$, the optimal forward-guidance policy function is as follows: irrespective of the signal that the central bank observes, it should send both the optimistic message and the pessimistic message with probability $\frac{1}{2}$. The message is completely uninformative because the posterior of the private sector remains equivalent to the prior. It can be summarized as in Table 5.

Effects of Optimal Uninformative Forward Guidance. When the central bank pre-commits to implement the above forward

Table 5. Optimal Uninformative Forward-Guidance Policy Function if $\theta_W - \theta_S - \gamma v > 0$ (hawkish)

$\sigma : S \rightarrow \Delta(M)$	m_{opt}	m_{pes}
s_S	1/2	1/2
s_W	1/2	1/2

guidance, i.e., $\rho_{opt} = 1/2,\ \rho_{pes} = 1/2$ to Equation (13) and Equation (14), the private sector forms the following expected inflation after receiving the optimistic and pessimistic message:

$$\pi^e(\sigma(\cdot), m_{opt}) = \pi^T + v \cdot \left[(1 - 2\psi_S)\psi_0 + (2\psi_W - 1)(1 - \psi_0)\right]$$
$$\pi^e(\sigma(\cdot), m_{pes}) = \pi^T + v \cdot \left[(1 - 2\psi_S)\psi_0 + (2\psi_W - 1)(1 - \psi_0)\right].$$

Under the optimal uninformative forward guidance, however, irrespective of the received message, the private sector sets the same expected inflation based only on the prior. It is because the private sector is given no additional information through the communication of the central bank. The inflation surprise in each state of the economy is, respectively,

$$\pi(\theta_S) - \pi^e(m_.) = -v \cdot \left[2\psi_0(1 - \psi_S) + 2(1 - \psi_0)\psi_W\right] < 0,$$
$$\pi(\theta_W) - \pi^e(m_.) = v \cdot \left[2\psi_0\psi_S + 2(1 - \psi_0)(1 - \psi_W)\right] > 0,$$

for any $m_. \in \{m_{opt}, m_{pes}\}$. Since $\psi_S, \psi_W \in (\frac{1}{2}, 1]$, $(2 - 2\psi_S) > 0$, and $(2 - 2\psi_W) > 0$, the sign of inflation surprise in the strong economy is negative, i.e., $\{\pi(\theta_S) - \pi^e(m_.)\} < 0$. By contrast, the sign of inflation surprise in the weak economy is positive, i.e., $\{\pi(\theta_W) - \pi^e(m_.)\} > 0$. Therefore, the cyclical unemployment will be lower than θ_W if the weak economy is realized, while it will rise higher than θ_S when the strong economy is realized. The inflation surprises contribute to a fall in cyclical unemployment from θ_W in the weak economy and to a rise in cyclical unemployment from θ_S in the strong economy. By doing so, inflation surprises reduce the variance of cyclical unemployment across the states of the economy. The underlying economic environment $v < \frac{\theta_W - \theta_S}{\gamma}$ helps that the

Table 6. Optimal Overly Pessimistic Forward-Guidance Plan if $\theta_W - \theta_S - \gamma v < 0$ (relatively unemployment fighting) and $\psi_0 \in [\frac{1}{2}, 1]$

$\sigma : S \to \Delta(M)$	m_{opt}	m_{pes}
s_S	$\frac{1}{2\psi_0}$	$(1 - \frac{1}{2\psi_0})$
s_W	0	1

overall ex ante social welfare is to be maximized with such amounts of inflation surprises under the uninformative forward guidance.[7]

Optimal Overly Pessimistic Forward Guidance. For the case where monetary policy regime is relatively unemployment fighting ($\theta_W - \theta_S - \gamma v < 0$) and the economy is considered robust so that it is more likely to have the strong economy ($\psi_0 \in \left[\frac{1}{2}, 1\right]$), the optimal forward guidance is as follows: when the central bank observes the signal s_W, it should send the pessimistic message m_{pes} with probability 1. When it observes the signal s_S, it should send the optimistic message and the pessimistic message with mixed probabilities such that $Pr(m_{opt} \mid s_S) = \frac{1}{2\psi_0}$ and $Pr(m_{pes} \mid s_S) = \left(1 - \frac{1}{2\psi_0}\right)$. This forward guidance is pre-committed, so the private sector knows the function, albeit the private sector does not know the realized signal. It can be summarized as in Table 6.

Effects of Optimal Overly Pessimistic Forward Guidance. When the central bank pre-commits to implement the above forward guidance, i.e., $\rho_{opt} = \frac{1}{2\psi_0}$, $\rho_{pes} = 1$, the private sector forms the expected inflation as follows after receiving optimistic or pessimistic message, respectively:

[7]The losses from the inflation gap $-\alpha \cdot v^2$ are not affected by the forward-guidance policy under the assumptions that the inflation-targeting bandwidth v is exogenously given and that the central bank has complete control over the inflation, without the influence of the private sector's expected inflation. The complete control assumption can be relaxed so that the ex post realized inflation is a convex combination of the policy implementation of the central bank and the expected inflation of the private sector ("incomplete control"). The results for the forward-guidance policy remain the same, while the welfare benefits from reduction in cyclical unemployment gain more weights in the welfare function. See Appendix Section C for more detail.

$$\pi^e(\sigma(\cdot), m_{opt}) = \pi^T + v \cdot (1 - 2\psi_S)$$
$$\pi^e(\sigma(\cdot), m_{pes}) = \pi^T + v \cdot [(1 - 2\psi_S)(2\psi_0 - 1) + (2\psi_W - 1)(2 - 2\psi_0)].$$

Since the expected inflation is formed before the state of the economy is realized, it only depends on the received message. Observe that the private sector will have a truthful signal when the optimistic message m_{opt} is realized. It is because when the central bank observes the signal s_W, it never sends the optimistic message. Therefore, the optimistic message means that the central bank observes the signal s_S. By contrast, the pessimistic message m_{pes} generates obfuscation: under the overly pessimistic forward guidance, the pessimistic message can be sent when the central bank observes either the signal s_S or s_W (with different probabilities). If the pessimistic message is received, the private sector will set the expected inflation considering that both cases are possible. That helps the expected inflation become lower than the truthful communication.

Given any message, either the strong economy or the weak economy can be realized as the state of the economy. Hence, there are four scenarios for inflation surprises as follows:

$$\pi(\theta_S) - \pi^e(m_{opt}) = -v \cdot (2 - 2\psi_S) < 0$$
$$\pi(\theta_S) - \pi^e(m_{pes}) = -v \cdot [(2 - 2\psi_S)(2\psi_0 - 1) + 2\psi_W(2 - 2\psi_0)] < 0$$
$$\pi(\theta_W) - \pi^e(m_{opt}) = v \cdot 2\psi_S > 0$$
$$\pi(\theta_W) - \pi^e(m_{pes}) = v \cdot [2\psi_S(2\psi_0 - 1) + (2 - 2\psi_W)(2 - 2\psi_0)] > 0.$$

The direction of influence on cyclical unemployment is the same as the previous case of uninformative forward-guidance policy: Since $\psi_S, \psi_W \in (\frac{1}{2}, 1]$ and $\psi_0 \in \left(\frac{1}{2}, 1\right]$, the sign of inflation surprises is negative when the strong economy is realized, i.e., $\{\pi(\theta_S) - \pi^e(m_.)\} < 0$, while the sign of inflation surprise is positive in the weak economy, i.e., $\{\pi(\theta_W) - \pi^e(m_.)\} > 0$. Therefore, the cyclical unemployment will be lower than θ_W if the weak economy is realized, while it will rise higher than θ_S when the strong economy is realized.

Recall that in the previous case, the exogenously given bandwidth $v < \frac{\theta_W - \theta_S}{\gamma}$ while the overly pessimistic forward guidance is applied to the environment $v > \frac{\theta_W - \theta_S}{\gamma}$. Therefore, it is unfair to compare the inflation surprises between the previous uninformative forward guidance and the overly pessimistic forward guidance. However, it can give us insight on the underlying mechanism to compare the coefficients of inflation surprises, except for the bandwidth v.

The average surprise in inflation of the weak economy is given by

$$v\psi_S(1-\psi_S) + v\left[\psi_S(2\psi_0 - 1) + (1-\psi_W)(2-2\psi_0)\right]$$
$$\left[(1-\psi_S)(2\psi_0 - 1) + \psi_W(2-2\psi_0)\right].$$

Then, the coefficient of v can be written as $\left[\psi_S(1-\psi_S) + \mathbb{A}_w (1-\mathbb{A}_w)\right]$, where $\mathbb{A}_w \equiv \psi_S(2\psi_0 - 1) + (1-\psi_W)(2-2\psi_0)$. Recall that under the uninformative forward guidance, the corresponding coefficient of v is $[2\psi_0\psi_S + 2(1-\psi_0)(1-\psi_W)]$, which is $\psi_S + \mathbb{A}_w$. To compare these two coefficients, as $(1-\psi_S) < 1$ and $(1-\mathbb{A}_w) < 1$,

$$\psi_S(1-\psi_S) + \mathbb{A}_w(1-\mathbb{A}_w) < \psi_S + \mathbb{A}_w.$$

Therefore, when the weak economy is realized, the coefficient of inflation surprise in uninformative forward guidance is greater than the one in overly pessimistic forward guidance. If the same inflation-targeting bandwidth v is applied, the reduction in cyclical unemployment in the weak economy will be greater under the uninformative forward-guidance policy. Then, why does the overly pessimistic forward guidance become optimal in certain economic environments? It is because such forward guidance reduces the variance of cyclical unemployment more effectively in those economic environments. Although there is a rise in cyclical unemployment in the strong economy, such a cost is less than the rise under the uninformative forward guidance. To compare the costs induced by the overly pessimistic forward guidance and the uninformative forward guidance more concretely, recall that the coefficient part of inflation surprise under uninformative forward guidance is given by $-[2\psi_0(1-\psi_S) + 2(1-\psi_0)\psi_W]$, which is $-(1-\psi_S) - (1-\mathbb{A}_w)$. The average surprise in inflation of the strong economy under overly pessimistic forward guidance is given by

$$-v\psi_S(1-\psi_S) - v\left[(1-\psi_S)(2\psi_0 - 1) + \psi_W(2-2\psi_0)\right]$$
$$\left[\psi_S(2\psi_0 - 1) + (1-\psi_W)(2-2\psi_0)\right].$$

