## Appendix 1. Proof of Proposition 1

The first-order condition with respect to $l_{i,\delta,t}$ implies that each agent sells only the fraction of the agent's capital whose shadow value is exceeded by the market price, $Q_t$:

$$l_{i,\delta,t} = \begin{cases} k_{i,t-1}(2\Delta)^{-1}, & \text{if } Q_t > \lambda_{i,t}(1-\delta), \\ 0, & \text{otherwise,} \end{cases} \tag{26}$$

where

$$\lambda_{i,t} = \begin{cases} \min\{\phi^{-1}, Q_t(1-\hat{\delta}_t)^{-1}\}, & \text{if } \phi_{i,t} = \phi, \\ Q_t(1-\hat{\delta}_t)^{-1}, & \text{if } \phi_{i,t} = 0. \end{cases} \tag{27}$$

The variable $\lambda_{i,t}$ denotes the shadow value of capital net of depreciation at the end of period $t$ (i.e., $k_{i,t}$), so that $\lambda_{i,t}(1-\delta)$ is the shadow value of capital with depreciation rate $\delta$. The envelope theorem implies that $\lambda_{i,t}$ equals current consumption, $c_{i,t}$, multiplied by the Lagrange multiplier for equation (5). Also, given the envelope theorem, the first-order condition with respect to $k_{i,t}$ yields equation (27), which implies that the value of $\lambda_{i,t}$ equals the marginal acquisition cost of capital net of depreciation for each agent. The minimum operator in equation (27) reflects that productive agents can choose the cheaper way to obtain capital net of depreciation between investing in new capital and buying existing capital in the secondary capital market. The first and the second options, respectively, cost the amounts $\phi^{-1}$ and $Q_t(1-\hat{\delta}_t)^{-1}$ of goods per capital net of depreciation. Unproductive agents buy capital in the secondary market for storing their wealth, as they cannot invest in new capital.

## Appendix 2. Proof of Proposition 2

Proposition 2 is a corollary of the following two propositions:

PROPOSITION 2A. *There exists unique equilibrium in the basic model. The values of $Q_t$ and $\mu_t$ in equilibrium are such that*

$$\begin{cases} \mu_t = \bar{\delta} + \Delta \ \textit{and} \ 1 - Q_t\phi \geq \mu_t, & \textit{if } \phi\beta\alpha_t \leq (1-\beta)(1-\bar{\delta}), \\ \mu_t = \bar{\delta} \ \textit{and} \ 1 - Q_t\phi = \bar{\delta}, & \textit{if } \phi\beta\alpha_t \in ((1-\beta)(1-\bar{\delta}), \Lambda) \\ & \quad \textit{and } \Delta = 0, \\ \mu_t \in \left(\Xi, \bar{\delta} + \Delta\right) \ \textit{and} & \textit{if } \phi\beta\alpha_t \in ((1-\beta)(1-\bar{\delta}), \Lambda) \\ \quad 1 - Q_t\phi \in (\bar{\delta} - \Delta, \mu_t), & \quad \textit{and } \Delta > 0, \\ \mu_t = \Xi \ \textit{and} \ 1 - Q_t\phi \leq \bar{\delta} - \Delta, & \textit{if } \phi\beta\alpha_t \geq \Lambda, \end{cases} \tag{28}$$

*where* $\Xi \equiv \bar{\delta} + (1 - \sqrt{\rho})(1 + \sqrt{\rho})^{-1}\Delta$ *and*

$$\Lambda \equiv \frac{1 - \bar{\delta} + \Delta}{1 - \Xi}\left[\frac{1 - \bar{\delta}}{1 - \rho} - \beta\left(\int_{\bar{\delta}-\Delta}^{\Xi} \frac{1 - \delta}{2\Delta} d\delta + \int_{\Xi}^{\bar{\delta}+\Delta} \frac{1 - \Xi}{2\Delta} d\delta\right)\right] > (1-\beta)(1-\bar{\delta}). \tag{29}$$

*Proof.* Equations (26) and (27) imply that each agent sells the fraction of capital whose depreciation rate is greater than $\delta_{i,t}$, which is defined as

$$\delta_{i,t} = \begin{cases} 1 - Q_t\phi, & \text{if } \phi_{i,t} = \phi, \\ \hat{\delta}_t, & \text{if } \phi_{i,t} = 0. \end{cases} \tag{30}$$

Substituting this into equation (7) yields

$$\hat{\delta}_t = \frac{\rho \int_{\max\{\bar{\delta}-\Delta,\ \min\{1-Q_t\phi,\ \hat{\delta}_t\}\}}^{\bar{\delta}+\Delta} \delta(2\Delta)^{-1}\, d\delta + (1-\rho)\int_{\hat{\delta}_t}^{\bar{\delta}+\Delta} \delta(2\Delta)^{-1}\, d\delta}{\rho \int_{\max\{\bar{\delta}-\Delta,\ \min\{1-Q_t\phi,\ \hat{\delta}_t\}\}}^{\bar{\delta}+\Delta} (2\Delta)^{-1}\, d\delta + (1-\rho)\int_{\hat{\delta}_t}^{\bar{\delta}+\Delta} (2\Delta)^{-1}\, d\delta}. \tag{31}$$

Due to the log-utility function, each agent consumes a fraction $1 - \beta$ of net worth and saves the rest in each period:

$$c_{i,t} = (1-\beta)\left(\alpha_t + \int_{\bar{\delta}-\Delta}^{\delta_{i,t}} \frac{\lambda_{i,t}(1-\delta)}{2\Delta}\, d\delta + \int_{\delta_{i,t}}^{\bar{\delta}+\Delta} \frac{Q_t}{2\Delta}\, d\delta\right) k_{i,t-1}, \tag{32}$$

$$\lambda_{i,t}k_{i,t} = \beta\left(\alpha_t + \int_{\bar{\delta}-\Delta}^{\delta_{i,t}} \frac{\lambda_{i,t}(1-\delta)}{2\Delta}\,d\delta + \int_{\delta_{i,t}}^{\bar{\delta}+\Delta} \frac{Q_t}{2\Delta}\,d\delta\right)k_{i,t-1}. \tag{33}$$

In these equations, capital that the agent keeps is evaluated by its shadow value, $\lambda_{i,t}$, and capital that the agent sells is evaluated by the secondary market price, $Q_t$. Combining equations (5), (8), (26), and (33) and normalizing the combined equation by unproductive agents' capital at the beginning of the period, $\int_{\{i|\phi_{i,t}=0\}} k_{i,t-1}\,di$, I can obtain

$$\begin{aligned}\frac{Q_t}{1-\hat{\delta}_t}\left\{\frac{1-\bar{\delta}}{1-\rho} - \frac{\rho}{1-\rho}\int_{\bar{\delta}-\Delta}^{\max\{\bar{\delta}-\Delta,\ \min\{1-Q_t\phi,\ \hat{\delta}_t\}\}} \frac{1-\delta}{2\Delta}\,d\delta\right\} \\ = \beta\left(\alpha_t + \frac{Q_t}{1-\hat{\delta}_t}\int_{\bar{\delta}-\Delta}^{\hat{\delta}_t} \frac{1-\delta}{2\Delta}\,d\delta + \int_{\hat{\delta}_t}^{\bar{\delta}+\Delta} \frac{Q_t}{2\Delta}\,d\delta\right).\end{aligned} \tag{34}$$

Given the normalization, the left-hand side is the shadow value of capital net of depreciation that must be held by unproductive agents at the end of the period for a given amount of capital sold by productive agents, and the right-hand side is the fraction of unproductive agents' net worth that is spent on capital net of depreciation for saving. The market clearing condition (equation (8)) requires both sides to be equal. Overall, equations (31) and (34) jointly determine the equilibrium values of $Q_t$ and $\hat{\delta}_t$ given the aggregate productive shock, $\alpha_t$.

