Optimal Negative Interest Rate under Uncertainty*

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I employ a simple overlapping-generations model of money and nominal bonds with Epstein-Zin preferences and study the optimal negative interest rate. A subzero lower bound can arise in the model due to the illiquidity of money as a savings instrument. This model of negative interest rates differentiates from conventional ones based on exogenous money holding costs in that the subzero lower bound as well as the optimal negative rate turn out to crucially depend upon agents' preferences for the timing of uncertainty resolution. Both the lower bound and the optimal interest rate for aggregate consumption can fall into a negative territory only if agents prefer late resolution of uncertainty. In the latter case, the lower bound and the optimal rate both decrease even further when aggregate output uncertainty rises. However, the optimal interest rate turns out to be non-negative and to have a positive relationship with the degree of aggregate uncertainty if agents prefer early resolution of uncertainty.

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1. Introduction

Since 2011 the world monetary system has experienced something that only a few had imagined in a hypothetical world, namely negative interest rates. As of May 2019, five central banks are now in the negative-rate club, including the European Central Bank, the Bank of Japan, the National Bank of Denmark, the Swiss National Bank, and Sweden's Riksbank. Radical as it sounds, this recent phenomenon has spurred public debates, topics of which can be basically categorized into two questions: (i) Can (substantially) negative rates be implemented? (ii) If yes, is it desirable to pursue such policy?

The former question largely revolves around how to substantially get rid of incentives to hoard cash under negative interest rates. Many possible mechanisms were already suggested. Examples include abolishing cash, levying a tax on paper currency, imposing a fee for converting cash deposits to electronic bank reserves at the central bank, etc.[1] Whether these suggested methods could really clear the path for negative interest rates remains to be seen.

Despite its importance, this paper will be agnostic about the former question. Instead, it aims to shed new light on the latter one. In particular, I am interested in how low (negative) rates should go once they are possible. In short, this paper doesn't aim to fully microfound the feasibility of negative rates. Instead, it aims to find the optimal negative interest rate, if any, in a hypothetical world where negative rates are already feasible. Two pivotal issues remain for this task. The first one is how to introduce a subzero lower bound for interest rates, and the second issue is about appropriately incorporating various costs and benefits associated with negative interest rates.

The first issue should be taken seriously because the Freidman rule (FR), i.e., a zero nominal interest rate, becomes the lower bound for interest rate policy in a standard monetary model. One conventional way to bypass this problem is to introduce some exogenous fixed costs associated with holding cash. Yet, this approach is problematic since it would usually imply both the negative lower bound and the optimal rate being effectively equivalent to the FR minus

[1] Interested readers may refer to Rogoff (2016) for an excellent review of these suggested mechanisms.

the exogenous holding cost. Therefore, I adopt another approach where money is used for exchange only, while agents use bonds to save (but not to exchange). The lower bound can be negative in this case to the extent that agents' intertemporal desire to save through bonds outweighs costs from their negative returns. While it is admittedly true that this way of introducing the negative lower bound is still a reduced form, the optimal result in my model turns out to be richer than and produce more far-reaching policy implications than the conventional approach. Detailed and intuitive explanation will follow shortly.

The second issue is even more important. The literature has already pointed out major pros and cons of negative interest rates. Possible benefits discussed so far cover various factors: more effective inflation expectations control, less distortion from intertemporal price fluctuations—especially given the long-term declining trend in real interest rates, and stimulation effects on lending and investment, etc. On the contrary, serious concerns have been raised as well: downward pressure on banks' net interest margin, threats of currency wars, and negative signaling effects on the effectiveness of monetary policy, etc. (See Agarwal and Kimball 2015, Goodfriend 2016, and references therein for detailed review.) Though certainly important, incorporating all these factors into a unified framework is not the purpose of this study. Instead, I only focus on one novel channel through which the optimality of negative rates can be analyzed in a tractable manner. This newly offered channel is "aggregate uncertainty," defined as second-moment changes in the distribution of aggregate output. Despite its narrow focus, many interesting and novel policy implications also emerge, as will be shown subsequently.

The model constructed in this paper is, therefore, meant to rigorously reflect upon the aforementioned ways of dealing with the two issues in a unified manner. It builds upon a standard two-period overlapping-generations (OLG) model with endowments. The young receive a fixed amount of goods, while the old face uncertain amounts of the same good. I also introduce decentralized trading in which agents acquire a separate second good through the use of a medium of exchange (MOE). Two financial assets are introduced: *fiat* money and nominal bonds. As mentioned earlier, I assume that money only serves an MOE role, while bonds only function as a savings instrument. Lastly, the Epstein-Zin (EZ) preferences are adopted to materialize the idea of uncertainty-driven optimal negative rates. A key and well-known innovation of the EZ preferences is that they allow agents to separate their degree of aversion to cross-sectional and intertemporal risks; see Epstein and Zin (1989), Bansal and Yaron (2004), and Jung (2017) for details.

The model delivers very interesting equilibrium allocations and welfare implications. First, the lower bound for interest rates is no longer zero. Intuition is not hard. Agents can still demand bonds as a savings instrument in light of negative interest rates, because intergenerational transfers of the endowment good are only possible through bonds. It is still true that interest rates should be bounded from below; otherwise, agents are better off simply with autarky under too-negative rates. The extent to which the lower bound can fall, therefore, depends upon the degree to which the intertemporal inequality in endowments prevails. In other words, the lower bound can go deep into a negative territory further when the average level of old-age endowments relatively decreases.

One novel prediction of the current model is that the lower bound also depends on the second moment of random old-age endowments, i.e., aggregate uncertainty, and an agent's preference for the timing of uncertainty resolution, i.e., the relative degree to which agents dislike intertemporal risks compared with the cross-sectional ones. As opposed to the standard constant relative risk aversion (CRRA) utility case, the uncertainty *directly* affects an agent's utility under EZ preferences because of its negative effect on the certainty equivalent value of old-age consumption. Accordingly, the uncertainty undoubtedly affects an agent's willingness to save using bonds in this framework. The point is that such incentives act differently depending on an agent's attitude towards cross-sectional and intertemporal risks.