Table 7. Truthful Forward-Guidance Policy Function if $\theta_W - \theta_S - \gamma v < 0$ (relatively unemployment fighting) and $\psi_0 \in [0, \frac{1}{2}]$

$\sigma : S \to \Delta(M)$	m_{opt}	m_{pes}
s_S	1	0
s_W	0	1

The coefficient of v can be written by $-\left[\psi_S(1-\psi_S) + \mathbb{A}_w(1-\mathbb{A}_w)\right]$. Then, the cost is less under the overly pessimistic forward guidance than the one under the uninformative forward guidance because

$$\psi_S(1-\psi_S) + \mathbb{A}_w(1-\mathbb{A}_w) < (1-\psi_S) + (1-\mathbb{A}_w).$$

Therefore, the overly pessimistic forward guidance can be advantageous in certain economic environments with relatively larger v, i.e., $v > \frac{\theta_W - \theta_S}{\gamma}$, while the uninformative forward guidance can be beneficial in the other economic environments, i.e., $v < \frac{\theta_W - \theta_S}{\gamma}$.

Intuitively, inflation surprises under overly pessimistic forward guidance contribute to reducing cyclical unemployment less than θ_W in the weak economy while it causes a rise in cyclical unemployment from θ_S in the strong economy: such inflation surprises reduce the variance of cyclical unemployment.

Optimal Truthful Forward Guidance. For the case where monetary policy regime is relatively unemployment fighting ($\theta_W - \theta_S - \gamma v < 0$) and the economy is considered strained so that it is more likely to have the weak economy ($\psi_0 \in \left[0, \frac{1}{2}\right]$), the optimization problem has a corner solution. The case requires direct comparison of ex ante expected payoffs among policy functions. The resulting optimal forward-guidance policy function prescribes the central bank to be truthful. Then, the private sector can infer exactly which signal the central bank observes. It can be summarized as in Table 7.

Effects of Truthful Forward Guidance. As the central bank pre-commits to implement the above forward guidance, plugging in

$\rho_{opt} = 1,\ \rho_{pes} = 1$, the private sector forms the expected inflation as follows:

$$\pi^e(\sigma(\cdot), m_{opt}) = \pi^T + v \cdot (1 - 2\psi_S)$$
$$\pi^e(\sigma(\cdot), m_{pes}) = \pi^T + v \cdot (2\psi_W - 1).$$

There are four scenarios for inflation surprises given by the following:

$$\pi(\theta_S) - \pi^e(m_{opt}) = -v \cdot (2 - 2\psi_S) < 0$$
$$\pi(\theta_S) - \pi^e(m_{pes}) = -v \cdot 2\psi_W < 0$$
$$\pi(\theta_W) - \pi^e(m_{opt}) = v \cdot 2\psi_S > 0$$
$$\pi(\theta_W) - \pi^e(m_{pes}) = v \cdot (2 - 2\psi_W) > 0.$$

The direction of influence on cyclical unemployment is the same to the previous cases of the uninformative forward guidance and the overly pessimistic forward guidance: When the weak economy is realized, the inflation surprises lower the cyclical unemployment than θ_W, whereas the inflation surprises raise cyclical unemployment higher than θ_S. This helps lower variance in cyclical unemployment across the states of the economy.

The inflation surprises under the optimistic messages are the same in both cases under the overly pessimistic forward guidance and the truthful forward guidance. However, the inflation surprises under the pessimistic message are different: given $\psi_0 \in [0, \frac{1}{2})$, the effects of inflation surprises under the truthful forward guidance are greater than the effects under the overly pessimistic forward guidance, with a smaller cost and a greater benefit: $2\psi_W < [(2 - 2\psi_S)(2\psi_0 - 1) + 2\psi_W(2 - 2\psi_0)]$ and $(2 - 2\psi_W) > [2\psi_S(2\psi_0 - 1) + (2 - 2\psi_W)(2 - 2\psi_0)]$ where $\psi_0 \in [0, \frac{1}{2}]$. By contrast, if $\psi_0 \in (\frac{1}{2}, 1]$, the inequality reverses, and the effects of the overly pessimistic forward guidance becomes greater, with a smaller cost and a greater benefit.

The expected benefit and expected cost of the overly pessimistic policy function can be computed with respect to the truthful policy function. First, the expected cost incurs when the signal s_S forecasting the future strong economy is realized:

$$
\begin{aligned}
&(\text{after } s_S)\\
&EW_{\text{overly pes}}(s_S) - EW_{\text{truthful}}(s_S) \equiv C\\
&\quad = -\left(1 - \frac{1}{2 \cdot \psi_0}\right) \cdot (2 - 2 \cdot \psi_0) \cdot (2\psi_W + 2\psi_W - 2) \cdot \gamma \cdot v\\
&\qquad \times \Big\{ 2\psi_S \cdot (2 \cdot \gamma \cdot v + \theta_S - \theta_W) + (2\theta_W - 4 \cdot \gamma \cdot v)\\
&\qquad - 2 \cdot \psi_0 \cdot (2\psi_W + 2\psi_S - 2) \Big\} < 0.
\end{aligned} \tag{16}
$$

The expected benefit is earned when the signal s_W forecasting the future weak economy is realized:

$$
\begin{aligned}
&(\text{after } s_W)\\
&EW_{\text{overly pes}}(s_W) - EW_{\text{truthful}}(s_W) \equiv B\\
&\quad = \Big[2 \cdot \theta_S + \{(2 \cdot \psi_0 - 1) \cdot (2 - 2 \cdot \psi_S) + (3 - 2 \cdot \psi_0) \cdot (2 \cdot \psi_W)\}\\
&\qquad \cdot \gamma \cdot v + 2\psi_W \cdot (2 \cdot \gamma \cdot v + \theta_S - \theta_W) \Big]\\
&\qquad \times (2 \cdot \psi_0 - 1) \cdot (2 \cdot \psi_W + 2 \cdot \psi_S - 2) \cdot \gamma \cdot v > 0.
\end{aligned} \tag{17}
$$

As long as the monetary policy regime is relatively unemployment fighting ($\theta_W - \theta_S < \gamma \cdot v$) and it is more likely to have the signal forecasting the strong economy a priori ($\psi_0 \in [\frac{1}{2}, 1]$), the overly pessimistic communication's expected benefit outweighs the expected cost multiplied by the probability of each signal being realized: $B \times (1 - \psi_0) + C \times \psi_0 \geq 0$. In such economic environment, the overly pessimistic communication function is better than the truthful communication function.

4. Discussion

In this paper, the monetary policy function of the central bank is given exogenously, and it is clearly understood by the private sector.

Hence, the monetary policy can be said to be transparent,[8] and it is well known to the private sector how high or low the realized inflation rate will be as soon as the state of the economy is realized; for instance, if the weak economy θ_W is realized, then the inflation rate will be realized as $\pi^T + v$. Given the monetary policy transparency, however, at issue is the transparency in communication or information symmetry between the central bank and the private sector. In this model, before the state of the economy is realized, the central bank is assumed to have only a probabilistic signal. Then, should the central bank communicate the information truthfully? The result that the uninformative forward guidance is optimal for the relatively inflation-fighting central bank may appear counterintuitive considering the emphasis on transparency by the central banks around the world.

The benefits from opaqueness in communication as opposed to transparency have long been acknowledged by the literature on central bank communication (See Geraats 2002, p. F547 for benefits from economic uncertainty). To see the benefits from inflation surprises under the uninformative forward guidance in the main model,[9] a central bank with strict inflation targeting can be compared with the relatively inflation-fighting central bank with flexible inflation targeting. In strict inflation targeting, the ex post realized inflation is π^T regardless of the realized state of the economy, and thus, the expected inflation is π^T irrespective of forward-guidance policy. As a result, there are zero surprises in inflation. It means there is no buffer for the supply shock: unemployment will fully swing between θ_W and θ_S, whereas the inflation rate is always at π^T. This case

[8] According to the classification of Geraats (2002, p. F540), the environment satisfies political (objective) transparency, procedural transparency, policy transparency, and operational transparency.

[9] In this model, it is assumed that the central bank completely controls realized inflation, and there is no role for the expected inflation in ex post realized inflation. However, in reality, the expected inflation may be self-fulfilling, and central banks use communication extensively to gravitate the expected inflation near the inflation target. In Appendix Section C, the model is extended to incorporate such reality that the central bank may have only incomplete control over the realized inflation while the private sector's expected inflation may have self-fulfilling effect. In addition, a central bank is allowed to choose its optimal inflation-targeting bandwidth endogenously. The inflation-targeting bandwidth can be asymmetric.

is suboptimal. Rather, the positive inflation-targeting bandwidth and uninformative forward guidance that generate inflation surprises contribute welfare improvement through decreases in the variance of cyclical unemployment across the states of the economy.[10]

The next question raised will be why overly pessimistic communication is better than truthful communication in certain unemployment-fighting environments (where parameters satisfy $\theta_W - \theta_S - \gamma v < 0$ and $\psi_0 \in [\frac{1}{2}, 1]$). The economic intuition stems from the idea that it is more useful to have a lower variance in cyclical unemployment across the states of the economy in order to maximize social welfare. To achieve such a lower variance in cyclical unemployment, it is better to lower the expected inflation to induce a greater inflation surprise in the weak economy. Assume the environment is given as $\theta_W - \theta_S - \gamma v < 0$, where $\psi_0 \in [\frac{1}{2}, 1]$. Compared with truthful communication, the overly pessimistic communication lowers the expected inflation more effectively when the central bank received the signal forecasting the future weak economy. If the weak economy is actually realized following the signal, then the inflation surprise will be larger than the truthful communication. The larger inflation surprise contributes to lower cyclical unemployment more effectively. This benefit, however, comes at a cost. To have the larger inflation surprise in the weak economy, the central bank will endure a smaller inflation surprise in the strong economy than the truthful communication. Both the benefit and the cost stem from the strategic part of the overly pessimistic communication that the central bank may send the pessimistic message strategically under both signals forecasting the future strong economy or forecasting the future weak economy.