Rewrite equations (31) and (34) as

$$\begin{aligned}0 &= \Psi(Q_t, \hat{\delta}_t) \\ &\equiv (1-\rho)(\hat{\delta}_t)^2 - 2\hat{\delta}_t(\bar{\delta}+\Delta - \rho\max\{\bar{\delta}-\Delta,\ \min\{1-Q_t\phi,\ \hat{\delta}_t\}\}) \\ &\quad + (\bar{\delta}+\Delta)^2 - \rho(\max\{\bar{\delta}-\Delta,\ \min\{1-Q_t\phi,\ \hat{\delta}_t\}\})^2,\end{aligned} \tag{35}$$

$$0 = \Gamma(Q_t, \hat{\delta}_t) \equiv 1 - \bar{\delta} - \rho \int_{\bar{\delta}-\Delta}^{\max\{\bar{\delta}-\Delta,\ \min\{1-Q_t\phi,\ \hat{\delta}_t\}\}} \frac{1-\delta}{2\Delta}\, d\delta$$
$$- \beta(1-\rho)\left[\frac{(1-\hat{\delta}_t)\alpha_t}{Q_t} + \int_{\bar{\delta}-\Delta}^{\hat{\delta}_t} \frac{1-\delta}{2\Delta}\, d\delta + \int_{\hat{\delta}_t}^{\bar{\delta}+\Delta} \frac{1-\hat{\delta}_t}{2\Delta}\, d\delta\right]. \quad (36)$$

Hereafter a "root" means a root for $\Psi(Q_t, \hat{\delta}_t) = 0$ and $\Gamma(Q_t, \hat{\delta}_t) = 0$ unless mentioned otherwise.

Given $\max\{\bar{\delta}-\Delta,\ \min\{1-Q_t\phi,\ \hat{\delta}_t\}\}$ in equations (35) and (36), it is convenient to split the domain of $Q_t$ into three regions, $[0, (1-\bar{\delta}-\Delta)\phi^{-1})$, $[(1-\bar{\delta}-\Delta)\phi^{-1}, (1-\bar{\delta}+\Delta)\phi^{-1}]$, and $((1-\bar{\delta}+\Delta)\phi^{-1}, \infty)$. For each of the three regions, I will derive the necessary and sufficient conditions under which $\Gamma(Q_t, \hat{\delta}_t) = 0$ and $\Psi(Q_t, \hat{\delta}_t) = 0$ have a root in the region, given $\hat{\delta}_t \in [\bar{\delta}-\Delta, \bar{\delta}+\Delta]$.

First, suppose $Q_t \in ((1-\bar{\delta}+\Delta)\phi^{-1}, \infty)$. In this case $\max\{\bar{\delta} - \Delta,\ \min\{1-Q_t\phi,\ \hat{\delta}_t\}\} = \bar{\delta} - \Delta$ as $\hat{\delta}_t \geq \bar{\delta} - \Delta > 1 - Q_t\phi$. Then equation (35) yields that $\hat{\delta}_t = \bar{\delta} + (1-\sqrt{\rho})(1+\sqrt{\rho})^{-1}\Delta$. Note that $\partial\Gamma(Q_t, \hat{\delta}_t)/\partial Q_t > 0$ given $\max\{\bar{\delta} - \Delta,\ \min\{1 - Q_t\phi,\ \hat{\delta}_t\}\} = \bar{\delta} - \Delta$. Thus, there exists a unique root that satisfies $Q_t \in ((1 - \bar{\delta} + \Delta)\phi^{-1}, \infty)$ and $\hat{\delta}_t \in [\bar{\delta} - \Delta, \bar{\delta} + \Delta]$ if and only if $\Gamma((1 - \bar{\delta} + \Delta)\phi^{-1}, \bar{\delta} + (1-\sqrt{\rho})(1+\sqrt{\rho})^{-1}\Delta) < 0$, which is equivalent to

$$\phi\beta\alpha_t > \frac{1-\bar{\delta}+\Delta}{1-\Xi}\left[\frac{1-\bar{\delta}}{1-\rho} - \beta\left(\int_{\bar{\delta}-\Delta}^{\Xi} \frac{1-\delta}{2\Delta}d\delta + \int_{\Xi}^{\bar{\delta}+\Delta} \frac{1-\Xi}{2\Delta}d\delta\right)\right], \quad (37)$$

where $\Xi \equiv \bar{\delta} + (1-\sqrt{\rho})(1+\sqrt{\rho})^{-1}\Delta$.

Second, suppose $Q_t \in [0, (1-\bar{\delta}-\Delta)\phi^{-1})$. In this case $\max\{\bar{\delta} - \Delta,\ \min\{1-Q_t\phi,\ \hat{\delta}_t\}\} = \hat{\delta}_t$ as $\hat{\delta}_t \leq \bar{\delta}+\Delta < 1-Q_t\phi$. Then equation (35) implies that $\hat{\delta}_t = \bar{\delta} + \Delta$. Equation (36) in turn implies that $Q_t = (1-\bar{\delta}-\Delta)\beta\alpha_t(1-\beta)^{-1}(1-\bar{\delta})^{-1}$. Thus, there exists a unique root that satisfies $Q_t \in [0, (1-\bar{\delta}-\Delta)\phi^{-1})$ and $\hat{\delta}_t \in [\bar{\delta}-\Delta, \bar{\delta}+\Delta]$ if and only if $(1-\bar{\delta}-\Delta)\beta\alpha_t(1-\beta)^{-1}(1-\bar{\delta})^{-1} < (1-\bar{\delta}-\Delta)\phi^{-1}$, which is equivalent to $\phi\beta\alpha_t < (1-\beta)(1-\bar{\delta})$ given the assumption that $\Delta \in [0, \min\{\bar{\delta}, 1-\bar{\delta}\})$. It can be shown that $\Gamma((1-\bar{\delta}+\Delta)\phi^{-1}, \bar{\delta} + (1-\sqrt{\rho})(1+\sqrt{\rho})^{-1}\Delta) < 0$ and $\phi\beta\alpha_t < (1-\beta)(1-\bar{\delta})$ are mutually exclusive, given the assumption that $\rho < 1$.

Third, suppose $Q_t \in [(1-\bar{\delta}-\Delta)\phi^{-1}, (1-\bar{\delta}+\Delta)\phi^{-1}]$. In this case, $\max\{\bar{\delta}-\Delta,\ \min\{1-Q_t\phi,\ \hat{\delta}_t\}\} = \min\{1-Q_t\phi,\ \hat{\delta}_t\}$, as $1-Q_t\phi \geq \bar{\delta}-\Delta$ and $\hat{\delta}_t \geq \bar{\delta}-\Delta$. Moreover, $\min\{1-Q_t\phi,\ \hat{\delta}_t\} = 1-Q_t\phi$ because, given equation (35), $1-Q_t\phi > \hat{\delta}_t$ would imply $\hat{\delta}_t = \bar{\delta}+\Delta$ as shown in the second case above, which contradicts $1-Q_t\phi \leq \bar{\delta}+\Delta$. Given $\max\{\bar{\delta}-\Delta,\ \min\{1-Q_t\phi,\ \hat{\delta}_t\}\} = 1-Q_t\phi$, equation (35) implies that $1-Q_t\phi = [1+(\sqrt{\rho})^{-1}]\hat{\delta}_t - (\sqrt{\rho})^{-1}(\bar{\delta}+\Delta)$. Substituting this equation in equation (36), denote $\Gamma(\phi^{-1}\{1-[1+(\sqrt{\rho})^{-1}]\hat{\delta}_t + (\sqrt{\rho})^{-1}(\bar{\delta}+\Delta)\}, \hat{\delta}_t)$ by $\Theta(\hat{\delta}_t)$. It is possible to show that $d\Theta(\hat{\delta}_t)/d\hat{\delta}_t < 0$, given the assumption that $\bar{\delta}+\Delta < 1$. Note that $Q_t \in [(1-\bar{\delta}-\Delta)\phi^{-1}, (1-\bar{\delta}+\Delta)\phi^{-1}]$ is equivalent to $\hat{\delta}_t \in [\bar{\delta}+(1-\sqrt{\rho})(1+\sqrt{\rho})^{-1}\Delta, \bar{\delta}+\Delta]$ given $1-Q_t\phi = [1+(\sqrt{\rho})^{-1}]\hat{\delta}_t - (\sqrt{\rho})^{-1}(\bar{\delta}+\Delta)$. Thus there exists a unique root that satisfies $Q_t \in [(1-\bar{\delta}-\Delta)\phi^{-1}, (1-\bar{\delta}+\Delta)\phi^{-1}]$ and $\hat{\delta}_t \in [\bar{\delta}-\Delta, \bar{\delta}+\Delta]$ if and only if $\Theta(\bar{\delta}+(1-\sqrt{\rho})(1+\sqrt{\rho})^{-1}\Delta) \geq 0$ and $\Theta(\bar{\delta}+\Delta) \leq 0$, which are equivalent to $\Gamma((1-\bar{\delta}+\Delta)\phi^{-1}, \bar{\delta}+(1-\sqrt{\rho})(1+\sqrt{\rho})^{-1}\Delta) \geq 0$ and $\phi\beta\alpha_t \geq (1-\beta)(1-\bar{\delta})$.