Intuitively, when an agent's aversion to intertemporal risks is relatively greater, they dislike intertemporal inequality in consumption more than the cross-sectional variation in old-age consumption. Under this case, a higher uncertainty means more intertemporal consumption inequality. Thus, it surely increases an agent's willingness to purchase bonds. As a result, the lower bound must also fall in response to a higher uncertainty. The exact opposite logic applies to the case where an agent's aversion to intertemporal risks is relatively smaller. In this opposite case, agents become relatively more averse to cross-sectional variations in old-age consumption, and therefore a higher uncertainty actually induces them to transfer consumption towards the young through less bond savings. This means the lower bound must increase in response to a rise in uncertainty, as opposed to the former case.

Main results of the paper, i.e., what the optimal negative rate is and how it is affected by uncertainty, also turn out to be interesting. A pure-utility-based welfare as a function of the nominal interest rate exhibits a hump-shaped pattern with the FR being a maximizer. This property is the same regardless of the agent's relative intertemporal risk aversion. The idea is that only the Friedman rule guarantees no distortion in the price level of the intergenerational transfer instrument, which is also common in other major OLG monetary models.

However, welfare measured in terms of aggregate consumption shows novel and richer patterns. The maximum aggregate consumption in this endowment economy is achieved only when the consumption of the three different goods, i.e., the (numeraire) good consumed in both periods and the special good, are equally distributed. Yet, changes in the rate of return on bonds certainly affect that distribution. Again, the point is that such interest rate effects crucially differ depending on the agent's relative aversion to intertemporal risks. For instance, when agents are indifferent to cross-sectional and intertemporal risks, i.e., agents have the CRRA utility, uncertainty has no effects on agents' consumption decision because the certainty equivalent and the average value of old-age consumption are always equal to each other. Thus, nominal interest rates distort the aggregate consumption distribution in the standard way, i.e., a higher interest rate leads to more old-age consumption. To avoid such a distortion, a zero nominal interest rate must be taken, i.e., the FR becomes the optimal rule to achieve the maximum aggregate consumption.

When agents have different attitudes towards cross-sectional and intertemporal risks, the FR breaks up. In light of a greater aversion to intertemporal risks, the aggregate consumption portfolio becomes more biased towards old-age consumption even with a zero nominal interest rate, for the reasons mentioned earlier. Thus, a subzero level of nominal interest rate is required to mitigate such negative distribution effects, meaning that the optimal interest rate should be negative in this case. Moreover, it naturally follows that the negative optimal interest rate would fall even further as uncertainty rises in this case. Nevertheless, these properties get totally reversed when agents dislike cross-sectional risks more. The aggregate consumption distribution this time is biased towards young-age consumption even under the FR. Therefore, a positive level of interest rate becomes an optimal policy to pursue. Finally, the (positive) optimal interest rate in this case should increase in uncertainty so as to further transfer consumption towards the old.

2. Related Literature

Full-fledged academic literature on negative interest rates has yet to be developed. Nevertheless, there are some excellent review papers on the feasibility, desirability, and implementation of negative interest rates. Interested readers should refer to Agarwal and Kimball (2015), Goodfriend (2016), and Williams (2016). The main theme of this paper is the optimal level of negative interest rate, and papers on this particular issue are still scarce to the best knowledge of the author. In what follows, I, therefore, briefly introduce existing studies that are related to the current paper in terms of methodology and contribution only.

Goodfriend (2000) was the first one to theoretically explore the possibility of a negative nominal interest rate policy. He largely focused on how to enable central banks to actually target negative interest rates. Suggested options are a carry tax on money, open market operations in long bonds, and monetary transfers. Unlike his emphasis on technical implementation issues, the current paper explores what might be the optimal negative interest rate once negative interest rates are implementable. Agarwal and Kimball (2015) also suggest a scheme for a changeable exchange rate between currency and reserves for negative interest rates to be implementable. The idea is that central banks can lower the rate at which reserves can be converted to cash so that the negative interest rate on reserves becomes arbitrage free. Haldane (2015) and Kocherlakota (2016) argue that the best way to make negative interest rates practically feasible is to abolish cash and move completely to electronic cash with any yield. Again, none of them aims to look for the optimal negative interest rate based on a general equilibrium monetary framework.

Brunnermeier and Koby (2016) is probably the study most related to the current paper in terms of finding the optimal interest rate subject to the negative lower bound. They explicitly take into account banking sectors to find what they call the "reversal interest rate," the rate at which accommodative monetary policy reverses its effect and becomes contractionary for lending. According to them, that rate critically depends upon various microstructures in banking sectors, and could well be negative. The current paper differs in that the effects of negative interest rates on aggregate variables and welfare are analyzed through an agent's general preferences for the timing of uncertainty resolution rather than banking intermediation channels.

Lastly, this paper is related to those using OLG monetary models. Schreft and Smith (2002) and Bhattacharya, Haslag, and Martin (2005) introduce financial intermediation and limited communication into the OLG framework, and show the suboptimality of the Friedman rule. In contrast to this line of research, the suboptimality arises in the current model from interactions between uncertainty and an agent's relative aversion to intertemporal risks. The current model follows most closely Jung (2018) in terms of methodology, where decentralized trading is explicitly incorporated into the OLG framework. Yet, it is a pure currency model, while the current model extends it to include nominal bonds.

3. The Model

The current model is a discrete-time and two-period overlapping-generations model with no time discounting and no population growth. The economy consists of one main island at the center and a unit measure of periphery islands around it. Each period, a unit measure of households is born in the main island, and lives only for two periods. When households are born, they get endowed with fixed units, i.e., x, of *numeraire* goods. By assumption, these goods are perishable such that carrying them across periods and outside the main island is not possible. When households move into the second period of their life, they also receive an identical endowment, ε units of *numeraire* goods, but this time it is random and follows a uniform distribution, $\mathcal{U}(y-b, y+b)$, where $y \geq b$ and $x > y$.

A key feature of the model is that the household is divided into two independent individuals, a *worker* and a *shopper*. The worker needs to consume *numeraire* goods in both periods, while a shopper only needs to consume in the second period of life. Furthermore, the shopper must consume something different from the *numeraire* good. We call it "special goods." The problem is households are never endowed with special goods, which only "sellers" living on periphery islands can produce. To be more precise, we assume that one seller is born every period on each periphery island, and lives only one period. Each seller is born with a homogeneous technology to produce the special good with a linear cost of labor disutility. However, they only get the utility from consuming *numeraire* goods. This framework basically gives rise to a trading motive between old shoppers and sellers each period.