This paper shows optimal tone of central bank forward guidance when a central bank has been confined with an exogenous monetary policy and tries to use forward guidance about future uncertainty as an additional tool. I also show in the appendix that when the inflation bandwidth is endogenously determined, global optimal is the relatively inflation-fighting monetary policy and the uninformative forward guidance.[11] Considering Woodford (2005) and the current emphasis on central bank communication in practice, the result that

[10]Please see the Appendix Section B for more detail.

[11]See Appendix Sections B and C for more detail.

global optimal forward guidance being uninformative one when the monetary policy is endogenous determined jointly with forward guidance may seem puzzling. As future research, the role of central bank forward guidance on the monetary policy transmission mechanism (for instance, see Leombroni et al. 2021) and other factors that affect the private sector's formation of expected inflation (see Binder and Makridis 2022, Coibion, Gorodnichenko, and Weber 2022, and others) may require more attention to further investigate the global optimal pair of endogenous monetary policy and forward guidance.

Appendix

In this appendix, I show the expected inflation of the private sector under each forward guidance in Section A. I derive the optimal symmetric inflation-targeting bandwidth paired under each forward guidance (Section B.1–3). I present generalized formulation incorporating possibly self-fulfilling effect of the private sector's expected inflation and allowing for asymmetric inflation-targeting bandwidth (Section C.1–5). Finally, I do welfare comparison among the pairs of forward guidance and monetary policy (Section B.4 and C.6, respectively).[12] All the mathematical derivations are elaborated in the online math appendix (Ko 2022).

A. Expected Inflation under Each Communication Policy

In this section, I demonstrate how the expected-inflation is computed in each forward-guidance regime. For the economic intuition, please see Section 3 of the main text.

A.1 $\psi_0 \in [\frac{1}{2}, 1]$ *and* $(\theta_W - \theta_S - \gamma v) < 0$

The optimal forward-guidance policy plan in this environment is shown in Table A.1.

Plugging in the values shown in Table A.1, the expected inflations of the private sector are given as follows: after the private sector receives the optimistic message from the central bank,

[12] I thank the anonymous referee and the editor for the suggestion to obtain forward guidance and inflation-targeting bandwidth jointly and compare welfare levels.

Table A.1. Optimal Overly Pessimistic Forward-Guidance Plan if $(\theta_W - \theta_S - \gamma v) < 0$ (relatively unemployment fighting) and $\psi_0 \in [\frac{1}{2}, 1]$

$\sigma : S \rightarrow \Delta(M)$	m_{opt}	m_{pes}
s_S	$\frac{1}{2\psi_0}$	$(1 - \frac{1}{2\psi_0})$
s_W	0	1

$$\begin{aligned}
&\pi^e(\sigma(\cdot), m_{opt}) \\
&\quad = \pi^T + v \cdot \left[\frac{(1-2\psi_S)\psi_0\rho_{opt} + (2\psi_W - 1)(1-\psi_0)(1-\rho_{pes})}{(1-\psi_0)(1-\rho_{pes}) + \psi_0\rho_{opt}}\right] \\
&\qquad \text{plugging in } \rho_{opt} = \frac{1}{2\psi_0},\ 1 - \rho_{pes} = 0 \qquad \text{(A.1)} \\
&\quad = \pi^T + v \cdot (1 - 2\psi_S).
\end{aligned}$$

That is, the private sector recognizes that m_{opt} is only sent when the central bank observes s_S with probability 1. By contrast, after the private sector receives the pessimistic message from the central bank, the private sector is not sure if the central bank observes s_S or s_W.

$$\begin{aligned}
&\pi^e(\sigma(\cdot), m_{pes}) \\
&\quad = \pi^T + v \cdot \left[\frac{(1-2\psi_S)\psi_0(1-\rho_{opt}) + (2\psi_W - 1)(1-\psi_0)\rho_{pes}}{(1-\psi_0)\rho_{pes} + \psi_0(1-\rho_{opt})}\right] \\
&\qquad \text{plugging in } 1 - \rho_{opt} = \left(1 - \frac{1}{2\psi_0}\right),\ \rho_{pes} = 1 \qquad \text{(A.2)} \\
&\quad = \pi^T + v \cdot \{(1-2\psi_S)(2\psi_0 - 1) + (2\psi_W - 1)(2 - 2\psi_0)\}
\end{aligned}$$

A.2 $\psi_0 \in [0, 1/2)$ *and* $(\theta_W - \theta_S - \gamma v) < 0$

The optimal forward-guidance policy plan in this environment is shown in Table A.2.

Under the truthful forward-guidance policy, after the private sector receives the optimistic message m_{opt}, the private sector is

Table A.2. Truthful Forward-Guidance Policy Function if $\theta_W - \theta_S - \gamma v < 0$ (relatively unemployment fighting) and $\psi_0 \in [0, \frac{1}{2}]$

$\sigma : S \to \Delta(M)$	m_{opt}	m_{pes}
s_S	1	0
s_W	0	1

assured that the central bank observes the signals for the future strong economy s_S. The expected inflation is updated as follows:

$$\begin{aligned}
&\pi^e(\sigma(\cdot), m_{opt}) \\
&= \pi^T + v \cdot \left[\frac{(1-2\psi_S)\psi_0\rho_{opt} + (2\psi_W - 1)(1-\psi_0)(1-\rho_{pes})}{(1-\psi_0)(1-\rho_{pes}) + \psi_0\rho_{opt}}\right] \\
&\quad \text{plugging in } \rho_{opt} = 1,\ 1-\rho_{pes} = 0 \qquad \text{(A.3)} \\
&= \pi^T + v \cdot (1-2\psi_S).
\end{aligned}$$

Notice that this is also the case under the above optimal overly pessimistic forward-guidance policy: receiving m_{opt} means the central bank observing s_S for sure. Therefore, the expected inflation $\pi^e(\sigma(\cdot), m_{opt})$ becomes equivalent.

When the private sector receives the pessimistic message m_{pes} from the central bank, the private sector is assured that the central bank observes the signals for the future weak economy s_W. The expected inflation is given by the following:

$$\begin{aligned}
&\pi^e(\sigma(\cdot), m_{pes}) \\
&= \pi^T + v \cdot \left[\frac{(1-2\psi_S)\psi_0(1-\rho_{opt}) + (2\psi_W - 1)(1-\psi_0)\rho_{pes}}{(1-\psi_0)\rho_{pes} + \psi_0(1-\rho_{opt})}\right] \\
&\quad \text{plugging in } 1-\rho_{opt} = 0,\ \rho_{pes} = 1 \qquad \text{(A.4)} \\
&= \pi^T + v \cdot (2\psi_W - 1).
\end{aligned}$$

A.3 $(\theta_W - \theta_S - \gamma v) > 0$

In the case of uninformative communication policy, the central bank announcement does not induce any updates in the expected inflation.

Table A.3. Optimal Uninformative Forward-Guidance Policy Function if $\theta_W - \theta_S - \gamma v > 0$ (relatively inflation fighting)

$\sigma : S \to \Delta(M)$	m_{opt}	m_{pes}
s_S	1/2	1/2
s_W	1/2	1/2

Hence, the expected inflation remains the same as under the prior irrespective of the messages.

$$\begin{aligned}\pi^e(\sigma(\cdot), m_{opt}) &= \pi^e(\sigma(\cdot), m_{pes}) \\ &= \pi^T + v \cdot [(1 - 2\psi_S)\psi_0 + (2\psi_W - 1)(1 - \psi_0)]\end{aligned}$$

B. Optimal Symmetric Inflation-Targeting Bandwidth v^*

In this section, the optimal pairs of forward guidance and inflation-targeting bandwidth are presented. For each optimal communication policy, a paired optimal inflation-targeting bandwidth can be computed.

B.1 $\psi_0 \in [1/2, 1]$ *and* $(\theta_W - \theta_S - \gamma v) < 0$

In this environment, by plugging in the optimal overly pessimistic forward-guidance policy, the ex ante social welfare is given as follows:

$$\begin{aligned}\max_v \quad & \mathbb{E}[SW] \\ &= \left\{-\left(\theta_S + \gamma \cdot v(2 - 2\psi_S)\right)^2 - \alpha v^2\right\} \cdot \frac{\psi_S}{2} \\ &\quad + \Big\{ - \Big(\theta_S + \gamma \cdot v(2 - 2\psi_S) + 4 \cdot v\gamma \cdot (1 - \psi_0) \\ &\quad \cdot (\psi_S + \psi_W - 1)\Big)^2 - \alpha v^2\Big\} \times \left(\phi - \frac{\psi_S}{2}\right) \\ &\quad + \left\{-\left(\theta_W + \gamma \cdot v(-2\psi_S)\right)^2 - \alpha v^2\right\} \cdot \frac{1 - \psi_S}{2} \\ &\quad + \Big\{ - \Big(\theta_W + \gamma \cdot v(-2\psi_S) + 4 \cdot v\gamma \cdot (1 - \psi_0) \\ &\quad \cdot (\psi_S + \psi_W - 1)\Big)^2 - \alpha v^2\Big\} \left(1 - \phi - \frac{1 - \psi_S}{2}\right), \end{aligned} \tag{B.1}$$

where $\phi = \psi_0\psi_S + (1-\psi_0)(1-\psi_W)$ and $(1-\phi_0) = \psi_0(1-\psi_S) + (1-\psi_0)\psi_W$.