Hence, combining the three cases for the value of $Q_t$ described above proves uniqueness of equilibrium.

PROPOSITION 2B. *Denote* $\int x_{i,t}di$ *by* $X_t$ *and* $\int y_{i,t}di$ *by* $Y_t$. *Then, in equilibrium in the basic model,*

$$\frac{X_t}{Y_t}\begin{cases} = 0, & \text{if } \phi\beta\alpha_t(1-\bar{\delta})^{-1} \leq 1-\beta, \\ < \beta - \dfrac{(1-\beta)(1-\bar{\delta})}{\phi\alpha_t}, & \text{if } \Delta > 0 \text{ and } \phi\beta\alpha_t(1-\bar{\delta})^{-1} > 1-\beta, \\ < \dfrac{\beta\rho}{1-\beta(1-\rho)}, & \text{if } \Delta > 0 \text{ and } \phi\beta\alpha_t(1-\bar{\delta})^{-1} \\ & \quad \geq (1-\rho)^{-1}-\beta, \\ = \beta - \dfrac{(1-\beta)(1-\bar{\delta})}{\phi\alpha_t}, & \text{if } \Delta = 0 \text{ and } \phi\beta\alpha_t(1-\bar{\delta})^{-1} \\ & \quad \in (1-\beta, (1-\rho)^{-1}-\beta), \\ = \dfrac{\beta\rho}{1-\beta(1-\rho)}, & \text{if } \Delta = 0 \text{ and } \phi\beta\alpha_t(1-\bar{\delta})^{-1} \\ & \quad \geq (1-\rho)^{-1}-\beta. \end{cases} \tag{38}$$

*Proof.* First, suppose $\phi\beta\alpha_t \leq (1-\beta)(1-\bar{\delta})$. The part of the proof of proposition 2A for this case shows $\hat{\delta}_t = \bar{\delta} + \Delta \leq 1 - Q_t\phi$ in this case. Similarly to equation (27), it can be shown that $x_{i,t} = 0$ for agents with $\phi_{i,t} = \phi$, if $\phi^{-1} > Q_t(1-\hat{\delta}_t)^{-1}$. Thus $X_t/Y_t = 0$.

Second, suppose $\phi\beta\alpha_t > (1-\beta)(1-\bar{\delta})$ and $\Delta > 0$. The part of the proof of proposition 2A for this case shows $\phi^{-1} < Q_t(1-\hat{\delta}_t)^{-1}$ in this case. Then, the equilibrium value of $X_t/Y_t$ is determined by

$$\frac{X_t}{Y_t} = \frac{\rho}{\alpha_t}\left[\beta\alpha_t + \beta\int_{\max\{\bar{\delta}-\Delta,\,1-Q_t\phi\}}^{\bar{\delta}+\Delta}\frac{Q_t}{2\Delta}\,d\delta\right.$$
$$\left. - (1-\beta)\int_{\bar{\delta}-\Delta}^{\max\{\bar{\delta}-\Delta,\,1-Q_t\phi\}}\frac{1-\delta}{2\Delta\phi}\,d\delta\right], \qquad (39)$$

which is derived from equations (5) and (33) for productive agents and $Y_t = \alpha_t \int k_{i,t-1} di$, given $\phi\beta\alpha_t > (1-\beta)(1-\bar{\delta})$.

Substituting equation (39) in the second line of equation (38), it can be shown that

$$\phi\alpha_t\left[\beta - \frac{(1-\beta)(1-\bar{\delta})}{\phi\alpha_t} - \frac{X_t}{Y_t}\right] = \phi\beta\alpha_t - (1-\beta)(1-\bar{\delta})$$
$$- \rho\left[\phi\beta\left(\alpha_t + \int_{\max\{\bar{\delta}-\Delta,\,1-Q_t\phi\}}^{\bar{\delta}+\Delta}\frac{Q_t}{2\Delta}\,d\delta\right)\right.$$
$$\left. - (1-\beta)\int_{\bar{\delta}-\Delta}^{\max\{\bar{\delta}-\Delta,\,1-Q_t\phi\}}\frac{1-\delta}{2\Delta}\,d\delta\right]$$
$$= (1-\rho)\phi\beta\alpha_t - \int_{\max\{\bar{\delta}-\Delta,\,1-Q_t\phi\}}^{\bar{\delta}+\Delta}\frac{\rho\phi\beta Q_t}{2\Delta}\,d\delta$$
$$- (1-\beta)\left[\rho\int_{\max\{\bar{\delta}-\Delta,\,1-Q_t\phi\}}^{\bar{\delta}+\Delta}\frac{1-\delta}{2\Delta}\,d\delta\right.$$
$$\left. + (1-\rho)\left(\int_{\hat{\delta}_t}^{\bar{\delta}+\Delta}\frac{1-\delta}{2\Delta}\,d\delta + \int_{\bar{\delta}-\Delta}^{\hat{\delta}_t}\frac{1-\delta}{2\Delta}\,d\delta\right)\right]$$

$$> (1-\rho)\left[\phi\beta\alpha_t - (1-\beta)\left(\int_{\bar{\delta}-\Delta}^{\hat{\delta}_t}\frac{1-\delta}{2\Delta}\,d\delta + \int_{\hat{\delta}_t}^{\bar{\delta}+\Delta}\frac{1-\hat{\delta}_t}{2\Delta}\,d\delta\right)\right]$$
$$-\int_{\max\{\bar{\delta}-\Delta,\,1-Q_t\phi\}}^{\bar{\delta}+\Delta}\frac{\rho\phi Q_t}{2\Delta}\,d\delta. \qquad (40)$$

The last inequality is obtained by substituting $\phi^{-1} < Q_t(1-\hat{\delta}_t)^{-1}$ and equation (31), which implies

$$\rho\int_{\max\{\bar{\delta}-\Delta,\,1-Q_t\phi\}}^{\bar{\delta}+\Delta}\frac{1-\hat{\delta}_t}{2\Delta}\,d\delta + (1-\rho)\int_{\hat{\delta}_t}^{\bar{\delta}+\Delta}\frac{1-\hat{\delta}_t}{2\Delta}\,d\delta$$
$$= \rho\int_{\max\{\bar{\delta}-\Delta,\,1-Q_t\phi\}}^{\bar{\delta}+\Delta}\frac{1-\delta}{2\Delta}\,d\delta + (1-\rho)\int_{\hat{\delta}_t}^{\bar{\delta}+\Delta}\frac{1-\delta}{2\Delta}\,d\delta. \qquad (41)$$