Two key trading characteristics are worth noting. First, trades between sellers and old shoppers must take place in a bilateral fashion due to spatial separation among islands. Second, any kind of credit arrangement and/or barters between sellers and old shoppers are also ruled out due to anonymity and limited commitment within a bilateral meeting along with the assumption that *numeraire* goods are perishable across periods and islands. In consequence, the medium of exchange (MOE) is required for this mutually beneficial trade to take place.

To that end, we introduce two potential candidates: an intrinsically useless object called "(fiat) money" and a one-period nominal government bond. Money in this economy is issued by the government and assumed to be in fixed supply. Therefore, we denote M as the total money supply every period. We rule out lump-sum money transfers by the government so as to introduce a nominal interest rate as the only available government policy instrument.[2] We denote φ_t as the real price of money in terms of *numeraire* goods at period t. Apart from the money, there exist one-period pure discount nominal bonds. They take the form of a book entry such as the U.S. Treasury bonds. The real price of one unit of the nominal bond at period t is denoted by ψ_t. This means that a unit of money at period $t+1$ can be redeemed from one unit of nominal bonds sold at the price level of ψ_t. Importantly, we assume that the nominal bond

[2]The constancy of money supply is chosen purely for the sake of simplicity. One could easily introduce a constant (gross) money growth rate, say μ, as an additional policy instrument. However, this would not change welfare results qualitatively, but only generate level effects.

price level at period t, i.e., ψ_t/φ_t, is a policy variable set by the government. Furthermore, we assume that the government is always on the balanced budget. That is the total real value of the bonds the government must redeem each period, i.e., $\varphi_t A_{t-1}$, and it ought to equal the total real value of the nominal bonds issued each period, i.e., $\psi_t A_t$, where A_t denotes the total amount of nominal bonds held by agents in period t.

Regarding bilateral trading frictions, we adopt a simple mechanism.[3] First, a perfect match between each old shopper and each seller is assumed. That is, every seller and old shopper gets to match and consume every period. Second, we adopt a *take-it-or-leave-it* offer by (old) shoppers to sellers as a pricing protocol within the pairwise trade. As in Jung (2018), bargaining solutions are trivial. Old shoppers always hand over all of their real balances to young sellers who produce exactly the same amount of special goods as the real money balances they receive. Lastly and most importantly, we assume that nominal bonds are perfectly illiquid, meaning that sellers never accept nominal bonds as a payment method within a pairwise trade. This means each old shopper's real balances consist of only money.

Old workers' real balances, on the other hand, can potentially become a portfolio of money and bonds. Technically speaking, money can potentially serve both as an MOE and as a savings instrument. Under positive nominal interest rates, it is obvious that workers will never use money as a savings instrument due to a higher rate of return on nominal bonds. The problem here is what happens in light of negative nominal interest rates. In such a case, nominal bonds whose return rate is lower than that of money will never be valued in equilibrium. Thus, this economy would return to a pure currency economy as in Jung (2018). This is an undesirable feature of the model since the very purpose of this paper is to search for optimal negative interests under an economy with both money and bonds being valued. For this reason, we take a shortcut, as we did with respect to the illiquidity of nominal bonds. Simply, we assume

[3] See search-based monetary theory literature, e.g., Lagos and Wright (2005) and Geromichalos and Jung (2018) for a detailed introduction to bilateral trading frictions.

that money can never be used as a savings instrument under any circumstances.[4]

Finally, a household born in period t has EZ-type preferences, $U(c_t, s_{t+1}, c_{t+1})$, given by the following form:

$$U(c_t, s_{t+1}, c_{t+1}) = \left[c_t^{1-\rho} + s_{t+1}^{1-\rho} + [R_t(c_{t+1})]^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

$$\text{where } R_t(c_{t+1}) = \left(E_t\left[c_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}, \rho > 0.$$

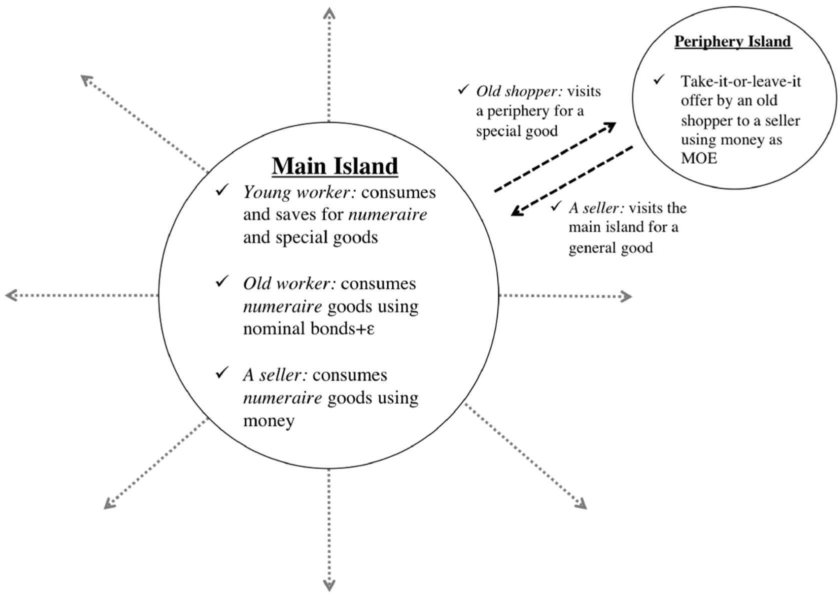
c_t and c_{t+1} denote the amount of *numeraire* goods consumed by the worker in period t and $t+1$, while s_{t+1} denotes the amount of special goods consumed by the shopper in period $t+1$. Figure 1 provides a graphical illustration of the timing of key events. Finally, one can refer to Jung (2018) for an intuitive interpretation of EZ preferences.