Then, by taking derivatives, we have the welfare-maximizing inflation-targeting bandwidth v:

$$v = \frac{\gamma(\theta_W - \theta_S)\left[\psi_S(1-\psi_S) + (2\phi - \psi_S)\left\{2(1-\phi) - (1-\psi_S)\right\}\right]}{2 \cdot \gamma^2\left[\psi_S(1-\psi_S) + (2\phi-\psi_S)\{2(1-\phi) - (1-\psi_S)\}\right] + \alpha}. \tag{B.2}$$

However, this value of v will not satisfy the condition $(\theta_W - \theta_S - \gamma v) < 0$. Therefore, the infimum of this bound $v_{threshold} = \frac{\theta_W - \theta_S}{\gamma}$ is to be paired as optimal inflation-targeting bandwidth with the optimally overly pessimistic forward-guidance policy.

B.2 $\psi_0 \in [0, \frac{1}{2})$ *and* $(\theta_W - \theta_S - \gamma v) < 0$

In this environment, by plugging in the optimal truthful forward-guidance policy, the ex ante social welfare is given as follows:

$$\begin{aligned}
\max_v \quad & \mathbb{E}[SW] \\
= & -\left[(\theta_S + \gamma \cdot v \cdot (2 - 2\psi_S))^2 + \alpha v^2\right]\psi_0\psi_S \\
& -\left[\left(\theta_S + \gamma \cdot v \cdot (2\psi_W)\right)^2 + \alpha v^2\right](1-\psi_0)(1-\psi_W) \\
& -\left[(\theta_W + \gamma \cdot v \cdot (-2\psi_S))^2 + \alpha v^2\right]\psi_0(1-\psi_S) \\
& -\left[(\theta_W + \gamma \cdot v \cdot (2\psi_W - 2))^2 + \alpha v^2\right](1-\psi_0)\psi_W.
\end{aligned} \tag{B.3}$$

Then, from the optimization condition, we have

$$v = \frac{2\gamma \cdot (\theta_W - \theta_S)\left(\psi_0(1-\psi_S)\psi_S + (1-\psi_0)\psi_W(1-\psi_W)\right)}{4\gamma^2 \cdot \left(\psi_0(1-\psi_S)\psi_S + (1-\psi_0)\psi_W(1-\psi_W)\right) + \alpha}. \tag{B.4}$$

This value of v also fails to satisfy the threshold condition $(\theta_W - \theta_S - \gamma v) < 0$. Therefore, the infimum of this bound $v_{threshold} = \frac{\theta_W - \theta_S}{\gamma}$ is to be paired as optimal inflation-targeting bandwidth with the optimal truthful forward-guidance policy.

B.3 $(\theta_W - \theta_S - \gamma v) > 0$

In this inflation-fighting environment, by plugging in the optimal uninformative forward-guidance policy, the ex ante social welfare is given as follows:

$$\begin{aligned}\max_v \quad & \mathbb{E}[SW] \\ &= -\left[(\theta_S + \gamma \cdot v \cdot (2\psi_W - Z\psi_0))^2 + \alpha v^2\right]\phi \\ &\quad - \left[(\theta_W + \gamma \cdot v \cdot (2\psi_W - Z\psi_0 - 2))^2 + \alpha v^2\right](1-\phi), \qquad \text{(B.5)}\end{aligned}$$

where $\phi = \big(\psi_0\psi_S + (1-\psi_0)(1-\psi_W)\big)$, $(1-\phi) = \big(\psi_0(1-\psi_S) + (1-\psi_0)\psi_W\big)$. Then, the optimal v^* is given as follows:

$$v^* = \frac{2\gamma\,(\theta_W - \theta_S)\,\phi(1-\phi)}{4 \cdot \gamma^2\phi(1-\phi) + \alpha}. \qquad \text{(B.6)}$$

Notice that as the weight on quadratic loss from inflation gap, α, increases, the optimal inflation-targeting bandwidth v^* will decrease. Also, optimal v^* is not zero for the relatively inflation-fighting central bank.

To understand why strict inflation targeting is not optimal, consider the scenario where v is set zero. Then ex post realized inflation is always π^T, irrespective of the state of the economy in this model. That means expected inflation of the private sector will be fixed at π^T for any communication policy. As a result, the economy will experience zero losses from inflation gap, whereas the economy shall take the welfare losses from cyclical unemployment shocks θ_W or θ_S, without any mitigation. By contrast, if inflation bandwidth v^* is set as positive, the central bank will be able to generate surprises in inflation which can mitigate the impact from the cyclical unemployment shocks.

B.4 Global Optimal among Pairs of Forward Guidance and Symmetric Inflation-Targeting Bandwidth

I have three local optimal pairs of forward-guidance policy and inflation-targeting bandwidth in different economic environments as in Table B.1: first, overly pessimistic forward-guidance policy

Table B.1. Optimal Pairs of Forward Guidance and Symmetric Inflation-Targeting Bandwidth (no self-fulfilling effect of private sector expected inflation)

	$\psi_0 \geq 1/2$	$\psi_0 < 1/2$	$\forall \psi_0$
Forward Guidance	Overly Pessimistic	Truthful	Uninformative
Inflation Bandwidth	$v_{threshold} = \frac{\theta_W - \theta_S}{\gamma}$		$v^* = \frac{2\gamma(\theta_W - \theta_S)\varphi(1-\varphi)}{4\cdot\gamma^2\varphi(1-\varphi)+\alpha}$
Welfare Comparison	Suboptimal	Suboptimal	Global Optimal Pair

and inflation-targeting bandwidth of $v_{threshold} = \frac{\theta_W - \theta_S}{\gamma}$; second, truthful forward-guidance policy and inflation-targeting bandwidth $v_{threshold} = \frac{\theta_W - \theta_S}{\gamma}$; third, uninformative forward guidance and optimal inflation-targeting bandwidth

$$v^* = \frac{(\theta_W - \theta_S)}{2\gamma + \frac{\alpha}{2\gamma(1-\phi)\phi}}, \tag{B.7}$$

where $\phi(1-\phi) = \{\psi_0\psi_S + (1-\psi_0)(1-\psi_W)\} \cdot \{\psi_0(1-\psi_S) + (1-\psi_0)\psi_W\}$.

Then, which one would give the highest ex ante expected social welfare? To examine the levels of ex ante expected social welfare among these threefold results, first the overly pessimistic forward-guidance policy and $\gamma \cdot v_{threshold} = \theta_W - \theta_S$ are plugged in:

$$\begin{aligned} &\mathbb{E}[SW_{pess}] \\ &\quad = -\left(\theta_S + \gamma \cdot v_{threshold} 4p_3\right)^2 \cdot p_1 - \left[\theta_S + \gamma \cdot v_{threshold} 4p_4\right]^2 \cdot p_2 \\ &\qquad - \left(\theta_W + \gamma \cdot v_{threshold}(-4p_1)\right)^2 \cdot p_3 \\ &\qquad - \left[\theta_W + \gamma \cdot v_{threshold}(-4p_2)\right]^2 \cdot p_4 - \alpha(v_{threshold})^2, \end{aligned} \tag{B.8}$$

where $p_1 = \frac{\psi_S}{2}$, $p_2 = \left(\phi - \frac{\psi_S}{2}\right)$, $p_3 = \frac{1-\psi_S}{2}$, $p_4 = \left(1 - \phi - \frac{1-\psi_S}{2}\right)$. Observe that $p_1 + p_2 = \phi$ and $p_3 + p_4 = 1 - \phi$. Also, $p_1 + p_3 = \frac{1}{2} = p_2 + p_4$.

Second, the truthful forward-guidance policy and $\gamma \cdot v_{threshold} = \theta_W - \theta_S$ are plugged in:

$$
\begin{aligned}
&\mathbb{E}[SW_{truth}] \\
&\quad = -\left(\theta_S + \gamma \cdot v_{threshold} \cdot (2 - 2\psi_S)\right)^2 \psi_0 \psi_S \\
&\qquad - \left(\theta_S + \gamma \cdot v_{threshold} \cdot (2\psi_W)\right)^2 (1 - \psi_0)(1 - \psi_W) \\
&\qquad - \left(\theta_W + \gamma \cdot v_{threshold} \cdot (-2\psi_S)\right)^2 \psi_0 (1 - \psi_S) \\
&\qquad - \left(\theta_W + \gamma \cdot v_{threshold} \cdot (2\psi_W - 2)\right)^2 (1 - \psi_0)\psi_W - \alpha(v_{threshold})^2.
\end{aligned} \tag{B.9}
$$

Comparing $\mathbb{E}[SW_{pess}]$ and $\mathbb{E}[SW_{truth}]$, the inequality depends on the value of ψ_0. If $\psi_0 \geq 1/2$, then $\mathbb{E}[SW_{pess}]$ gives greater ex ante expected social welfare; otherwise, $\mathbb{E}[SW_{truth}]$ is greater.

