Estimating Inflation Expectations with a Limited Number of Inflation-Indexed Bonds*

Richard Finlay and Sebastian Wende
Reserve Bank of Australia

We develop a novel technique to estimate inflation expectations and inflation risk premia when only a limited number of inflation-indexed bonds are available. The method involves pricing coupon-bearing inflation-indexed bonds directly in terms of an affine term structure model, and avoids the usual requirement of estimating zero-coupon real yield curves. We estimate the model using a non-linear Kalman filter and apply it to Australia. The results suggest that long-term inflation expectations in Australia are well anchored within the Reserve Bank of Australia's inflation target range of 2 to 3 percent, and that inflation expectations are less volatile than inflation risk premia.

JEL Codes: E31, E43, G12.

1. Introduction

Reliable and accurate estimates of inflation expectations are important to central banks, given the role of these expectations in influencing inflation and economic activity. Inflation expectations may also indicate over what horizon individuals believe that a central bank will achieve its inflation target, if at all.

A common measure of inflation expectations based on financial market data is the break-even inflation yield, referred to simply as the inflation yield. The inflation yield is given by the difference in

*The authors thank Rudolph van der Merwe for help with the central difference Kalman filter, as well as Adam Cagliarini, Jonathan Kearns, Christopher Kent, Frank Smets, Ian Wilson, and an anonymous referee for useful comments and suggestions. Responsibility for any remaining errors rests with the authors. The views expressed in this paper are those of the authors and are not necessarily those of the Reserve Bank of Australia. E-mail: FinlayR@rba.gov.au.

yields of nominal and inflation-indexed zero-coupon bonds of equal maturity. That is,

$$y^i_{t,\tau} = y^n_{t,\tau} - y^r_{t,\tau},$$

where $y^i_{t,\tau}$ is the inflation yield between time t and $t+\tau$, $y^n_{t,\tau}$ is the nominal yield, and $y^r_{t,\tau}$ is the real yield.[1] But the inflation yield may not give an accurate reading of inflation expectations. Inflation expectations are an important determinant of the inflation yield but are not the only determinant; the inflation yield is also affected by inflation risk premia, which is the extra compensation required by investors who are exposed to the risk that inflation will be higher than expected (we assume that other factors that may affect the inflation yield, such as liquidity premia, are absorbed into risk premia in our model). By treating inflation as a random process, we are able to model expected inflation and the cost of the uncertainty associated with inflation separately.

Inflation expectations and inflation risk premia have been estimated for the United Kingdom and the United States using models similar to the one used in this paper. Beechey (2008) and Joyce, Lildholdt, and Sorensen (2010) find that inflation risk premia decreased in the United Kingdom, first after the Bank of England adopted an inflation target and then again after it was granted independence. Using U.S. Treasury Inflation-Protected Securities (TIPS) data, Durham (2006) estimates expected inflation and inflation risk premia, although he finds that inflation risk premia are not significantly correlated with measures of the uncertainty of future inflation or monetary policy. Also using TIPS data, D'Amico, Kim, and Wei (2008) find inconsistent results due to the decreasing liquidity premia in the United States, although their estimates are improved by including survey forecasts and using a sample over which the liquidity premia are constant.

In this paper we estimate a time series for inflation expectations at various horizons, taking into account inflation risk premia, using a latent factor affine term structure model which is widely

[1] To fix terminology, all yields referred to in this paper are gross, continuously compounded zero-coupon yields. So, for example, the nominal yield is given by $y^n_{t,\tau} = -\log(P^n_{t,\tau})$, where $P^n_{t,\tau}$ is the price at time t of a zero-coupon nominal bond paying one dollar at time $t+\tau$.

used in the literature. Compared with the United Kingdom and the United States, there are a very limited number of inflation-indexed bonds on issue in Australia. This complicates the estimation but also highlights the usefulness of our approach. In particular, the limited number of inflation-indexed bonds means that we cannot reliably estimate a zero-coupon real yield curve and so cannot estimate the model in the standard way. Instead we develop a novel technique that allows us to estimate the model using the *price* of coupon-bearing inflation-indexed bonds instead of zero-coupon real yields. The estimation of inflation expectations and risk premia for Australia, as well as the technique we employ to do so, is the chief contribution of this paper to the literature.