Substituting equation (41) into equation (34) yields that

$$\beta\alpha_t = \frac{Q_t}{1-\hat{\delta}_t}\left[\frac{\rho}{1-\rho}\int_{\max\{\bar{\delta}-\Delta,\,1-Q_t\phi\}}^{\bar{\delta}+\Delta}\frac{1-\hat{\delta}_t}{2\Delta}\,d\delta\right.$$
$$\left. + (1-\beta)\left(\int_{\bar{\delta}-\Delta}^{\hat{\delta}_t}\frac{1-\delta}{2\Delta}\,d\delta + \int_{\hat{\delta}_t}^{\bar{\delta}+\Delta}\frac{1-\hat{\delta}_t}{2\Delta}\,d\delta\right)\right]. \qquad (42)$$

Then substituting this into the right-hand side of inequality (40) implies that the right-hand side of inequality (40) equals

$$(1-\rho)(1-\beta)\left(\frac{\phi Q_t}{1-\hat{\delta}_t} - 1\right)\left[\int_{\bar{\delta}-\Delta}^{\hat{\delta}_t}\frac{1-\delta}{2\Delta}\,d\delta + \int_{\hat{\delta}_t}^{\bar{\delta}+\Delta}\frac{1-\hat{\delta}_t}{2\Delta}\,d\delta\right], \qquad (43)$$

which is positive given $\phi^{-1} < Q_t(1-\hat{\delta}_t)^{-1}$. Thus $\beta - (1-\beta)(1-\bar{\delta})(\phi\alpha_t)^{-1} > X_t/Y_t$ if $\phi\beta\alpha_t > (1-\beta)(1-\bar{\delta})$ and $\Delta > 0$.

Third, suppose $\phi\beta\alpha_t \geq [(1-\rho)^{-1}-\beta](1-\bar{\delta})$ and $\Delta > 0$. Equation (39) implies that

$$\begin{aligned}
&\frac{\phi\alpha_t}{\rho}\left[\frac{\beta\rho}{1-\beta(1-\rho)} - \frac{X_t}{Y_t}\right] \\
&= \frac{(1-\rho)\beta^2\phi\alpha_t}{1-\beta(1-\rho)} - \left[\phi\beta\left(\alpha_t + \int_{\max\{\bar{\delta}-\Delta,\, 1-Q_t\phi\}}^{\bar{\delta}+\Delta} \frac{Q_t}{2\Delta}\, d\delta\right)\right. \\
&\quad \left. - (1-\beta)\int_{\bar{\delta}-\Delta}^{\max\{\bar{\delta}-\Delta,\, 1-Q_t\phi\}} \frac{1-\delta}{2\Delta}\, d\delta\right]. \qquad (44)
\end{aligned}$$

Solving equation (42) for $Q_t(1-\hat{\delta}_t)^{-1}$ and substituting it in the right-hand side of equation (44), it can be shown that $\beta\rho[1-\beta(1-\rho)]^{-1} - X_t/Y_t > 0$ if and only if

$$\begin{aligned}
&\frac{\beta^2\phi\alpha_t(1-\rho)}{1-\beta(1-\rho)}\left(\int_{\bar{\delta}-\Delta}^{\hat{\delta}_t} \frac{1-\delta}{2\Delta} d\delta - \int_{\max\{\bar{\delta}-\Delta,\, 1-Q_t\phi\}}^{\hat{\delta}_t} \frac{1-\hat{\delta}_t}{2\Delta} d\delta\right) \\
&\quad + \int_{\bar{\delta}-\Delta}^{\max\{\bar{\delta}-\Delta,\, 1-Q_t\phi\}} \frac{1-\delta}{2\Delta} d\delta \left[\frac{\rho}{1-\rho}\int_{\max\{\bar{\delta}-\Delta,\, 1-Q_t\phi\}}^{\bar{\delta}+\Delta} \frac{1-\hat{\delta}_t}{2\Delta}\, d\delta\right. \\
&\quad \left. + (1-\beta)\left(\int_{\bar{\delta}-\Delta}^{\hat{\delta}_t} \frac{1-\delta}{2\Delta}\, d\delta + \int_{\hat{\delta}_t}^{\bar{\delta}+\Delta} \frac{1-\hat{\delta}_t}{2\Delta}\, d\delta\right)\right] > 0. \qquad (45)
\end{aligned}$$

This strict inequality holds, as the part of the proof of proposition 2A for $\phi\beta\alpha_t \geq [(1-\rho)^{-1}-\beta](1-\bar{\delta})$ and $\Delta > 0$ shows that $\hat{\delta}_t > \bar{\delta}-\Delta$, given $\Delta > 0$.

Fourth, suppose $\phi\beta\alpha_t > (1-\beta)(1-\bar{\delta})$ and $\Delta = 0$. The part of the proof of proposition 2A for this case shows $\hat{\delta}_t = \bar{\delta} \geq 1 - Q_t\phi$ in this case. Aggregating equations (5) and (33) for each type of agent yields that

$$\phi^{-1}\int_{\{i|\phi_{i,t}=\phi\}} k_{i,t}di = \beta(\alpha_t + Q_t)\rho\int k_{i,t-1}di, \qquad (46)$$

$$Q_t(1-\bar{\delta})^{-1}\int_{\{i|\phi_{i,t}=0\}} k_{i,t}di = \beta(\alpha_t + Q_t)(1-\rho)\int k_{i,t-1}di, \qquad (47)$$

$$\int k_{i,t}di = \phi X_t + (1-\bar{\delta})\int k_{i,t-1}di, \tag{48}$$

$$\int_{\{i|\phi_{i,t}=0\}} k_{i,t}di \leq (1-\bar{\delta})\int k_{i,t-1}di. \tag{49}$$

It is straightforward to show that, if $\bar{\delta} = 1 - Q_t\phi$, then $X_t/Y_t = \beta - (1-\beta)(1-\bar{\delta})(\phi\alpha_t)^{-1}$. Inequality (49) implies that $\phi\beta\alpha_t \leq [(1-\rho)^{-1} - \beta](1-\bar{\delta})$ must hold. (Note that $X_t > 0$ is satisfied given $\phi\beta\alpha_t > (1-\beta)(1-\bar{\delta})$.) On the other hand, if $\bar{\delta} > 1 - Q_t\phi$, then productive agents sell all of their capital, given equation (26). Thus inequality (49) must hold in equality, which implies $X_t/Y_t = \beta\rho[1-\beta(1-\rho)]^{-1}$. In this case, $\bar{\delta} > 1 - Q_t\phi$ is equivalent to $\phi\beta\alpha_t > [(1-\rho)^{-1} - \beta](1-\bar{\delta})$.

## Appendix 3. The Market Clearing Condition for and the Definitions of $\hat{\delta}_t$, $\Lambda_{V,t}$, and $\Lambda_{R,t}$ in the Model of Banking

The definition of the average depreciation rate of capital sold in the secondary market, $\hat{\delta}_t$, is

$$\hat{\delta}_t = \frac{\int \int_{\bar{\delta}-\Delta}^{\bar{\delta}+\Delta} \delta \, l_{i,\delta,t} \, d\delta \, di + \bar{\delta} L_{B,t}}{\int \int_{\bar{\delta}-\Delta}^{\bar{\delta}+\Delta} l_{i,\delta,t} \, d\delta \, di + L_{B,t}}. \tag{50}$$

Note that the average depreciation rate of capital sold by each bank is the unconditional average, $\bar{\delta}$. Thus, $\bar{\delta}L_{B,t}$ is the total depreciation of capital sold by banks.

The market clearing condition for the secondary market price of capital, $Q_t$, is

$$\int h_{i,t} \, di + H_{B,t} = \int \int_{\bar{\delta}-\Delta}^{\bar{\delta}+\Delta} l_{i,\delta,t} \, d\delta \, di + L_{B,t}, \tag{51}$$

which adds the sale and the purchase of capital by banks to equation (8) in the basic model.