Third, the uninformative forward-guidance policy and paired optimal inflation-targeting bandwidth $v^*_{uninfo} = \frac{(\theta_W - \theta_S)}{2\gamma + \frac{\alpha}{2\gamma(1-\phi)\phi}}$ are plugged in. Then, $\gamma \cdot v^*_{uninfo}$ part is replaced by $\left(\frac{\theta_W - \theta_S}{2} - \epsilon\right)$ while $\epsilon = \frac{\alpha(\theta_W - \theta_S)}{2\alpha + 8\gamma^2\phi(1-\phi)}$.

$$
\begin{aligned}
&\mathbb{E}[SW_{uninfo}] \\
&\quad = -\left(\theta_S + \gamma \cdot v^*_{uninfo} \cdot 2(1 - \phi)\right)^2 \cdot \phi \\
&\qquad - \left(\theta_W + \gamma \cdot v^*_{uninfo} \cdot (-2)\phi\right)^2 \cdot (1 - \phi) - \alpha(v^*_{uninfo})^2
\end{aligned} \tag{B.10}
$$

Assume that $\psi_0 > \frac{1}{2}$. To compare Equation (B.10) of $\mathbb{E}[SW_{uninfo}]$ with Equation (B.8) of $\mathbb{E}[SW_{pess}]$, first, a fixed inflation-targeting bandwidth v is plugged in, and next, the fact that (B.10) of $\mathbb{E}[SW_{uninfo}]$ is local optimal is utilized.

Fix $v = v_{threshold}$ and observe that $p_1 > p_2$ and $p_3 < p_4$.

$$
\begin{aligned}
&\mathbb{E}[SW_{pess};\ v = v_{threshold}] - \mathbb{E}[SW_{uninfo};\ v = v_{threshold}] \\
&\quad = -(\theta_W - \theta_S - \gamma \cdot v_{threshold}) 4\gamma \cdot v_{threshold}(p_1 - p_2)(p_4 - p_3) \\
&\qquad + (p_4 - p_3 - p_1 + p_2) 4\gamma^2 (v_{threshold})^2 \left\{2p_1p_3 - 2p_2p_4\right\}
\end{aligned} \tag{B.11}
$$

Since $(p_4 - p_3 - p_1 + p_2) = 0$, it is clear that $\mathbb{E}[SW_{pess};\ v = v_{threshold}] = \mathbb{E}[SW_{uninfo};\ v = v_{threshold}]$ by plugging in

$\gamma \cdot v_{threshold} = (\theta_W - \theta_S)$. Then, as local optimal pair for the uninformative forward-guidance policy is given by $v = v^*_{uninfo}$, it is obvious that $\mathbb{E}[SW_{uninfo};\ v = v^*_{uninfo}] > \mathbb{E}[SW_{uninfo};\ v = v_{threshold}]$. Therefore, $\mathbb{E}[SW_{pess};\ v = v_{threshold}] < \mathbb{E}[SW_{uninfo};\ v = v^*_{uninfo}]$. In conclusion, if $\psi_0 \geq 1/2$, the global optimal pair of forward-guidance policy and symmetric inflation-targeting policy will be the uninformative forward-guidance and inflation bandwidth v^*_{uninfo}.

Now assume that $\psi_0 < \frac{1}{2}$. To compare Equation (B.10) of $\mathbb{E}[SW_{uninfo}]$ with Equation (B.9) of $\mathbb{E}[SW_{truth}]$, first, a fixed inflation-targeting bandwidth v is plugged in, and next, the fact that (B.10) of $\mathbb{E}[SW_{uninfo}]$ is local optimal is utilized.

Fix $v = v_{threshold}$ and observe that $\psi_S - \phi = \psi - \psi_0\psi_S - (1-\psi_0)(1-\psi_W) = (1-\psi_0)(\psi_S + \psi_W - 1)$ and $1 - \phi - \psi_W = \psi_0(1-\psi_S) + (1-\psi_0)\psi_W - \psi_W = -\psi_0(\psi_S + \psi_W - 1)$.

$$\begin{aligned}
&\mathbb{E}[SW_{truth}] - \mathbb{E}[SW_{uninfo}] \\
&\quad = -\left(\theta_S + \gamma \cdot v_{threshold} \cdot (2 - 2\psi_S)\right)^2 \psi_0\psi_S \\
&\qquad + \left(\theta_S + \gamma \cdot v_{threshold} \cdot 2(1-\phi)\right)^2 \psi_0\psi_S \\
&\qquad - \left(\theta_S + \gamma \cdot v_{threshold} \cdot (2\psi_W)\right)^2 (1-\psi_0)(1-\psi_W) \\
&\qquad + \left(\theta_S + \gamma \cdot v_{threshold} \cdot 2(1-\phi)\right)^2 (1-\psi_0)(1-\psi_W) \\
&\qquad - \left(\theta_W + \gamma \cdot v_{threshold} \cdot (-2\psi_S)\right)^2 \psi_0(1-\psi_S) \\
&\qquad + \left(\theta_W + \gamma \cdot v_{threshold} \cdot (-2)\phi\right)^2 \psi_0(1-\psi_S) \\
&\qquad - \left(\theta_W + \gamma \cdot v_{threshold} \cdot (2\psi_W - 2)\right)^2 (1-\psi_0)\psi_W \\
&\qquad + \left(\theta_W + \gamma \cdot v_{threshold} \cdot (-2)\phi\right)^2 (1-\psi_0)\psi_W \\
&\qquad - \alpha(v_{threshold})^2 + \alpha(v_{threshold})^2 \\
&\quad = -4\gamma \cdot v_{threshold} \cdot \psi_0(1-\psi_0)(\psi_S + \psi_W - 1)^2 \\
&\qquad (\theta_W - \theta_S - \gamma \cdot v_{threshold}).
\end{aligned} \tag{B.12}$$

It is clear that $\mathbb{E}[SW_{truth};\ v = v_{threshold}] = \mathbb{E}[SW_{uninfo};\ v = v_{threshold}] < \mathbb{E}[SW_{uninfo};\ v = v_{threshold}]$. Hence, if $\psi_0 < 1/2$, the global optimal pair of forward guidance and inflation-targeting policy will be the uninformative forward guidance with the bandwidth v^*_{uninfo}.

The welfare comparison results imply that if a central bank is allowed to choose a pair of forward-guidance policy and a symmetric inflation-targeting policy, in order to maximize the ex ante expected social welfare, the central bank should follow the uninformative forward guidance and inflation targeting with a relatively narrow inflation bandwidth for all ψ_0. Additionally, notice that $v^*_{uninfo} > 0$: the strict inflation targeting is not optimal.

C. Generalization: Asymmetric Inflation-Targeting Bandwidth under Incomplete Control of Central Bank over Inflation

C.1 Incomplete Control of Central Bank over Inflation

In the main model, the central bank has complete control over the realized inflation. The monetary policy function $\pi : \Theta \to \mathbb{R}$ is given as the actually realized inflation function. That is, when the state of the economy is θ_S, the central bank implements monetary policy toward $\pi(\theta_S) = \pi^T - v$, and the realized inflation is actually $\pi^T - v$. Similarly, when the state of the economy is θ_W, the central bank implements its monetary policy toward $\pi(\theta_W) = \pi^T + v$, and the realized inflation is as the central bank desires. In this sense, inflation is under complete control of the central bank in the main model, and the expected inflation of the private sector does not affect the realized inflation at all. Therefore, inflation gap between realized inflation and the target π^T is either $\pm v$.

Although the assumption that the central bank completely control the ex post realized inflation is not uncommon in the literature (see Geraats 2002, p. F537), a more realistic assumptions would be that the central bank has incomplete control over the realized inflation. The expected inflation of the private sector can affect the realized inflation, such as

$$\pi^{ex.post}(\theta_., m_.) = \xi \cdot \pi^{policy}(\theta_.) + (1 - \xi) \cdot \pi^e(\sigma(\cdot), m_.),$$

where ξ is the degree of control of the central bank over the ex post realized inflation. If $\xi = 1$, the central bank has complete control over inflation, which is the main model. As ξ declines, the private sector's expected inflation begins to affect the realized inflation, and if ξ approaches zero, it is so called self-fulfilling expectation. Observe that expected inflation is set as the private sector forecasts the future inflation as follows:

$$\begin{aligned}\pi^e(\sigma(\cdot), m_.) &= \widehat{E}\left[\pi^{ex.post}(\theta_., m_.) \mid m_.\right] \\ &= \xi \cdot \widehat{E}\left[\pi^{policy}(\theta_.) \mid m_.\right] + (1-\xi)\cdot \pi^e(\sigma(\cdot), m_.).\end{aligned}$$

Hence, expected inflation is the solution of the above fixed-point problem, and the expected inflation remains the same to the formulation under the case where the central bank has complete control over realized inflation.

$$\pi^e(\sigma(\cdot), m_.) = \widehat{E}\left[\pi^{policy}(\theta_.) \mid m_.\right]$$

As a result, the threefold results of the optimal forward-guidance policy under exogenously given inflation targeting do not change. However, the optimal inflation-targeting bandwidth paired with each optimal forward guidance needs modification because the losses from inflation gap contributes less to the ex ante expected social welfare as much as $\alpha\xi$. In the model of incomplete control of the central bank over inflation, the social welfare function is defined by

$$\begin{aligned}&W(\pi^e(\sigma(\cdot), m_.),\ \pi^{ex.post}(\theta, m), \theta) \\ &\quad = -(u - u^{NAR})^2 - \alpha\cdot(\pi^{ex.post}(\theta_., m_.) - \pi^T)^2 \\ &\quad = -[\theta_. + \gamma\cdot(\pi^e(\sigma(\cdot), m_.) - \pi^{ex.post}(\theta_., m_.))]^2 \\ &\qquad - \alpha\cdot(\pi^{ex.post}(\theta_., m_.) - \pi^T)^2 \\ &\quad = -[\theta_. + \gamma\cdot\xi\cdot(\pi^e(\sigma(\cdot), m_.) - \pi^{policy}(\theta_., m_.))]^2 \\ &\qquad - \alpha\cdot\left[(1-\xi)\left\{\pi^e(\sigma(\cdot), m_.) - \pi^T\right\} + \xi(\pi^{policy}(\theta_.) - \pi^T)\right]^2. \qquad \text{(C.1)}\end{aligned}$$

Observe that inflation gap is measured by the difference between ex post realized inflation and the inflation target π^T. The quadratic loss from inflation gap is given by $-\alpha(\pi^{ex.post}(\theta_., m_.) - \pi^T)^2$, weighed by $\alpha > 0$.