To better identify model parameters, we also incorporate inflation forecasts from Consensus Economics in the estimation. Inflation forecasts provide shorter-maturity information (for example, forecasts exist for inflation next quarter) as well as information on inflation expectations that is separate from risk premia. Theoretically the model is able to estimate inflation expectations and inflation risk premia purely from the nominal and inflation-indexed bond data; inflation risk premia compensate investors for exposure to variation in inflation, which should be captured by the observed variation in prices of bonds at various maturities. This is, however, a lot of information to extract from a limited amount of data. Adding forecast data helps to better anchor the model estimates of inflation expectations and so improves model fit.

Inflation expectations as estimated in this paper have a number of advantages over using the inflation yield to measure expectations. For example, five-year-ahead inflation expectations as estimated in this paper (i) account for risk premia and (ii) are expectations of the inflation rate *in five years time*. In contrast, the five-year inflation yield ignores risk premia and gives an *average* of inflation rates *over the next five years*.[2] The techniques used in the paper are potentially

[2]In addition, due to the lack of zero-coupon real yields in Australia's case, yields-to-maturity of coupon-bearing nominal and inflation-indexed bonds have historically been used when calculating the inflation yield. This restricts the horizon of inflation yields that can be estimated to the maturities of the existing inflation-indexed bonds, and is not a like-for-like comparison due to the differing coupon streams of inflation-indexed and nominal bonds.

useful for other countries with a limited number of inflation-indexed bonds on issue.

In section 2 we outline the model. Section 3 describes the data, estimation of the model parameters and latent factors, and how these are used to extract our estimates of inflation expectations. Results are presented in section 4 and conclusions are drawn in section 5.

2. Model

2.1 Affine Term Structure Model

Following Beechey (2008), we assume that the inflation yield can be expressed in terms of an inflation stochastic discount factor (SDF). The inflation SDF is a theoretical concept, which for the purpose of asset pricing incorporates all information about income and consumption uncertainty in our model. Appendix 1 provides a brief overview of the inflation, nominal, and real SDFs.

We assume that the inflation yield can be expressed in terms of an inflation SDF, M_t^i, according to

$$y_{t,\tau}^i = -\log\left(\mathbb{E}_t\left(\frac{M_{t+\tau}^i}{M_t^i}\right)\right).$$

We further assume that the evolution of the inflation SDF can be approximated by a diffusion equation,

$$\frac{\mathrm{d}M_t^i}{M_t^i} = -\pi_t^i \mathrm{d}t - \boldsymbol{\lambda}_t^{i\prime}\, \mathrm{d}\mathbf{B}_t. \tag{1}$$

According to this model, $\mathbb{E}_t(\mathrm{d}M_t^i/M_t^i) = -\pi_t^i \mathrm{d}t$, so that the instantaneous inflation rate is given by π_t^i. The inflation SDF also depends on the term $\boldsymbol{\lambda}_t^{i\prime}\mathrm{d}\mathbf{B}_t$. Here $\mathbf{B}_t$ is a Brownian motion process and $\boldsymbol{\lambda}_t^i$ relates to the market price of this risk. $\boldsymbol{\lambda}_t^i$ determines the risk premium, and this setup allows us to separately identify inflation expectations and inflation risk premia. This approach to bond pricing is standard in the literature and has been very successful in capturing the dynamics of nominal bond prices (see Kim and Orphanides 2005, for example).