The definitions of the stochastic discount factors for the agents holding bank equity and deposits, $\Lambda_{V,t+1}$ and $\Lambda_{R,t+1}$, are

$$\Lambda_{V,t+1} \equiv \frac{\beta c_{i^*,t}}{c_{i^*,t+1}}, \quad i^* \equiv \underset{i\in[0,1]}{\operatorname{argmax}} \, E_t \left[ \frac{\beta c_{i,t}(D_{t+1} + V_{t+1})}{(1+\zeta)c_{i,t+1}} \right], \tag{52}$$

$$\Lambda_{R,t+1} \equiv \frac{\beta c_{i^{**},t}}{c_{i^{**},t+1}}, \quad i^{**} \equiv \underset{i\in[0,1]}{\operatorname{argmax}} \, E_t \left[ \frac{\beta c_{i,t}\tilde{R}_{t+1}}{c_{i,t+1}} \right], \tag{53}$$

respectively. These definitions imply that the buyers of bank deposits and equity are those who value them most. Otherwise, there would exist some agents whose first-order conditions with respect to $s_{i,t}$ or $b_{i,t}$ do not hold with equality. In such a case, the optimum condition for the agents' utility-maximization problem would be violated.

## Appendix 4. Proof of Proposition 3

I solve the banks' profit-maximization problem (15) to prove proposition 3. As assumed in the main text, the number of exogenous states is two in each period. Given the values of period-$t$ variables, denote the smaller value of $\alpha_{t+1} + Q_{t+1}$ by $\underline{\omega}_{t+1}$ and the larger value by $\bar{\omega}_{t+1}$. The conditional probability that $\alpha_{t+1} + Q_{t+1} = \bar{\omega}_{t+1}$ is denoted by $P_t(\bar{\omega}_{t+1})$, and the one for $\alpha_{t+1} + Q_{t+1} = \underline{\omega}_{t+1}$ is denoted by $P_t(\underline{\omega}_{t+1})$.

I start by proving the following lemma:

LEMMA 1. *Suppose that $\Omega_{t+1}$ satisfies equations (58)–(60) for period $t+1$. Split the constraint set of the maximization problem (15) into three regions: $\bar{R}_t B_{B,t} \leq \underline{\omega}_{t+1} K_{B,t}$; $\bar{R}_t B_{B,t} \in (\underline{\omega}_{t+1} K_{B,t}, \bar{\omega}_{t+1} K_{B,t}]$; and $\bar{R}_t B_{B,t} > \bar{\omega}_{t+1} K_{B,t}$. Then, in equilibrium, $\bar{R}_t B_{B,t}$ equals $\underline{\omega}_{t+1} K_{B,t}$ at optimum in the first region and $\bar{\omega}_{t+1} K_{B,t}$ at optimum in the second region.*

*Proof.* Use the Lagrange method to solve the maximization problem in the first and the second region. For the second region, solve the maximization problem in the closure of the region and suppose that $\Omega_{t+1}$ takes the limit value when $\bar{R}_t B_{B,t} = \underline{\omega}_{t+1} K_{B,t}$. This makes the function $\Omega_{t+1}$ differentiable in each region. This expansion of the second region does not affect the solution to the maximization problem, since it will be shown that $\bar{R}_t B_{B,t} = \bar{\omega}_{t+1} K_{B,t}$ at optimum in the second region.

In the first region, $\bar{R}_t$ is determined solely by equation (11) and can be taken as exogenous for a bank. Equation (11) implies that $\bar{R}_t > 0$, since agents never choose zero consumption with the time-separable log-utility function in equilibrium. The first-order condition with respect to $B_{B,t}$ is

$$1 - \frac{1}{1+\zeta} E_t \left[ \frac{\beta c_{i,t} \bar{R}_t}{c_{i,t+1}} \middle| \phi_{i,t} = 0 \right] - \bar{\theta}_{rgn1,t} \bar{R}_{t+1} = 0, \qquad (54)$$

where $\bar{\theta}_{rgn1,t}$ is the Lagrange multiplier for the upper bound of the first region ($\bar{R}_t B_{B,t} \leq \underline{\omega}_{t+1} K_{B,t}$). Thus, $\bar{\theta}_{rgn1,t} = \zeta(1+\zeta)^{-1}(\bar{R}_t)^{-1} > 0$, given $\zeta > 0$ and $\bar{R}_t > 0$. Hence, $\bar{R}_t B_{B,t} = \underline{\omega}_{t+1} K_{B,t}$ at optimum in the first region.

For the second region, if $K_{B,t} = 0$, then the claim in the lemma is automatically satisfied, since equation (11) implies that $B_{B,t}$ must be 0, given that $K_{B,t}(B_{B,t})^{-1}$ in the equation is replaced with infinity if $B_{B,t} = 0$. Hereafter suppose $K_{B,t} > 0$ in the second region. In equilibrium, $Q_t$ is always positive and thus $\underline{\omega}_t > 0$ for all $t$, since otherwise each agent would demand an infinite amount of capital in the secondary market, which would violate the market clearing condition for the secondary capital market. In the second region, $K_{B,t} > 0$ and $\underline{\omega}_{t+1} > 0$ imply that $B_{B,t} > 0$ and $\bar{R}_t > 0$, since $B_{B,t}$ must be non-negative by the non-negativity constraint. The first-order conditions with respect to $B_{B,t}$ and $\bar{R}_t$ in the second region are, respectively,

$$\begin{aligned} 1 &- \frac{P_t(\bar{\omega}_{t+1})}{1+\zeta} E_t \left[ \frac{\beta c_{i,t} \bar{R}_t}{c_{i,t+1}} \middle| \phi_{i,t} = 0,\ \alpha_{t+1} + Q_{t+1} = \bar{\omega}_{t+1} \right] \\ &+ (\underline{\theta}_{rgn2,t} - \bar{\theta}_{rgn2,t}) \bar{R}_t - \theta_{PC,t} P_t(\underline{\omega}_{t+1}) \\ &\times E_t \left[ \frac{\beta c_{i,t} \underline{\omega}_{t+1} K_{B,t}}{c_{i,t+1} (B_{B,t})^2} \middle| \phi_{i,t} = 0,\ \alpha_{t+1} + Q_{t+1} = \underline{\omega}_{t+1} \right] = 0, \end{aligned} \qquad (55)$$

$$\begin{aligned} &- \frac{P_t(\bar{\omega}_{t+1})}{1+\zeta} E_t \left[ \frac{\beta c_{i,t} B_{B,t}}{c_{i,t+1}} \middle| \phi_{i,t} = 0,\ \alpha_{t+1} + Q_{t+1} = \bar{\omega}_{t+1} \right] \\ &+ (\underline{\theta}_{rgn2,t} - \bar{\theta}_{rgn2,t}) B_{B,t} + \theta_{PC,t} P_t(\bar{\omega}_{t+1}) \\ &\times E_t \left[ \frac{\beta c_{i,t}}{c_{i,t+1}} \middle| \phi_{i,t} = 0,\ \alpha_{t+1} + Q_{t+1} = \bar{\omega}_{t+1} \right] = 0, \end{aligned} \qquad (56)$$

where $\bar{\theta}_{rgn2,t}$ is the Lagrange multiplier for the upper bound of the closure of the second region ($\bar{R}_t B_{B,t} \leq \bar{\omega}_{t+1} K_{B,t}$), $\underline{\theta}_{rgn2,t}$ is the Lagrange multiplier for the lower bound of the closure of the second region ($\bar{R}_t B_{B,t} \geq \underline{\omega}_{t+1} K_{B,t}$), and $\bar{\theta}_{PC,t}$ is the Lagrange multiplier for equation (11). Equations (55) and (56) imply that $\theta_{PC,t} = B_{B,t}$. Substituting this into equation (55) leads to

$$(\bar{\theta}_{rgn2,t} - \underline{\theta}_{rgn2,t})\bar{R}_t = \frac{\zeta P_t(\bar{\omega}_{t+1})}{1+\zeta} E_t \left[ \frac{\beta c_{i,t} \bar{R}_t}{c_{i,t+1}} \middle| \phi_{i,t} = 0,\ \alpha_{t+1} + Q_{t+1} = \bar{\omega}_{t+1} \right], \quad (57)$$

which in turn indicates that $\bar{\theta}_{rgn2,t} > 0$ and $\underline{\theta}_{rgn2,t} = 0$, given $\zeta > 0$ and $\bar{R}_t > 0$. Thus, $\bar{R}_t B_{B,t} = \bar{\omega}_{t+1} K_{B,t}$ at optimum in the second region.