C.2 Optimal Forward Guidance under Exogenous Asymmetric Inflation-Targeting Bandwidth

How would the results change if the inflation-targeting bandwidth is allowed to be asymmetric?[13] The monetary policy function is now set to $\pi^{policy}(\theta_S) = \pi^T - v_S$ and $\pi^{policy}(\theta_W) = \pi^T + v_W$. Assuming the possible incomplete control of the central bank over realized inflation, the private sector's expected inflation under each message is given as follows:

$$\pi^e(\sigma(\cdot), m_{opt}) = \pi^T + \left[\frac{\{(1-\psi_S)v_W - \psi_S v_S\}\psi_0 \rho_{opt} + \{(\psi_W - 1)v_S + \psi_W v_W\}(1-\psi_0)(1-\rho_{pes})}{(1-\psi_0)(1-\rho_{pes}) + \psi_0 \rho_{opt}} \right]$$

$$\pi^e(\sigma(\cdot), m_{pes}) = \pi^T + \left[\frac{\{(1-\psi_S)v_W - \psi_S v_S\}\psi_0 (1-\rho_{opt}) + \{(\psi_W - 1)v_S + \psi_W v_W\}(1-\psi_0)\rho_{pes}}{(1-\psi_0)\rho_{pes} + \psi_0 (1-\rho_{opt})} \right]. \quad \text{(C.2)}$$

Then the forward-guidance policy is derived from the following first-order conditions (FOCs) with respect to ρ_{opt} and d:

$$\frac{\partial \mathbb{E}_{s.,m.,\theta.}(SW)}{\partial \rho_{opt}} = \left(\frac{(v_S + v_W)(\psi_S + \psi_W - 1)^2 \psi_0^2 (1-\psi_0)^2 d^2 (1-2X)}{X^2(1-X)^2} \right) \times \left[2\gamma\xi(\theta_W - \theta_S) + \left\{ \alpha(1-\xi^2) - \gamma^2\xi^2 \right\} (v_S + v_W) \right] \quad \text{(C.3)}$$

$$\frac{\partial \mathbb{E}_{s.,m.,\theta.}(SW)}{\partial d} = \left\{ \frac{(v_S + v_W)(\psi_S + \psi_W - 1)^2 \psi_0^2 (1-\psi_0)^2 \cdot d}{X^2(1-X)^2} [X(1-\rho_{opt}) + (1-X)\rho_{opt}] \right\} \times \left[-2\gamma\xi(\theta_W - \theta_S) - \left\{ \alpha(1-\xi^2) - \gamma^2\xi^2 \right\} (v_S + v_W) \right], \quad \text{(C.4)}$$

[13] I thank the anonymous referee for encouraging me to develop this section.

Table C.1. Optimal Overly Pessimistic Forward-Guidance Plan if $[2\gamma\xi(\theta_W - \theta_S) + \{\alpha(1-\xi^2) - \gamma^2\xi^2\}(v_S + v_W)] < 0$ and $\psi_0 \in [\frac{1}{2}, 1]$

$\sigma : S \to \Delta(M)$	m_{opt}	m_{pes}
s_S	$\frac{1}{2\psi_0}$	$(1 - \frac{1}{2\psi_0})$
s_W	0	1

where $X \stackrel{def}{=} (1-\psi_0)(\rho_{opt} - d) + \psi_0\rho_{opt} = \rho_{opt} - d(1-\psi_0)$, $(1 - X) \stackrel{def}{=} (1-\psi_0)(1-\rho_{opt}+d) + \psi_0(1-\rho_{opt}) = (1-\rho_{opt}) + d(1-\psi_0)$, $Z \stackrel{def}{=} 2\psi_W + 2\psi_S - 2$. The threefold results of the optimal forward-guidance policy function remain unchanged, but the threshold condition changes. The threshold condition is now $\left[2\gamma\xi(\theta_W - \theta_S) + \left\{\alpha(1-\xi^2) - \gamma^2\xi^2\right\}(v_S + v_W)\right] \gtrless 0$. For the paired optimal inflation-targeting bandwidth, again, plug in the optimal forward-guidance policy for each environment, take the first-derivative w.r.t. v_S and v_W, and find the values that maximize the ex ante expected social welfare: $\dfrac{\partial \mathbb{E}_{s.,m.,\theta.}(SW)}{\partial v} = 0$. The optimal pairs of forward guidance and asymmetric inflation-targeting bandwidth are presented below by the types of forward-guidance policy.

C.3 Optimal Asymmetric Inflation Bandwidth Paired with Overly Pessimistic Forward Guidance

When the parameters about the economic fundamental satisfy $\psi_0 \in [\frac{1}{2}, 1]$ and $\left[2\gamma\xi(\theta_W - \theta_S) + \left\{\alpha(1-\xi^2) - \gamma^2\xi^2\right\}(v_S + v_W)\right] < 0$, the optimal forward-guidance policy is as the optimal overly pessimistic forward-guidance plan (see Table C.1).

By plugging the forward-guidance policy function from Table C.1 into the ex ante expected social welfare function (for computational details, please see Ko 2022), the paired optimal inflation-targeting bandwidth is derived as follows:

$$v_S^* = \frac{(\theta_W - \theta_S)\,\gamma\xi\,(C_1 + (1-\psi_S))\left\{(1-\psi_S)\psi_S + 2A_1\left(1 - 2A_1\right)\right\}}{B_1},$$

$$v_W^* = \frac{(\theta_W - \theta_S)\,\gamma\xi \cdot C_1\left\{(1-\psi_S)\psi_S + 2A_1\left(1 - 2A_1\right)\right\}}{B_1},$$

where $2A_1 \overset{def}{=} 2(1-\psi_0)(\psi_S+\psi_W-1)+(1-\psi_S) = 2\left(\frac{2\psi_0-1}{2}\cdot(1-\psi_S)+(1-\psi_0)\cdot\psi_W\right)$, $(1-2A_1) \overset{def}{=} -2(1-\psi_0)(\psi_S+\psi_W-1)+\psi_S = 2\left(\frac{2\psi_0-1}{2}\cdot\psi_S+(1-\psi_0)\cdot(1-\psi_W)\right)$, $B_1 \overset{def}{=} 2\alpha(1-\psi_0)(\psi_S+\psi_W-1)\Big\{(1-\psi_0)(\psi_S+\psi_W-1)+(1-\psi_S)\Big\}+\Big\{\gamma^2\xi^2-\alpha(1-\xi^2)\Big\}2A_1\Big\{(1-\psi_S)\psi_S+2A_1\left(1-2A_1\right)\Big\}$, $C_1 \overset{def}{=} (1-\psi_0)(\psi_S+\psi_W-1)$.

However, the value above $(v_W^*+v_S^*)$ does not satisfy the threshold condition about $\left[2\gamma\xi(\theta_W-\theta_S)+\left\{\alpha(1-\xi^2)-\gamma^2\xi^2\right\}(v_W^*+v_S^*)\right]$. Then, the optimal asymmetric inflation-targeting bandwidth paired with the overly pessimistic forward guidance is determined by the threshold:

$$v_S'+v_W' = \frac{2\gamma\xi(\theta_W-\theta_S)}{\gamma^2\xi^2-\alpha(1-\xi^2)}$$
$$v_S' = \frac{\{C_1+(1-\psi_S)\}\,2\gamma\xi(\theta_W-\theta_S)}{A_1\{\gamma^2\xi^2-\alpha(1-\xi^2)\}}, \quad v_W' = \frac{C_1\cdot 2\gamma\xi(\theta_W-\theta_S)}{A_1\{\gamma^2\xi^2-\alpha(1-\xi^2)\}}. \tag{C.5}$$

Clearly, $v_W' < v_S'$, where $v_W' = \frac{C_1}{C_1+(1-\psi_S)}\cdot v_S'$ following the optimal relationship given by $v_W^* = \frac{C_1}{C_1+(1-\psi_S)}\cdot v_S^*$. As the monetary policy is constructed as $\pi^{policy}(\theta_W) = \pi+v_W$ and $\pi^{policy}(\theta_S) = \pi - v_S$, the bandwidth above the inflation target is relatively narrow while the bandwidth beneath the inflation target is relatively wide.

Recall that α is the weight which the central bank puts on the inflation gap. Clearly, as α increases, the threshold increases that bifurcates whether the overly pessimistic forward guidance being optimal or the uninformative forward guidance being optimal $\left(\frac{\partial(v_S'+v_W')}{\partial\alpha}>0\right)$ when $\psi_0 \geq 1/2$. Concurrently, the constrained optimal asymmetric inflation-targeting bandwidth (v_S', v_W') paired with the overly pessimistic forward guidance increases.