4. Constrained Efficiency

We first study efficient allocations by solving a social planner problem. In doing so, we restrict our attention to constrained efficiency. The planner is prevented from achieving the first best because, like private agents, she is assumed to be unable to provide full insurance for old-age consumption. The rationale goes as follows. Suppose she was allowed to achieve complete risk sharing between old agents; she would no longer face EZ preferences due to no uncertainty on old-age consumption, i.e., the certainty equivalent value of old-age consumption would always be maximized to its expected value. Hence, a fully efficient allocation can be achieved. It is, however, important to note that private agents in competitive equilibrium and the planner would face different objective functions, i.e., a constant elasticity of substitution (CES) aggregate of c_y, s_o, and $R(c_o)$ for the former and a CES

[4] It is important to note that the perfect illiquidity of money as a savings instrument is assumed for simplicity. For instance, one could instead introduce a partial illiquidity of money as a savings instrument, say by assuming that only a fixed portion θ of agents can use money to save every period (think of $1-\theta$ as a portion of the population that has no bank accounts or cash storage technology). This relaxation wouldn't change the results qualitatively. So, technically speaking, one would only need some degree of illiquidity of money as a savings instrument to get my results. In this sense, the restriction on the money as a store of value may not be as ad hoc as it first seems.

Figure 1. Timing of Key Events



aggregate of c_y, s_o, and c_o for the latter. Thus, this (fully efficient) allocation would not be a fair benchmark to compare with the competitive equilibrium allocation. Furthermore, if agents are not able to set up an insurance arrangement that allows them to share their endowment risk in old age, it is not clear why the planner should be able to do so.

Second, we only focus on stationary allocations. That is, the social planner only gets to choose stationary allocations for *numeraire* goods consumed by young workers, c_y^*; special goods consumed by old shoppers, s^*; *numeraire* goods consumed by old workers, c_o^*; and *numeraire* goods consumed by sellers, n^*. Then, the planner's solution solves for the following problem:

$$\max_{c_y^*, c_o^*} \left\{ \left[(c_y^*)^{1-\rho} + (s^*)^{1-\rho} + [R(c_o^*)]^{1-\rho} \right]^{\frac{1}{1-\rho}} + [n^* - s^*] \right\}, \qquad (1)$$

$$\text{s.t.} \quad c_y^* + c_o^* + n^* = x + y,$$

$$\text{and} \quad s^* = n^*,$$

where $R(c_o^*) = \left(E\left[(c_o^*)^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}}, \rho > 0$.

The first aggregate (resource) constraint implies that total *numeraire* goods consumed by households and sellers must be the same as total endowments of the *numeraire* good in each period. The second aggregate (resource) constraint simply implies that the planner also faces the bilateral trading friction on each island as private agents in competitive equilibrium. Specifically, it tells that the amount of total special goods consumed by (old) shoppers is equal to the total special goods produced by sellers each period. Note that the amount of total special goods produced by sellers is equal to the total *numeraire* goods consumed by sellers due to the *take-it-or-leave-it* offer, which also explains the second linear part in the objective function. The following lemma summarizes the socially optimal stationary allocations of consumptions by households and sellers.

LEMMA 1. *The constrained efficient stationary allocations can be expressed as* $c_y^* = x - T^c - T^s$; $c_o^* = T^c + y$; $s^* = n^* = T^s$, *where* T^c *and* T^s *must meet the following two conditions:*

$$(i)\, Q(T^c, T^s) = 1 \quad \forall t,$$

$$(ii)\, T^s = [x - T^c - T^s]^{\gamma/\rho} \left[E\left[(T^c + \varepsilon)^{1-\gamma}\right]^{\frac{1}{1-\gamma}} \right]^{(\rho-\gamma)/\rho} \quad \forall t,$$

$$\textit{where} \quad Q(T^c, T^s) = \left[\frac{T^c + \varepsilon}{x - T^c - T^s} \right]^{-\rho} \left[\frac{T^c + \varepsilon}{E\left[(T^c + \varepsilon)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}} \right]^{\rho-\gamma},$$

$$\textit{and} \quad E\left[(T^c + \varepsilon)^{1-\gamma}\right] = \frac{(T^c + y + b)^{(2-\gamma)} - (T^c + y - b)^{(2-\gamma)}}{2b(2-\gamma)}.$$

Proof. See the appendix in Jung (2018).

Intuition for these results is exactly the same as the one in Jung (2018).

5. Competitive Equilibrium

While the constrained efficient outcome in this economy is identical to Jung (2018), a competitive equilibrium in this economy is different from it due to the introduction of nominal bonds. A key difference is that households need to decide how much real money balances to acquire for special good consumption, m_t, and how much nominal bonds are used to purchase for *numeriare* good consumption, a_t.[5] Then, a young household's choice problem can be given by

$$\max_{m_t, a_t} \left[(c_t)^{1-\rho} + (s_{t+1})^{1-\rho} + R_t(c_{t+1})^{1-\rho} \right]^{\frac{1}{1-\rho}}, \tag{2}$$

$$\text{s.t.} \quad c_t + \varphi_t m_t + \psi_t a_t = x, \quad c_{t+1} = \varphi_{t+1} a_t + \varepsilon_{t+1}, \quad \text{and}$$

$$s_{t+1} = \varphi_{t+1} m_t,$$

where $R_t(c_{t+1}) = \left(E_t \left[c_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}}$, $\rho > 0$, and ε_{t+1} follows a uniform distribution of $\mathcal{U}(y-b, y+b)$.

Intuitively, households face uncertainty with regard to old-age endowment. Thus, nominal bonds serve as a savings instrument for households to consume *numeraire* goods in the second period of their life. Again, nominal bonds here are assumed to be perfectly illiquid in a pairwise trade between old shoppers and sellers. Consequently, they serve only as a store of value.

Intuitive explanation for the three constraints in problem (2) can be provided as well. The first one refers to a budget constraint for young households. The second one simply says that *numeraire* goods consumption in old age must be financed by nominal bond savings from the previous period and the current endowment. The third constraint simply follows from two crucial assumptions: perfectly illiquid nominal bonds and the *take-it-or-leave-it* offer.

Using some properties of the EZ preferences, the following lemma summarizes individual optimal choice by the young household.