Given this lemma, the following proposition holds:

PROPOSITION 4. *Suppose equations (18)–(20) hold in equilibrium. Then*

$$\tilde{R}_t B_{B,t-1} + (D_t + V_t) S_{B,t-1} = \begin{cases} \left[\alpha_t + \lambda_{B,t}(1-\bar{\delta})\right] K_{B,t-1}, & \text{if } \bar{R}_{t-1} B_{B,t-1} \leq (\alpha_t + Q_t) K_{B,t-1}, \\ (\alpha_t + Q_t) K_{B,t-1}, & \text{if } \bar{R}_{t-1} B_{B,t-1} > (\alpha_t + Q_t) K_{B,t-1}, \end{cases} \quad (58)$$

*where* $\lambda_{B,t} \equiv \max\{\lambda'_{B,t},\ \lambda''_{B,t}\}$ *and*

$$\lambda'_{B,t} \equiv E_t \left\{ \frac{\beta c_{i,t} \left[\alpha_{t+1} + \lambda_{B,t+1}(1-\bar{\delta}) - \underline{\omega}_{t+1}\right]}{(1+\zeta) c_{i,t+1}} + \frac{\beta c_{i,t}\, \underline{\omega}_{t+1}}{c_{i,t+1}} \middle| \phi_{i,t} = 0 \right\}, \quad (59)$$

$$\lambda''_{B,t} \equiv P_t(\bar{\omega}_{t+1}) E_t \times \left\{ \frac{\beta c_{i,t} \left[\alpha_{t+1} + \lambda_{B,t+1}(1-\bar{\delta}) - \bar{\omega}_{t+1}\right]}{(1+\zeta) c_{i,t+1}} \middle| \begin{matrix} \phi_{i,t} = 0 \\ \alpha_{t+1} + Q_{t+1} = \bar{\omega}_{t+1} \end{matrix} \right\} + E_t \left[ \frac{\beta c_{i,t} \left(\alpha_{t+1} + Q_{t+1}\right)}{c_{i,t+1}} \middle| \phi_{i,t} = 0 \right], \quad (60)$$

$$\bar{R}_t B_{B,t} = \begin{cases} \underline{\omega}_{t+1} K_{B,t}, & \textit{if } \lambda'_{B,t} > \lambda''_{B,t}, \\ \bar{\omega}_{t+1} K_{B,t}, & \textit{if } \lambda'_{B,t} < \lambda''_{B,t}. \end{cases} \tag{61}$$

*Also,*

$$\lambda_{B,t} = \frac{Q_t}{1-\hat{\delta}_t}, \quad \textit{if } H_{B,t} > 0, \tag{62}$$

$$L_{B,t} = 0, \quad \textit{if } \hat{\delta}_t > \bar{\delta} \textit{ and } H_{B,t} > 0. \tag{63}$$

*Proof.* Suppose that $\Omega_{t+1}$ satisfies equations (58)–(60) for period $t+1$. Note that equation (58) satisfies equations (13) and (14).

To verify equation (58), split the constraint set of the maximization problem (15) into three regions: $\bar{R}_t B_{B,t} \leq \underline{\omega}_{t+1} K_{B,t}$; $\bar{R}_t B_{B,t} \in (\underline{\omega}_{t+1} K_{B,t}, \bar{\omega}_{t+1} K_{B,t}]$; and $\bar{R}_t B_{B,t} > \bar{\omega}_{t+1} K_{B,t}$. First of all, any point in the third region, $\bar{R}_t B_{B,t} > \bar{\omega}_{t+1} K_{B,t}$, is weakly dominated by $\bar{R}_t B_{B,t} = \bar{\omega}_{t+1} K_{B,t}$, since the feasible set of the choice variables is identical and the value of $\Omega_{t+1}$ is always 0 in the third region while it can be positive with $\bar{R}_t B_{B,t} = \bar{\omega}_{t+1} K_{B,t}$. Thus, the third region can be ignored.

By lemma 1, $\bar{R}_t B_{B,t} = \underline{\omega}_{t+1} K_{B,t}$ and $\bar{R}_t B_{B,t} = \bar{\omega}_{t+1} K_{B,t}$ at optimum in the first and the second region, respectively. Denote the maximum values of the objective function of the maximization problem (15) in the first and the second region by $\Omega'_t$ and $\Omega''_t$, respectively. Given that $\Omega_{t+1}$ satisfies equations (58)–(60) for period $t+1$, substituting the optimal values of $\bar{R}_t B_{B,t}$ in the first and the second region and equations (11), (13), and (14) into the objective function of the maximization problem (15) yields

$$\Omega'_t = \alpha_t K_{B,t-1} - Q_t(H_{B,t} - L_{B,t}) - \tilde{R}_t B_{B,t-1} + \lambda'_{B,t} K_{B,t}, \tag{64}$$

$$\Omega''_t = \alpha_t K_{B,t-1} - Q_t(H_{B,t} - L_{B,t}) - \tilde{R}_t B_{B,t-1} + \lambda''_{B,t} K_{B,t}. \tag{65}$$

The global solution to the maximization problem (15) can be obtained by maximizing the values of $\Omega'_t$ and $\Omega''_t$ with satisfying equation (16), $L_{B,t} \in [0, K_{B,t-1}]$, and $H_{B,t} \geq 0$. Since the first and the second region have the same feasible set of $H_{B,t}$ and $L_{B,t}$, $\Omega_t = \Omega'_t$ if $\lambda'_{B,t} \geq \lambda''_{B,t}$ and $\Omega_t = \Omega''_t$ if $\lambda'_{B,t} \leq \lambda''_{B,t}$. This result proves equations (59)–(61).

Given this result, now prove equations (62) and (63). The maximization problem (15) can be rewritten as

$$\Omega_t = \max_{\{H_{B,t}, L_{B,t}\}} \alpha_t K_{B,t-1} - Q_t(H_{B,t} - L_{B,t}) - \tilde{R}_t B_{B,t-1} + \lambda_{B,t} K_{B,t},$$
$$\text{s.t. equations (13), (14), and (16), } L_{B,t} \in [0, K_{B,t-1}],\ H_{B,t} \geq 0, \tag{66}$$

where $\lambda_{B,t} = \max\{\lambda'_{B,t},\ \lambda''_{B,t}\}$. Note that equation (11) is already incorporated by the definitions of $\lambda'_{B,t}$ and $\lambda''_{B,t}$. The maximization problem (66) implies that the equilibrium value of $\lambda_{B,t}$ satisfies

$$\lambda_{B,t} \quad \begin{cases} = Q_t(1-\hat{\delta}_t)^{-1}, & \text{if } H_{B,t} > 0, \\ = Q_t(1-\bar{\delta})^{-1}, & \text{if } L_{B,t} \in (0, K_{B,t-1}), \\ \leq Q_t(1-\bar{\delta})^{-1}, & \text{if } L_{B,t} = K_{B,t-1}, \\ \in [Q_t(1-\bar{\delta})^{-1},\ Q_t(1-\hat{\delta}_t)^{-1}], & \text{if } H_{B,t} = 0 \text{ and } L_{B,t} = 0. \end{cases} \tag{67}$$

When $\hat{\delta}_t > \bar{\delta}$, equation (67) implies that $L_{B,t} = 0$ if $H_{B,t} > 0$ and that $H_{B,t} = 0$ if $L_{B,t} > 0$. Thus equations (62) and (63) are proved. Substituting equations (16) and (67) into the objection function in the maximization problem (66) proves equation (58).