It is worth noting that as ξ decreases, i.e., the private sector plays a key role in realization of ex post inflation, the denominator decreases rapidly since $\xi^2 < \xi \leq 1$. Hence, if the central bank has weaker influence over ex post realized inflation and the private sector's influence becomes more significant, i.e., when ξ decreases, then the optimal inflation-targeting bandwidth $(v_S'+v_W')$ increases

Table C.2. Truth Forward-Guidance Policy Function if $\psi_0 \in [0, \frac{1}{2}]$ and $[2\gamma\xi(\theta_W - \theta_S) + \{\alpha(1-\xi^2) - \gamma^2\xi^2\}(v_S + v_W)] < 0$

$\sigma : S \to \Delta(M)$	m_{opt}	m_{pes}
s_S	1	0
s_W	0	1

$\left(\frac{\partial(v'_S+v'_W)}{\partial\xi} < 0\right)$. This result seems intuitive that when the private sector influences more significantly, the role of the expected inflation of the private sector to the social welfare function becomes more significant: the economy is better off when the central bank has more room to adjust cyclical unemployment through the expected inflation.

C.4 Optimal Asymmetric Inflation Bandwidth Paired with Truthful Forward Guidance

When the parameters about the economic fundamental satisfies $\psi_0 \in [0, \frac{1}{2})$ and $\left[2\gamma\xi(\theta_W - \theta_S) + \left\{\alpha(1-\xi^2) - \gamma^2\xi^2\right\}(v_S + v_W)\right] < 0$, the optimal forward-guidance policy plan is as the truthful policy (see Table C.2).

By plugging in the truthful policy from Table C.2, the optimal asymmetric inflation-targeting bandwidth is derived as follows:

$$v_S^{**} = \frac{(\theta_W - \theta_S)\,\gamma\cdot\xi\cdot(1-\phi)}{\gamma^2\xi^2 - \alpha\xi(1-\xi) + \alpha\xi(1-\phi)},$$
$$v_W^{**} = \frac{(\theta_W - \theta_S)\,\gamma\cdot\xi\cdot\phi}{\gamma^2\xi^2 - \alpha\xi(1-\xi) + \alpha\xi(1-\phi)}, \tag{C.6}$$

where $\phi = \psi_0\psi_S + (1-\psi_0)(1-\psi_W)$, $(1-\phi) = \psi_0(1-\psi_S) + (1-\psi_0)\psi_W$.

Unfortunately, the values above (v_S^{**}, v_W^{**}) do not satisfy the threshold condition: $\left[2\gamma\xi(\theta_W - \theta_S) + \left\{\alpha(1-\xi^2) - \gamma^2\xi^2\right\}(v_S^{**}+v_W^{**})\right] < 0$.

Therefore, the optimal bandwidth paired with the truthful forward guidance is determined by the threshold:

$$v_S'' + v_W'' = \frac{2\gamma\xi(\theta_W - \theta_S)}{\gamma^2\xi^2 - \alpha(1-\xi^2)} \tag{C.7}$$

$$v_S'' = \frac{(1-\phi)2\gamma\xi(\theta_W - \theta_S)}{\gamma^2\xi^2 - \alpha(1-\xi^2)}, \quad v_W'' = \frac{\phi 2\gamma\xi(\theta_W - \theta_S)}{\gamma^2\xi^2 - \alpha(1-\xi^2)}.$$

The above values are determined by $v_S'' + v_W'' = \frac{1}{\phi}v_W'' = \frac{1}{1-\phi}v_S''$. This relationship follows the optimal condition that $v_S'' + v_W'' = \frac{1}{\phi}v_W'' = \frac{1}{1-\phi}v_S''$. Observe that, since $\psi_0 < 1/2$, it is more likely that $\phi < (1-\phi)$, and thus $v_W'' = \frac{\phi}{1-\phi}v_S'' < v_S''$. However, the other case $\phi > (1-\phi)$ is not impossible, and $v_W'' = \frac{\phi}{1-\phi}v_S'' > v_S''$ holds if $\phi > (1-\phi)$. Observe that if $\psi_0 = 1/3,\ \psi_S = 8/9,\ \psi_W = 5/9$, then $\phi > (1-\phi)$ holds even if $\psi_0 = 1/3 < 1/2$.

For changes in α and ξ, observe that $\frac{\partial(v_S'' + v_W'')}{\partial\alpha} > 0$ and $\frac{\partial(v_S'' + v_W'')}{\partial\xi} < 0$. Similarly to the overly pessimistic case, as α increases, the threshold for $(v_S + v_W)$ rises that bifurcates whether the truthful forward guidance being optimal or the uninformative forward guidance being optimal when $\psi_0 < 1/2$. Also, when the private sector influences more significantly, i.e., $(1-\xi)$ rises, the role of the expected inflation of the private sector to the social welfare function becomes more significant and the central bank should have more room to adjust the expected inflation by increasing the values of (v_W'', v_S'') accordingly.

C.5 Optimal Asymmetric Inflation Bandwidth Paired with Uninformative Forward Guidance

When the parameters satisfy $\left[2\gamma\xi(\theta_W - \theta_S) + \left\{\alpha(1-\xi^2) - \gamma^2\xi^2\right\}(v_S + v_W)\right] > 0$, the optimal forward-guidance policy is the uninformative one (see Table C.3).

Given the forward-guidance policy shown in Table C.3 as optimal, the paired optimal asymmetric inflation-targeting bandwidth is as follows:

Table C.3. Optimal Uninformative Forward-Guidance Policy Function if $[2\gamma\xi(\theta_W - \theta_S) + \{\alpha(1-\xi^2) - \gamma^2\xi^2\}(v_S + v_W)] > 0$

$\sigma : S \to \Delta(M)$	m_{opt}	m_{pes}
s_S	1/2	1/2
s_W	1/2	1/2

Table C.4. Pairs of Forward Guidance and Asymmetric Inflation-Targeting Bandwidth

	$\psi_0 \geq 1/2$	$\psi_0 < 1/2$	$\forall\psi_0$
Threefold Forward Guidance	OverlyPessimistic	Truthful	Uninformative
	$v'_S + v'_W = v''_S + v''_W = \frac{2\gamma\xi(\theta_W-\theta_S)}{\gamma^2\xi^2-\alpha(1-\xi^2)}$		$v^{***}_S + v^{***}_S = \frac{(\theta_W-\theta_S)\gamma}{\xi(\gamma^2+\alpha)}$

$$v^{***}_S = \frac{(\theta_W - \theta_S)\gamma\,(1-\phi)}{\xi(\gamma^2+\alpha)}, \quad v^{***}_W = \frac{(\theta_W - \theta_S)\gamma\phi}{\xi(\gamma^2+\alpha)} \tag{C.8}$$

where $\phi = (\psi_0\psi_S + (1-\psi_0)(1-\psi_W))$, $(1-\phi) = \big(\psi_0(1-\psi_S) + (1-\psi_0)\psi_W\big)$.

The sum $v^{***}_S + v^{***}_W = \frac{(\theta_W-\theta_S)\gamma}{\xi(\gamma^2+\alpha)}$ satisfies the threshold condition. If α is sufficiently large, the values of v^{***}_S and v^{***}_W are decreasing in α $\left(\frac{\partial(v^{***}_S+v^{***}_W)}{\partial\alpha} < 0\right)$. Clearly, if the private sector's expectation plays a bigger role in the realization of ex post inflation, i.e., ξ declines, then the values of v^{***}_S, v^{***}_W increases $\left(\frac{\partial(v^{***}_S+v^{***}_W)}{\partial\xi} < 0\right)$.

C.6 Welfare Comparison

So far three distinct local optimal pairs of forward-guidance policy and asymmetric inflation-targeting band are derived as in Table C.4. In this section, I compare the welfare levels of these three distinct local optimal pairs. The notations are introduced as $L_S \stackrel{def}{=} (1-\psi_S)v_W - \psi_S v_S$, $L_W = \psi_W v_W - (1-\psi_W)v_S$, $\phi = \psi_0\psi_S + (1-\psi_0)(1-\psi_W)$, $(1-\phi) = \psi_0(1-\psi_S) + (1-\psi_0)\psi_W$, $p_1 = \frac{\psi_S}{2}$, $p_2 = \left(\phi - \frac{\psi_S}{2}\right)$, $p_3 = \frac{1-\psi_S}{2}$, $p_4 = \left(1 - \phi - \frac{1-\psi_S}{2}\right)$. Observe that $p_1 + p_2 = \phi$ and $p_3 + p_4 = 1 - \phi$. Also, $p_1 + p_3 = \frac{1}{2} = p_2 + p_4$. Finally, $\overline{v} \equiv (\psi_0(1-\psi_S) + (1-\psi_0)\psi_W)\, v_W - (\psi_0\psi_S + (1-\psi_0)(1-\psi_W))\, v_S$.