LEMMA 2. *Given aggregate real prices $\{\varphi_t, \varphi_{t+1}, \psi_t, \psi_{t+1}\}$ and old-age endowment shocks ε_{t+1}, the young household's optimal portfolio choice of $\{m_t, a_t\}$ must satisfy the following conditions:*

$$(i)\ \frac{\psi_t}{\varphi_{t+1}} = Q_{t,t+1}(m_t, a_t, \varepsilon_{t+1}) \quad \forall t,$$

[5] As in Jung (2018), we assume the endowment shocks are realized after shoppers leave the main island in order for a household to choose a portfolio ex ante.

$$(ii)\ \varphi_{t+1}m_t = [x - \varphi_t m_t - \psi_t a_t]^{\gamma/\rho} \times \left[E_t\left[(\varphi_{t+1}a_t + \varepsilon_{t+1})^{1-\gamma}\right]^{\frac{1}{1-\gamma}}\right]^{(\rho-\gamma)/\rho} \quad \forall t,$$

$$\textit{where} \quad Q_{t,t+1}(m_t, a_t, \varepsilon_{t+1}) = \left[\frac{\varphi_{t+1}a_t + \varepsilon_{t+1}}{x - \varphi_t m_t - \psi_t a_t}\right]^{-\rho} \times \left[\frac{\varphi_{t+1}a_t + \varepsilon_{t+1}}{E_t\left[(\varphi_{t+1}a_t + \varepsilon_{t+1})^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right]^{\rho-\gamma},$$

$$\textit{and} \quad E_t\left[(\varphi_{t+1}a_t + \varepsilon_{t+1})^{1-\gamma}\right] = \frac{(\varphi_{t+1}a_t + y + b)^{(2-\gamma)} - (\varphi_{t+1}a_t + y - b)^{(2-\gamma)}}{2b(2-\gamma)}.$$

Proof. The proof is the same as the one for lemma 4 of Jung (2018). One just needs to replace m_t^c, m_t^s, $h(m_t^s)$, and $h(m_t^c)$ in Jung (2018) with a_t, m_t, $\varphi_{t+1}m_t$, and $\varphi_{t+1}a_t$, respectively.

Interpretation of lemma 2 follows similarly from Jung (2018). The first condition refers to an intertemporal optimality between c_t and c_{t+1}, while the second one represents an intratemporal optimality between s_{t+1} and c_{t+1}. Note that agents' preferences for the timing of the uncertainty resolution, i.e., whether $\rho > \gamma$ or $\rho < \gamma$, along with the level of future endowment uncertainty, b, critically affect both optimal conditions.

Now we can define competitive equilibrium. As in Jung (2018), we restrict attention to the symmetric, monetary, and stationary equilibrium.

Definition 1. *A competitive, symmetric, monetary, and stationary equilibrium is a list* $\{Z, W, n, s, c_y, c_o\}$, *where* $Z \equiv Z_t \equiv Z_{t+1} \equiv \varphi_t m_t \equiv \varphi_{t+1}m_{t+1} \equiv \bar{\varphi}M$, $\forall t$, *where* $\varphi_t = \bar{\varphi}, \forall t$. $W \equiv W_t \equiv \psi_t a_t = \psi_{t+1}a_{t+1}$, *where the last equality follows from the government budget constraint, i.e.,* $\varphi_t a_{t-1} = \psi_t a_t$, $\forall t$ *and an aggregate resource constraint,* $a_t = A_t\ \forall t$. *Lastly,* $\{c_y, c_o, s, n\} = \{x - Z - W, W + y, Z, Z\}$. *The equilibrium real money and bond balances* $\{Z, W\}$ *satisfy lemma 2 given that* $\varphi_t/\varphi_{t+1} = 1$, $\varepsilon_{t+1} = E[\varepsilon] = y$, *and lastly a (gross) nominal interest rate set by the government, i.e.,* $i = \varphi_t/\psi_t$, $\forall t$.

This definition results in a system of two log-linearized equations that $\{Z, W\}$ must satisfy as below:

$$-\ln i - \rho \ln(x - Z - W) + \gamma \ln(W + y) = (\gamma - \rho)\ln(R(W + y)), \tag{3}$$

$$\ln(Z) = (\gamma/\rho)\ln(x - Z - W) + \{(\rho - \gamma)/\rho\}\ln(R(W + y)), \tag{4}$$

where $\ln(R(W + y))$

$$= \frac{1}{1-\gamma}\left\{\ln\left[\frac{(W+y+b)^{2-\gamma}}{2b(2-\gamma)} - \frac{(W+y-b)^{2-\gamma}}{2b(2-\gamma)}\right]\right\}.$$

Now, we conduct comparative static analyses based on the three cases regarding agents' preferences for the timing of the uncertainty resolution. Most importantly, we study the stationary welfare in two versions as in Jung (2018). The first one is in terms of pure utility, i.e., the CES aggregate of c_y, s, and $R(c_o)$ in the unique stationary monetary equilibrium. The second welfare measure shows aggregate consumption equivalents, i.e., the CES aggregate of c_y, s, and c_o in the unique stationary monetary equilibrium. We use $1 - \rho$ as a CES aggregate parameter for both cases. For notational convenience, we denote the former (latter) as W^1 (W^2). The following proposition discusses characteristics of equilibrium when agents are indifferent to the timing of uncertainty resolution.

PROPOSITION 1. *Consider the case where $\rho = \gamma$ and/or $b = 0$. Let Z_{EZ} and W_{EZ} denote real money and bond balances, respectively, in stationary equilibrium. $\exists!\, Z_{EZ}$ and $\exists!\, W_{EZ}$, only if $i \geq 2y^\gamma/x$. Then, the following holds true in the unique stationary monetary equilibrium:*

(i) $\partial Z_{EZ}/\partial i < 0$ *and* $\partial W_{EZ}/i > 0$.

(ii) $W^1 = W^2$ *and the Friedman rule achieves the constrained efficiency both in terms of pure utility and aggregate consumption.*

Proof. See the proof for proposition 3.

The first case admits intuitive welfare implications. First, i must be bounded from below to guarantee a unique equilibrium where bonds coexist with money. Intuition is straightforward. A too-low rate of return on bonds makes households lose incentives to carry bonds over periods. That lower bound is positively (negatively) related to the old-age (young-age) endowment, as the formula, i.e., $2y^\gamma/x$, in the proposition indicates.