We model both the instantaneous inflation rate and the market price of inflation risk as affine functions of three latent factors. The instantaneous inflation rate is given by

$$\pi_t^i = \rho_0 + \boldsymbol{\rho}'\mathbf{x}_t, \tag{2}$$

where $\mathbf{x}_t = [x_t^1, x_t^2, x_t^3]'$ are our three latent factors.[3] Since the latent factors are unobserved, we normalize $\boldsymbol{\rho}$ to be a vector of ones, $\mathbf{1}$, so that the inflation rate is the sum of the latent factors and a constant, ρ_0. We assume that the price of inflation risk has the form

$$\boldsymbol{\lambda}_t^i = \boldsymbol{\lambda}_0 + \Lambda\mathbf{x}_t, \tag{3}$$

where $\boldsymbol{\lambda}_0$ is a vector and Λ is a matrix of free parameters.

The evolution of the latent factors $\mathbf{x}_t$ is given by an Ornstein-Uhlenbeck process (a continuous-time mean-reverting stochastic process),

$$\mathrm{d}\mathbf{x}_t = K(\boldsymbol{\mu} - \mathbf{x}_t)\mathrm{d}t + \Sigma\ \mathrm{d}\mathbf{B}_t, \tag{4}$$

where $K(\boldsymbol{\mu} - \mathbf{x}_t)$ is the drift component, K is a lower triangular matrix, $\mathbf{B}_t$ is the same Brownian motion used in equation (1), and Σ is a diagonal scaling matrix. In this instance we set $\boldsymbol{\mu}$ to zero so that $\mathbf{x}_t$ is a zero-mean process, which implies that the average instantaneous inflation rate is ρ_0.

Equations (1)–(4) can be used to show how the latent factors affect the inflation yield (see appendix 2 for details). In particular, one can show that

$$y_{t,\tau}^i = \alpha_\tau^* + \boldsymbol{\beta}_\tau^{*\prime}\mathbf{x}_t, \tag{5}$$

where α_τ^* and $\boldsymbol{\beta}_\tau^*$ are functions of the underlying model parameters. In the standard estimation procedure, when a zero-coupon inflation yield curve exists, this function is used to estimate the values of $\mathbf{x}_t$.

[3]Note that one can specify models in which macroeconomic series take the place of latent factors—as done, for example, in Hördahl (2008). Such models have the advantage of simpler interpretation but, as argued in Kim and Wright (2005), tend to be less robust to model misspecification and generally result in a worse fit of the data.

2.2 Pricing Inflation-Indexed Bonds in the Latent Factor Model

We now derive the price of an inflation-indexed bond as a function of the model parameters, the latent factors, and nominal zero-coupon bond yields, denoted $H1(\mathbf{x}_t)$. This function will later be used to estimate the model as described in section 3.2.

As is the case with any bond, the price of an inflation-indexed bond is the present value of its stream of coupons and its par value. In an inflation-indexed bond, the coupons are indexed to inflation so that the real value of the coupons and principal is preserved. In Australia, inflation-indexed bonds are indexed with a lag of between 4½ and 5½ months, depending on the particular bond in question. If we denote the lag by Δ and the historically observed increase in the price level between $t-\Delta$ and t by $I_{t,\Delta}$, then at time t the implicit nominal value of the coupon paid at time $t+\tau_s$ is given by the real (at time $t-\Delta$) value of that coupon, C_s, adjusted for the historical inflation that occurred between $t-\Delta$ and t, $I_{t,\Delta}$, and further adjusted by the current market-implied change in the price level between periods t and $t+\tau_s-\Delta$ using the inflation yield. So the implied nominal coupon paid becomes $C_s I_{t,\Delta}\exp(y^i_{t,\tau_s-\Delta})$. The present value of this nominal coupon is then calculated using the nominal discount factor between t and $t+\tau_s$, $\exp(-y^n_{t,\tau_s})$. So if an inflation-indexed bond pays a total of m coupons, where the par value is included in the set of coupons, then the price at time t of this bond is given by

$$P^r_t = \sum_{s=1}^{m}(C_s I_{t,\Delta} e^{y^i_{t,\tau_s-\Delta}})e^{-y^n_{t,\tau_s}} = \sum_{s=1}^{m} C_s I_{t,\Delta} e^{y^i_{t,\tau_s-\Delta}-y^n_{t,\tau_s}}.$$