This proposition is sufficient to prove proposition 3. In this proposition, $\lambda'_{B,t}$ and $\lambda''_{B,t}$ denote the presented discounted values of marginal income from capital net of depreciation to a bank, when $\bar{R}_t B_{B,t} = \underline{\omega}_{t+1} K_{B,t}$ and $\bar{R}_t B_{B,t} = \bar{\omega}_{t+1} K_{B,t}$, respectively. The proposition implies that a bank chooses the face value of bank deposits, $\bar{R}_t B_{B,t}$, to maximize the present discounted value of its future income. Also, the total market value of bank securities, $\tilde{R}_t B_{B,t-1} + (D_t + V_t) S_{B,t-1}$, equals the present discounted value of the current and future income from the bank's capital, $[\alpha_t + \lambda_{B,t}(1-\bar{\delta})] K_{B,t-1}$, given no bank run in the current period. Note that a bank maximizes bank deposits given the risk of a bank run that the bank chooses to take, because the bank equity holding cost, $\zeta$, makes equity financing costly.

Also, equation (62) is a standard arbitrage-free condition for a bank, in which the right-hand side of the equation is the marginal acquisition cost of capital net of depreciation in the secondary capital market. If the equality were violated, then the quantity of capital bought by each bank in the market, $H_{B,t}$, would be either infinity or 0, which would violate the market clearing condition for the secondary capital market, or would contradict $H_{B,t} > 0$.

Finally, equation (63) implies that a bank is worse off by selling its capital randomly without knowing the depreciation rate of each unit of its capital, if it buys capital in the secondary capital market with an higher average depreciation rate ($\hat{\delta}_t$) than the average depreciation rate of its own capital ($\bar{\delta}$).

## Appendix 5. The Equilibrium Laws of Motion for Aggregate Variables in the Model of Banking

I show the equilibrium laws of motion in the model of banking. Suppose equations (18)–(20) hold in equilibrium.

*The Fraction of Capital Sold by Each Agent in the Secondary Capital Market*

Equation (26) for the basic model holds in the model of banking. Thus

$$\lambda_{i,t} = \begin{cases} \phi^{-1}, & \text{if } \phi_{i,t} = \phi, \\ \lambda_{U,t}, & \text{if } \phi_{i,t} = 0, \end{cases} \tag{68}$$

where $\lambda_{U,t}$ denotes the common value of $\lambda_{i,t}$ for unproductive agents, which satisfies

$$\begin{cases} \lambda_{U,t} = Q_t(1 - \hat{\delta}_t)^{-1}, & \text{if } h_{i,t} > 0 \text{ for all } i \text{ s.t. } \phi_{i,t} = 0, \\ h_{i,t} = 0 \text{ for all } i \text{ s.t. } \phi_{i,t} = 0, & \text{if } \lambda_{U,t} < Q_t(1 - \hat{\delta}_t)^{-1}. \end{cases} \tag{69}$$

Substituting the value of $\lambda_{i,t}$ for each type of agent into equation (26) yields the lower bound of the depreciation rates of capital sold by each agent, $\delta_{i,t}$:

$$\delta_{i,t} = \begin{cases} \delta_{P,t} \equiv \max\left\{\bar{\delta} - \Delta,\ 1 - \phi Q_t\right\}, & \text{if } \phi_{i,t} = \phi, \\ \delta_{U,t} \equiv \max\left\{\bar{\delta} - \Delta,\ 1 - Q_t(\lambda_{U,t})^{-1}\right\}, & \text{if } \phi_{i,t} = 0. \end{cases} \tag{70}$$

The maximum operator ensures that the value of $\delta_{i,t}$ is within the range of the distribution of depreciation rates. Equations (18) and (70) indicate that $\delta_{P,t} < \hat{\delta}_t$ and $\delta_{U,t} \le \hat{\delta}_t$. Thus, $\delta_{i,t} \le \bar{\delta} + \Delta$ for all $i$. Also, substituting equation (18) into equation (70) yields $\delta_{P,t} < \hat{\delta}_t$, which implies $\hat{\delta}_t < \bar{\delta} + \Delta$ given equation (50).

*Consumption and Saving by Each Agent*

Given the log-utility function, each agent consumes a fraction $1 - \beta$ of net worth and saves the rest in each period:

$$c_{i,t} = (1 - \beta) w_{i,t}, \tag{71}$$

$$\lambda_{i,t} k_{i,t} + b_{i,t} + (1 + \zeta) V_t s_{i,t} = \beta w_{i,t}, \tag{72}$$

where $w_{i,t}$ is the agent's net worth defined by

$$w_{i,t} \equiv \left( \alpha_t + \int_{\bar{\delta}-\Delta}^{\delta_{i,t}} \frac{\lambda_{i,t}(1-\delta)}{2\Delta} \, d\delta + \int_{\delta_{i,t}}^{\bar{\delta}+\Delta} \frac{Q_t}{2\Delta} \, d\delta \right) k_{i,t-1} + \tilde{R}_t b_{i,t-1} + (D_t + V_t) s_{i,t-1}. \tag{73}$$

In equations (72) and (73), the fractions of capital kept and sold by the agent are evaluated by the shadow value of capital net of depreciation for the agent, $\lambda_{i,t}$, and the secondary market price of capital, $Q_t$, respectively.

*The Equilibrium Laws of Motion for Aggregate Variables*

Given equations (18)–(20),

$$x_{i,t} > 0, h_{i,t} = b_{i,t} = s_{i,t} = 0, \text{ if } \phi_{i,t} = \phi, \tag{74}$$

as described in section 4.2. Also, in proposition 4, suppose that $\lambda'_{B,t} > \lambda''_{B,t}$, $H_{B,t} > 0$, and $\lambda_{U,t} < Q_t/(1 - \hat{\delta}_t)$ for all $t$. Note that $\lambda'_{B,t} > \lambda''_{B,t}$ implies $\hat{\delta}_t > \bar{\delta}$. Thus, $\bar{R}_t B_{B,t} = \underline{\omega}_{t+1} K_{B,t}$, $L_{B,t} = 0$, and $\lambda_{B,t} = Q_t/(1 - \hat{\delta}_t)$. Also, $h_{i,t} = 0$ and $b_{i,t} + s_{i,t} > 0$ if $\phi_{i,t} = 0$, given equation (69).