Next, the *Friedman rule* achieves the constrained efficiency in both pure-utility and aggregate consumption terms. This can be easily verified from the two equations in lemma 2 and equations (3) and (4). Since uncertainty does not affect households' preferences, all that matters is the relative returns on bonds. A positive rate of return on bond holdings would induce households to bias their portfolio towards bonds, i.e., part (i) of the proposition. In turn, households would spend on *numeraire* goods more than the constrained efficient amount. The exact opposite analysis could apply in the case of negative returns on bonds. A zero nominal interest rate is the only way to achieve the social optimum in this case.

Next, we consider the second case, $\rho > \gamma$, which brings about a much richer set of comparative static analyses on stationary allocations. The next proposition summarizes the results.

PROPOSITION 2. *Consider the second case, where $\rho > \gamma$. Let $\tilde{Z}$ equal Z such that $\ln Z = (\gamma/\rho)\ln(x - Z) + (\rho-\gamma)/\rho\ln([R(y)])$. $\exists!\, Z_{EZ}$ and $\exists!\, W_{EZ}$, only if*

$$i \geq \underline{i} \equiv \frac{y^\gamma}{(x-\tilde{Z})[R(y)]^{\gamma-\rho}}.$$

The following holds true in the unique stationary monetary equilibrium:

- *(i)* $\partial\underline{i}/\partial b < 0$.
- *(ii)* $\partial Z_{EZ}/\partial b < 0$ *and* $\partial W_{EZ}/\partial b > 0$.
- *(iii)* $\partial Z_{EZ}/\partial i < 0$ *and* $\partial W_{EZ}/\partial i > 0$.
- *(iv) The Friedman rule achieves the same W^1 that the planner does.*
- *(v) The Friedman rule usually does not maximize W^2. Define the optimal (gross) nominal interest rate that maximizes the*

W^2 as i^*. If $\underline{i} \geq 1$, then $i^* = \underline{i}$. Otherwise, $\underline{i} \leq i^* < 1$ and $\partial i^* / \partial b < 0$.

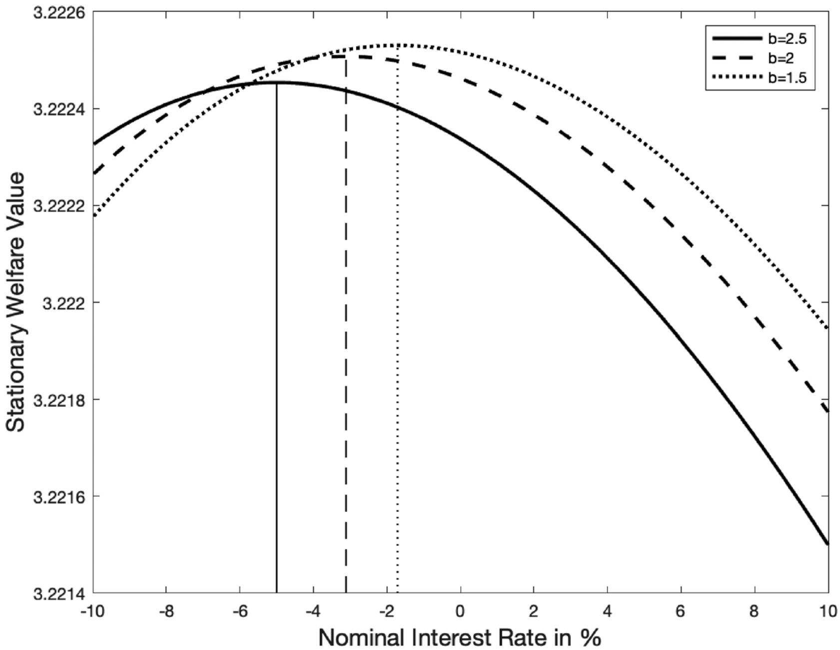
Proof. See the proof for proposition 3.

Unlike the case in proposition 1, the lower bound for nominal interest rates is now negatively affected by uncertainty, i.e., b. Intuition follows from households' preferences for the timing of the uncertainty resolution. Households in this case have a greater relative dislike for intertemporal inequality. Since a higher degree of uncertainty means more intertemporal inequality, households' incentives to save are strengthened. This eventually would reduce the lower bound.

Because households dislike intertemporal inequality in terms of general good consumption to a greater extent, they accumulate more bond holdings in response to a higher b, i.e., $\partial W_{EZ} / \partial b > 0$. This in turn means that young workers underconsume general goods in equilibrium. Given that households equalize the marginal utility from consuming general goods at a young age and special goods in old age, cash holdings for special goods must fall too when b goes up, i.e., $\partial Z_{EZ} / \partial b < 0$. As before, changes in nominal interest rates render qualitatively the same substitution effects. A higher rate of return on bond holdings would induce households to bias their portfolio towards bonds, i.e., $\partial Z_{EZ} / \partial i < 0$, and $\partial W_{EZ} / \partial i > 0$.

Welfare implications are also richer. In terms of pure utility, the Friedman rule is the unique optimal policy, i.e., part (iv) in proposition 2. Again, this is hardly surprising since the intergenerational transfer instrument price is never distorted only under the Friedman rule. What is interesting is that the optimal inflation rate for aggregate consumption is usually not the *Friedman rule* and negative. This follows from two important facts: (i) an equal division of c_y, c_o, and s achieves the maximum aggregate consumption, and (ii) uncertainty distorts that distribution. Intuitively, a higher degree of uncertainty would bias the aggregate consumption basket towards old-age consumption, i.e., $\partial Z_{EZ} / \partial b < 0$ and $\partial W_{EZ} / \partial b > 0$. To mitigate this effect, a subzero level of i is required, subject to the subzero lower bound (because a lower rate of return on bonds always leads to undersavings and, thus, lower old-age consumptions). Lastly, the higher aggregate output uncertainty gets, the bigger bond holdings

Figure 2. Numerical Examples for W^2 with $\rho > \gamma$



become. Thus, an even lower negative interest rate is required to bring down the overbond holdings, i.e., $\partial i^*/\partial b < 0$.