First, the difference between the ex ante expected social welfare associated with the overly pessimistic forward-guidance policy and the one for the uninformative forward-guidance policy is given as follows (when $\psi_0 \in [1/2, 1]$):

$$
\begin{aligned}
&\mathbb{E}\left[SW_{pess};\ (v_W + v_S) = \frac{2\gamma\xi(\theta_W - \theta_S)}{\gamma^2\xi^2 - \alpha(1-\xi^2)}\right] \\
&\quad - \mathbb{E}\left[SW_{uninfo};\ (v_W + v_S) = \frac{2\gamma\xi(\theta_W - \theta_S)}{\gamma^2\xi^2 - \alpha(1-\xi^2)}\right] \\
&= -\left(\theta_S + \gamma\xi(v_S + v_W)\cdot 2p_3\right)^2 \cdot p_1 \\
&\quad + \left\{\theta_S + \gamma\cdot\xi\cdot(p_3+p_4)(v_W+v_S)\right\}^2 p_1 \\
&\quad - \left[\theta_S + \gamma\xi(v_S+v_W)\cdot 2p_4\right]^2 \cdot p_2 \\
&\quad + \left\{\theta_S + \gamma\cdot\xi\cdot(p_3+p_4)(v_W+v_S)\right\}^2 p_2 \\
&\quad - \left(\theta_W + \gamma\xi(v_S+v_W)\cdot(-2p_1)\right)^2 \cdot p_3 \\
&\quad + \left\{\theta_W + \gamma\cdot\xi\cdot(-p_1-p_2)(v_W+v_S)\right\}^2 p_3 \\
&\quad - \left[\theta_W + \gamma\xi(v_S+v_W)\cdot(-2p_2)\right]^2 \cdot p_4 \\
&\quad + \left\{\theta_W + \gamma\cdot\xi\cdot(-p_1-p_2)(v_W+v_S)\right\}^2 p_4 \\
&\quad - \alpha\left\{(1-\xi)L_S + \xi(-v_S)\right\}^2 p_1 \\
&\quad + \alpha\left\{(1-\xi)(1-\psi_0)(L_W-L_S) + (1-\xi)\cdot L_S + \xi(-v_S)\right\}^2 p_1 \\
&\quad - \alpha\left\{(1-\xi)2(1-\psi_0)(L_W-L_S) + (1-\xi)\cdot L_S + \xi(-v_S)\right\}^2 p_2 \\
&\quad + \alpha\left\{(1-\xi)(1-\psi_0)(L_W-L_S) + (1-\xi)\cdot L_S + \xi(-v_S)\right\}^2 p_2 \\
&\quad - \alpha\left\{(1-\xi)L_S + \xi v_W\right\}^2 p_3 \\
&\quad + \alpha\left\{(1-\xi)(1-\psi_0)(L_W-L_S) + (1-\xi)\cdot L_S + \xi v_W\right\}^2 p_3 \\
&\quad - \alpha\left\{(1-\xi)2(1-\psi_0)(L_W-L_S) + (1-\xi)\cdot L_S + \xi v_W\right\}^2 p_4 \\
&\quad + \alpha\left\{(1-\xi)(1-\psi_0)(L_W-L_S) + (1-\xi)\cdot L_S + \xi v_W\right\}^2 p_4 \\
&= -\left\{2\gamma\xi(\theta_W-\theta_S) - \gamma^2\xi^2(v_W+v_S) + \alpha(1-\xi^2)(v_S+v_W)\right\} \\
&\quad (v_S+v_W)(1-\psi_0)^2(\psi_W+\psi_S-1)^2 \\
&= 0.
\end{aligned}
$$

That is, $\mathbb{E}\left[SW_{pess};\ (v_W + v_S) = \frac{2\gamma\xi(\theta_W - \theta_S)}{\gamma^2\xi^2 - \alpha(1-\xi^2)}\right] = \mathbb{E}\left[SW_{uninfo};\ (v_W + v_S) = \frac{2\gamma\xi(\theta_W - \theta_S)}{\gamma^2\xi^2 - \alpha(1-\xi^2)}\right]$. For computation details, see the mathematical appendix (Ko 2022). If $\alpha > \frac{\xi^2(1-2\xi)}{1-\xi^2+2\xi^3}\gamma^2$, then the uninformative forward guidance and its paired asymmetric inflation-targeting bandwidth achieves the global optimum, because by construction of $(v_W^{***} + v_S^{***})$, $\mathbb{E}\left[SW_{uninfo};\ (v_W + v_S) = \frac{2\gamma\xi(\theta_W - \theta_S)}{\gamma^2\xi^2 - \alpha(1-\xi^2)}\right]$ $< \mathbb{E}\left[SW_{uninfo};\ (v_W^{***} + v_S^{***}) = \frac{\gamma(\theta_W - \theta_S)}{\xi(\gamma^2+\alpha)}\right]$.

Second, assuming $\psi_0 < 1/2$, the ex ante expected social welfare associated with the truthful forward guidance is now compared with the one associated with the uninformative forward guidance.

$$
\begin{aligned}
&\mathbb{E}[SW_{truth};\ (v_W + v_S)] - \mathbb{E}[SW_{uninfo};\ (v_W + v_S)] \\
&\quad = -\left\{\theta_S + \gamma\cdot\xi\cdot(L_S + v_S)\right\}^2 \psi_0\psi_S \\
&\qquad + \left\{\theta_S + \gamma\cdot\xi\cdot(\overline{v} + v_S)\right\}^2 \psi_0\psi_S \\
&\qquad - \alpha\left\{(1-\xi)\cdot L_S + \xi(-v_S)\right\}^2 \psi_0\psi_S \\
&\qquad + \alpha\left\{(1-\xi)\cdot\overline{v} + \xi(-v_S)\right\}^2 \psi_0\psi_S \\
&\qquad - \left\{\theta_S + \gamma\cdot\xi(L_W + v_S)\right\}^2 (1-\psi_0)(1-\psi_W) \\
&\qquad + \left\{\theta_S + \gamma\cdot\xi\cdot(\overline{v} + v_S)\right\}^2 (1-\psi_0)(1-\psi_W) \\
&\qquad - \alpha\left\{(1-\xi)\cdot L_W + \xi(-v_S)\right\}^2 (1-\psi_0)(1-\psi_W) \\
&\qquad + \alpha\left\{(1-\xi)\cdot\overline{v} + \xi(-v_S)\right\}^2 (1-\psi_0)(1-\psi_W) \\
&\qquad - \left\{\theta_W + \gamma\cdot\xi\cdot(L_S - v_W)\right\}^2 \psi_0(1-\psi_S) \\
&\qquad + \left\{\theta_W + \gamma\cdot\xi\cdot(\overline{v} - v_W)\right\}^2 \psi_0(1-\psi_S) \\
&\qquad - \alpha\left\{(1-\xi)\cdot L_S + \xi v_W\right\}^2 \psi_0(1-\psi_S) \\
&\qquad + \alpha\left\{(1-\xi)\cdot\overline{v} + \xi(+v_W)\right\}^2 \psi_0(1-\psi_S) \\
&\qquad - \left\{\theta_W + \gamma\cdot\xi\cdot(L_W - v_W)\right\}^2 (1-\psi_0)\psi_W \\
&\qquad + \left\{\theta_W + \gamma\cdot\xi\cdot(\overline{v} - v_W)\right\}^2 (1-\psi_0)\psi_W \\
&\qquad - \alpha\left\{(1-\xi)\cdot L_W + \xi v_W\right\}^2 (1-\psi_0)\psi_W \\
&\qquad + \alpha\left\{(1-\xi)\cdot\overline{v} + \xi(+v_W)\right\}^2 (1-\psi_0)\psi_W
\end{aligned}
$$

$$= -\psi_0(1-\psi_0)(\psi_S+\psi_W-1)^2(v_W+v_S)\big[2\gamma\xi(\theta_W-\theta_S) - \left\{\gamma^2\xi^2-\alpha(1-\xi^2)\right\}(v_W+v_S)\big] \tag{C.9}$$

For any given value of $(v_W + v_S)$, the above holds. If $(v_W + v_S) = \frac{2\gamma\xi(\theta_W-\theta_S)}{\gamma^2\xi^2-\alpha(1-\xi^2)}$ is plugged in, then it is easy to show that $\mathbb{E}\left[SW_{truth};\ (v''_W + v''_S) = \frac{2\gamma\xi(\theta_W-\theta_S)}{\gamma^2\xi^2-\alpha(1-\xi^2)}\right] = \mathbb{E}\left[SW_{uninfo};\ (v''_W + v''_S) = \frac{2\gamma\xi(\theta_W-\theta_S)}{\gamma^2\xi^2-\alpha(1-\xi^2)}\right]$. When $\alpha > \frac{\xi^2(1-2\xi)}{1-\xi^2+2\xi^3}\gamma^2$, the $(v_W^{***} + v_S^{***})$ gives a better level of the ex ante expected social welfare than the one achieved by $(v''_W + v''_S) = \frac{2\gamma\xi(\theta_W-\theta_S)}{\gamma^2\xi^2-\alpha(1-\xi^2)}$. Therefore, the uninformative forward guidance and its paired asymmetric inflation-targeting bandwidth achieves the global optimum under the aforementioned condition.

C.7 Discussion

When the ex post realized inflation is influenced by private expectation with a weight of $(1-\xi)$ and inflation-targeting bandwidth is allowed to be asymmetric by $\pi^{policy}(\theta_S) = \pi^T - v_S$ and $\pi^{policy}(\theta_W) = \pi^T + v_W$, the forward-guidance threshold is determined by the sum of the bandwidth around the inflation target, $(v_W + v_S)$ or $2v$. That is, irrespective of asymmetric inflation targeting or symmetric one, the threshold for forward-guidance is at the point where the sum of positive and negative side bandwidth, $(v_W + v_S)$ or $2v$ is given as $\frac{2\gamma\xi(\theta_W-\theta_S)}{\gamma^2\xi^2-\alpha(1-\xi^2)}$. Also, the threefold optimal forward-guidance results remain the same. However, the paired inflation-targeting bandwidth becomes different from the symmetric case. It is because, for example, when the optimality condition for v_W is formed via the first-order derivative with respect to v_W, the terms with v_S and its associated probabilities are treated as constant. Similarly, for the optimality condition for v_S, the terms with v_W and the associated probability are treated as constant. Then, the optimal conditions for v_S and v_W are given differently, i.e., $\frac{\partial\mathbb{E}[SW]}{\partial v_W} \neq \frac{\partial\mathbb{E}[SW]}{\partial v_S}$. In the symmetric inflation-targeting bandwidth case, by contrast, there is only one FOC $\frac{\partial\mathbb{E}[SW]}{\partial v}$ w.r.t. v. Due to this distinction between symmetric and asymmetric inflation targeting, the optimal values of v_W and v_S of the asymmetric case do not converge to the optimal value of v

even if the private sector's self-fulfilling effect is ignored ($\xi \rightarrow 1$). And the asymmetric inflation-targeting bandwidth achieves higher social welfare level than the symmetric one: otherwise, the results of asymmetric inflation targeting would have converged to the symmetric inflation-targeting bandwidth.

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