Now aggregate equation (72) for productive and unproductive agents separately. Then substitute the equalities described in the previous paragraph and equations (22), (68), (69), (70), and (74) into equations (5), (16), (50), and (51) and equation (72) after aggregating these equations for each type of agent. It holds that

$$\frac{K_{P,t}}{\phi} = \beta\rho\left\{\left(\alpha_t + \int_{\bar{\delta}-\Delta}^{\delta_{P,t}} \frac{1-\delta}{\phi\cdot 2\Delta}\,d\delta + \int_{\delta_{P,t}}^{\bar{\delta}+\Delta} \frac{Q_t}{2\Delta}\,d\delta\right)\right.$$
$$\left.\times\left(K_{P,t-1}+K_{U,t-1}\right) + \left[\alpha_t + \frac{Q_t(1-\bar{\delta})}{1-\hat{\delta}_t}\right]K_{B,t-1}\right\}, \qquad (75)$$

$$\lambda_{U,t}K_{U,t} + \left[\frac{(1+\zeta)Q_t}{1-\hat{\delta}_t} - \frac{\zeta\underline{\omega}_{t+1}}{\bar{R}_t}\right]K_{B,t}$$
$$= \beta(1-\rho)\left\{\left[\alpha_t + \int_{\bar{\delta}-\Delta}^{\delta_{U,t}} \frac{\lambda_{U,t}(1-\delta)}{2\Delta}\,d\delta + \int_{\delta_{U,t}}^{\bar{\delta}+\Delta} \frac{Q_t}{2\Delta}\,d\delta\right]\right.$$
$$\left.\times\left(K_{P,t-1}+K_{U,t-1}\right) + \left[\alpha_t + \frac{Q_t(1-\bar{\delta})}{1-\hat{\delta}_t}\right]K_{B,t-1}\right\}, \qquad (76)$$

$$K_{P,t} = \phi X_t + \rho\left(K_{P,t-1}+K_{U,t-1}\right)\int_{\bar{\delta}-\Delta}^{\delta_{P,t}} \frac{1-\delta}{2\Delta}d\delta, \qquad (77)$$

$$K_{U,t} = (1-\rho)\left(K_{P,t-1}+K_{U,t-1}\right)\int_{\bar{\delta}-\Delta}^{\delta_{U,t}} \frac{1-\delta}{2\Delta}d\delta, \qquad (78)$$

$$K_{B,t} = (1-\hat{\delta}_t)H_{B,t} + (1-\bar{\delta})K_{B,t-1}, \qquad (79)$$

$$K_{P,t}+K_{U,t}+K_{B,t} = \phi X_t + (1-\bar{\delta})(K_{P,t-1}+K_{U,t-1}+K_{B,t-1}), \qquad (80)$$

$$\hat{\delta}_t = \frac{\rho\int_{\delta_{P,t}}^{\bar{\delta}+\Delta}\delta\,d\delta \; + (1-\rho)\int_{\delta_{U,t}}^{\bar{\delta}+\Delta}\delta\,d\delta}{\rho(\bar{\delta}+\Delta-\delta_{P,t}) + (1-\rho)(\bar{\delta}+\Delta-\delta_{U,t})}, \qquad (81)$$

$$\frac{Q_t}{1-\hat{\delta}_t} = E_t\left\{\frac{\beta c_{i,t}\left[\alpha_{t+1} + \frac{Q_{t+1}}{1-\hat{\delta}_{t+1}}(1-\bar{\delta}) - \underline{\omega}_{t+1}\right]}{(1+\zeta)c_{i,t+1}} + \frac{\beta c_{i,t}\,\underline{\omega}_{t+1}}{c_{i,t+1}}\,\middle|\,\phi_{i,t}=0\right\}, \qquad (82)$$

where $K_{P,t} \equiv \int_{\{i|\phi_{i,t}=\phi\}} k_{i,t}\;di$ and $K_{U,t} \equiv \int_{\{i|\phi_{i,t}=0\}} k_{i,t}\;di$, and $B_{U,t}$ and $S_{U,t}$ are similarly defined. Also, given the value of $\alpha_{t+1}$, equations (71)–(73) imply

$$\frac{\beta c_{i,t}}{c_{i,t+1}} = \frac{\lambda_{U,t} K_{U,t} + \left[\frac{(1+\zeta)Q_t}{1-\hat{\delta}_t} - \frac{\zeta \underline{\omega}_{t+1}}{\bar{R}_t}\right] K_{B,t}}{(\alpha_{t+1} + \Psi_{t+1}) K_{U,t} + \left[\alpha_{t+1} + \frac{Q_{t+1}(1-\bar{\delta})}{1-\hat{\delta}_{t+1}}\right] K_{B,t}}, \tag{83}$$

where

$$\Psi_{t+1} \equiv \begin{cases} \int_{\bar{\delta}-\Delta}^{\delta_{P,t+1}} \frac{1-\delta}{\phi \cdot 2\Delta}\, d\delta + \int_{\delta_{P,t+1}}^{\bar{\delta}+\Delta} \frac{Q_t}{2\Delta}\, d\delta, & \text{if } \phi_{i,t+1} = \phi, \\ \int_{\bar{\delta}-\Delta}^{\delta_{U,t+1}} \frac{\lambda_{U,t+1}(1-\delta)}{2\Delta}\, d\delta + \int_{\delta_{U,t+1}}^{\bar{\delta}+\Delta} \frac{Q_t}{2\Delta}\, d\delta, & \text{if } \phi_{i,t+1} = 0. \end{cases} \tag{84}$$

Given equations (11) and (53) for $\bar{R}_t$, equation (70) for $\delta_{P,t}$ and $\delta_{U,t}$, and the definition of $\underline{\omega}_{t+1}$, equations (75)–(84) determine the equilibrium dynamics of ($K_{P,t}$, $K_{U,t}$, $K_{B,t}$, $H_{B,t}$, $X_t$, $\delta_{P,t}$, $\delta_{U,t}$, $\hat{\delta}_t$, $Q_t$, $\underline{\omega}_{t+1}$, $\bar{R}_t$, $\lambda_{U,t}$) recursively. Once the dynamics is obtained, the values of $\lambda'_{B,t}$, $\lambda''_{B,t}$ and $V_t S_{B,t}/(B_{B,t} + V_t S_{B,t})$ can be derived from equations (59), (60), and (25), in order.

## Appendix 6. The Numerical Solution Method for Dynamic Equilibrium in the Model of Banking

I solve the dynamic equilibrium with the set of parameter values specified in section 4 by approximating the equilibrium laws of motion, equations (75)–(82), by the following projection method:

Step 0. Because the equilibrium laws of motion are homogeneous of degree 1 with respect to $K_{P,t-1}$, $K_{U,t-1}$, and $K_{B,t-1}$, set grid points on the state space for $K_{P,t-1}$, $K_{U,t-1}$, and the aggregate productivity shock, $\alpha_t$. The value of $K_{B,t-1}$ is set to $1 - K_{P,t-1} - K_{U,t-1}$ on each grid point. Guess the equilibrium values of endogenous variables on each grid point, including $\bar{\omega}_{t+1}$ and $\underline{\omega}_{t+1}$. Call this correspondence between state variables and endogenous variables a "candidate array."

Step 1. Suppose the candidate array returns the correct equilibrium values for each state of $K_{P,t}$, $K_{U,t}$, $K_{B,t}$, and the aggregate productivity shock in the next period. The points between the grid points in the state space are approximated by linear interpolation. Given this, derive another candidate array for the current period that satisfies the equilibrium laws of motion.

Step 2. Compare the candidate array for the current period and the one for the next period. If the ratio of each element between the two arrays becomes sufficiently close to 1, then take the candidate array as the equilibrium correspondence. Otherwise, update the candidate array by a linear combination of the two arrays and go back to step 1.

In the numerical examples in this paper, I set grid points in the $\pm 5$ percent range of the deterministic steady-state values of $K_{P,t-1}$ and $K_{U,t-1}$. The number of grid points are twenty for each of the two variables. The convergence criterion in step 2 is 1e-03. In updating the candidate array in step 2, the weight on the candidate array for the current period is 0.001. The initial guess in step 0 is obtained through homotopy starting from the set of parameter values with which the deterministic steady state provides a successful initial guess of the candidate array that leads to convergence.

The equilibrium laws of motion, equations (75)–(82), are valid if equations (18)–(20), $\lambda'_{B,t} > \lambda''_{B,t}$, $H_{B,t} > 0$, and $\lambda_{U,t} < Q_t/(1-\hat{\delta}_t)$, as described in appendix 5, and also if all variables are non-negative. These inequalities are checked for each element of the converged candidate array. Starting from the deterministic steady state, I run random simulations of the dynamics for 5,000 periods to confirm that the equilibrium dynamics move within the grid points satisfying the inequalities.