In what follows, a simple numerical example is illustrated. Figure 2 shows how W^2 (aggregate consumption level) responds to changes in nominal interest rates, i, under the case where $\rho > \gamma$. For various reasons of tractability, the model is not rigorously calibrated; see Jung (2018) for a detailed explanation on why two-period OLG models are problematic for calibration. In order to justify the parameter values as much as possible, nevertheless, the following measures are taken. First, the intertemporal risk-aversion parameter, ρ, is chosen to equal 2, i.e., intertemporal elasticity of substitution approximately equals 0.5. This is within reach of the usual values in the macro-finance literature that heavily rely on the EZ preferences for quantitative work. Note that the γ value used here (1.5) is somewhat lower than the usual in the literature. However, that γ value, along with $x = 10$ and $y = 4$, is chosen to make sure that the lower bound interest rate ($\underline{i}$) is sufficiently negative. Please check the formula in proposition 2. Finally, I also illustrate W^2 under three different values for b, i.e., 1.5, 2, and 2.5, so as to analyze the effects of uncertainty simultaneously. This example clearly shows that the lower bound is negative, i.e., –10 percent, and the optimal negative interest rate decreases in aggregate output uncertainty, consistent with predictions in proposition 2.

Using the intuition so far, it follows easily that the third case, $\rho < \gamma$, brings about opposite comparative static analyses on stationary allocations in general. First, proposition 3 summarizes such results.

PROPOSITION 3. *Consider the third case, where* $\rho < \gamma$. $\exists!\, Z_{EZ}$ *and* $\exists!\, W_{EZ}$*, only if* $i \geq \underline{i}$*. The following holds true in the unique stationary monetary equilibrium:*

(i) $\partial \underline{i} / \partial b > 0$.

(ii) $\partial Z_{EZ} / \partial b > 0$ *and* $\partial W_{EZ} / \partial b < 0$.

(iii) $\partial Z_{EZ} / \partial i < 0$ *and* $\partial W_{EZ} / \partial i > 0$.

(iv) The Friedman rule achieves the same W^1 *that the planner does.*

(v) The Friedman rule does not maximize W^2 *unless uncertainty disappears. Under* $b > 0$, $i^* > 1$ *and* $\partial i^* / \partial b > 0$.

Proof. See the appendix.

Uncertainty effects on the stationary allocation are exactly opposite to the second case. Now, households dislike cross-sectional variation in old-age consumption relatively more. Therefore, their incentives to hold nominal bonds get weaker. As a consequence, a higher degree of uncertainty effectively pushes up the lower bound for nominal interest rates, i.e., $\partial \underline{i} / \partial b > 0$. Again, since households are very averse to cross-section variation in the old-age general good consumption, they accumulate *fewer* bonds in response to a higher b, i.e., $\partial W_{EZ} / \partial b < 0$. This in turn means that young workers overconsume general goods in equilibrium. Given that households equalize the marginal utility from consuming general goods at a young age and special goods in old age, cash holdings for special goods must increase as well when b goes up, i.e., $\partial Z_{EZ}/\partial b > 0$. Interest rate effects on the stationary allocation are same as before, i.e., $\partial Z_{EZ}/\partial i < 0$, and $\partial W_{EZ}/\partial i > 0$, for obvious reasons.

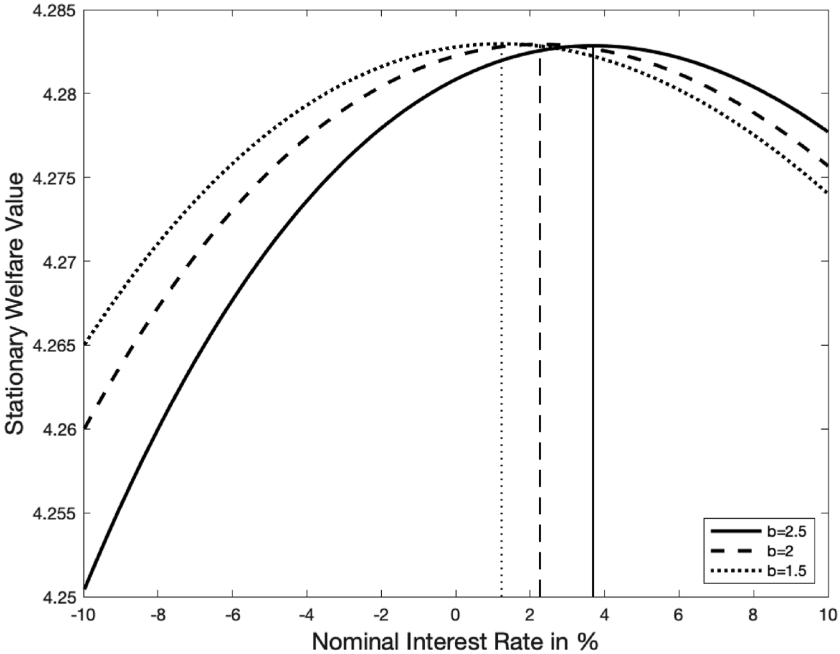
Welfare effects are also reversed except for the fact that the Friedman rule still achieves the constrained efficiency in terms of pure-utility-based welfare. In particular, the optimal inflation rate for aggregate consumption is positive this time. Unlike the second case, a higher degree of uncertainty would bias the aggregate consumption basket towards young-age general good consumption, i.e., $\partial Z_{EZ}/\partial b > 0$ and $\partial W_{EZ}/\partial b < 0$. To mitigate this effect, a positive level of i is required because a higher rate of return on bonds always brings about oversavings and, thus, greater old-age general good consumptions. Lastly, the higher aggregate output uncertainty gets, the smaller bond holdings become. Thus, an even more positive interest rate is required to boost the underbond holdings, i.e., $\partial i^*/\partial b > 0$.

Similar to the second case, a numerical example is shown in figure 3. All the parameter values except for $\gamma = 1.1$ and $\rho = 0.7$ are the same as in the second case. Consistent with predictions from proposition 3, the optimal interest rate happens to be positive this time, and the latter has a strictly positive relationship with the degree of uncertainty, as opposed to the second case.

6. Conclusion

To sum up, the current model delivers important policy implications. Negative interest rates can be beneficial only if an economic agent's aversion to intertemporal inequality in consumption is relatively greater. Under such a case, monetary policy authorities should target a negative interest rate that moves in the direction opposite to that of aggregate output uncertainty. It also leaves a completely reversed set of policy recommendations under the case where agents mind cross-sectional consumption inequality relatively more. In that case, a strictly positive interest rate should be targeted and positively tied to the degree of aggregate output uncertainty. Limitations of the current model for policy recommendations certainly exist, e.g.,

Figure 3. Numerical Examples for W^2 with $\rho < \gamma$



too-low transactional frequency inherent in two-period OLG models, etc. However, the current model can be used as a benchmark upon which more realistic and complex features of the economy can build.

For instance, much debate on negative interest rates recently revolves around their potential adverse effects on banking sectors; see Brunnermeier and Koby (2016). Many policymakers are concerned about financial instability that negative interest rates might cause through squeezing banks' net interest margin, which is completely absent in the current model. Embedding financial intermediaries into the current OLG structure following Schreft and Smith (2002) and/or Duffie, Gârleanu, and Pedersen (2005), therefore, might be a useful future research avenue. One could also extend the current model into a two-country environment to analyze the effects of negative interest rates on capital flows, currency markets, international trade, etc. I leave all these fruitful exercises to future research.

Appendix

Proof for Proposition 3

Proofs for proposition 1 and 2 are simply subcases of what follows. One just needs to impose $\rho = \gamma$ and $\rho < \gamma$, respectively. First, $Z \equiv Z(W)$ from equation (3). Then, it can be shown that $\partial Z/\partial W < 0$. Proof for the latter follows from the fact that

$$-\ln i - \rho \ln (x - Z - W) + \gamma \ln \left[\frac{W + y}{R(W) + y}\right] = -\rho \ln R(W + y),$$

which is from equation (3). Also note that $Z(x) = 0$ and $Z(0) = \tilde{Z}$, where

$$\tilde{Z} = \left\{Z : \ln Z = \frac{\gamma}{\rho} \ln (x - Z) + \frac{\rho - \gamma}{\rho} \ln[R(y)]\right\}.$$

Also note that $\tilde{Z} < x$ since $\ln x = \gamma/\rho(-\infty) + \text{constant}$.

Next, one can also derive $W \equiv W(Z)$ from equation (4). $\partial W/\partial Z < 0$. Proof is given through applying the implicit function theorem to equation (4).

$$\partial W/\partial Z = -\frac{\rho/(x - Z - W)}{\frac{\rho}{x-Z-W} + \gamma\left(\frac{1}{W+y} - \frac{\partial\gamma/\partial W}{R(W+y)}\right) + \rho\frac{\partial\gamma/\partial W}{R(W+y)}} < 0,$$

which again follows from $1/W + y > (\partial\gamma/\partial W)/R(W + y)$ due to the concavity of the R function. Also, $W(0) = \tilde{W}$, where

$$\begin{aligned}\tilde{W} = \big\{W &: -\ln i - \rho \ln (x - W) + \gamma \ln (W + y) \\ &= (\gamma - \rho) \ln[R(W + y)]\big\}.\end{aligned}$$

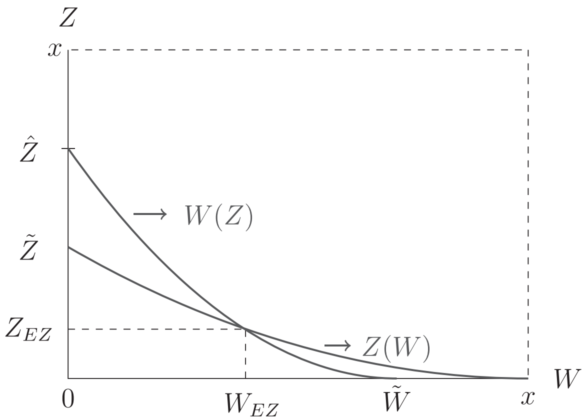
It's easy to show that $\tilde{W} < x$ since if $\tilde{W} = x$, then

$$-\ln i - \rho \ln (x - x) + \gamma \ln (W + y) > (\gamma - \rho) \ln[R(W + y)].$$

Thus, $\tilde{W}$ must be below x to lower the left-hand side of the above equation. Lastly, $\hat{Z}$ such that $W(\hat{Z}) = 0$ must satisfy the following:

$$\hat{Z} = \{-\ln i - \rho \ln (x - Z) + \gamma \ln y = (\gamma - \rho) \ln[R(y)]\}.$$

Figure 4. A Unique Stationary Equilibrium



So to ensure $\exists!(Z_{EZ}, W_{EZ})$, one must make sure $\hat{Z} \geq \tilde{Z}$, which is equivalent to

$$-\rho \ln (x - \tilde{Z}) + \gamma \ln y - (\gamma - \rho) \ln[R(y)] < \ln i.$$

This means $\exists!$ minimum $\underline{i}$ so that $\exists!(Z_{EZ}, W_{EZ})$, and

$$\underline{i} = \left\{ \underline{i} : \ln i = -\rho \ln (x - \tilde{Z}) + \gamma \ln y - (\gamma - \rho) \ln[R(y)] \right\}.$$

This proves for the $\underline{i}$ for all three cases, i.e., $\rho = \gamma$, $\rho < \gamma$, and $\rho < \gamma$.

One can finally visualize all of these in the diagram shown in figure 4. Given figure 4, one can easily check the effects of changes in i on Z_{EZ} and W_{EZ}. $i \uparrow \rightarrow W(Z)$ shifts out for all γ and ρ values. Hence, $\partial Z/\partial i < 0$ and $\partial W/\partial i > 0$ for all three cases. The effects of changes in b differ depending on the relative size of γ and ρ. When $\gamma > \rho$, $b \uparrow \rightarrow R(\cdot) \downarrow$. Thus, $Z(W)\,(W(Z))$ shifts up (down). This also proves why $i^* > 1$ and $\partial i^*/\partial b > 0$ under $\gamma > \rho$. On the contrary, when $\gamma < \rho$, $b \uparrow \rightarrow R(\cdot) \downarrow$. Thus, $Z(W)\,(W(Z))$ shifts down (up). This also proves why $i^* < 1$ and $\partial i^*/\partial b < 0$ under $\gamma > \rho$. Q.E.D.

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