We noted earlier that the inflation yield is given by $y^i_{t,\tau} = \alpha^*_\tau + \boldsymbol{\beta}^{*\prime}_\tau \mathbf{x}_t$, so the bond price can be written as

$$P^r_t = \sum_{s=i}^{m} C_s I_{t,\Delta} e^{-y^n_{t,\tau_s}+\alpha^*_{\tau_s-\Delta}+\boldsymbol{\beta}^*_{\tau_s-\Delta}{}'\mathbf{x}_t} = H1(\mathbf{x}_t). \tag{6}$$

Note that $\exp(-y^n_{t,\tau_s})$ can be estimated directly from nominal bond yields (see section 3.1). So the price of a coupon-bearing inflation-indexed bond can be expressed as a function of the latent factors $\mathbf{x}_t$

as well as the model parameters, nominal zero-coupon bond yields, and historical inflation. We define $H1(\mathbf{x}_t)$ as the non-linear function that transforms our latent factors into bond prices.

2.3 Inflation Forecasts in the Latent Factor Model

In the model, inflation expectations are a function of the latent factors, denoted $H2(\mathbf{x}_t)$. Inflation expectations are not equal to expected inflation yields since yields incorporate risk premia, whereas forecasts do not. Inflation expectations as reported by Consensus Economics are expectations at time t of how the CPI will increase between time s in the future and time $s+\tau$ and are therefore given by

$$\mathbb{E}_t\left(\exp\left(\int_s^{s+\tau}\pi_u^i \mathrm{d}u\right)\right) = H2(\mathbf{x}_t),$$

where π_t^i is the instantaneous inflation rate at time t. In appendix 2 we show that one can express $H2(\mathbf{x}_t)$ as

$$H2(\mathbf{x}) = \exp\Big(-\bar{\alpha}_\tau - \bar{\boldsymbol{\beta}}'_\tau(e^{-K(s-t)}\mathbf{x}_t + (I - e^{-K(s-t)})\boldsymbol{\mu}) + \frac{1}{2}\bar{\boldsymbol{\beta}}'_\tau\Omega_{s-t}\bar{\boldsymbol{\beta}}_\tau\Big). \quad (7)$$

The parameters $\bar{\alpha}_\tau$ and $\bar{\boldsymbol{\beta}}_\tau$ (and Ω_{s-t}) are defined in appendix 2, and are similar to α^*_τ and $\boldsymbol{\beta}^*_\tau$ from equation (5).

3. Data and Model Implementation

3.1 Data

Four types of data are used: nominal zero-coupon bond yields derived from nominal Australian Commonwealth Government bonds, Australian Commonwealth Government inflation-indexed bond prices, inflation forecasts from Consensus Economics, and historical inflation.

Nominal zero-coupon bond yields are estimated using the approach of Finlay and Chambers (2009). These nominal yields correspond to y^n_{t,τ_s} and are used in computing our function $H1(\mathbf{x}_t)$

from equation (6). Note that the Australian nominal yield curve has maximum maturity of roughly twelve years. We extrapolate nominal yields beyond this by assuming that the nominal and real yield curves have the same slope. This allows us to utilize the prices of all inflation-indexed bonds, which have maturities of up to twenty-four years (in practice, the slope of the real yield curve beyond twelve years is very flat, so that if we instead hold the nominal yield curve constant beyond twelve years, we obtain virtually identical results).

We calculate the real prices of inflation-indexed bonds using yield data.[4] Our sample runs from July 1992 to December 2010, with the available data sampled at monthly intervals up to June 1994 and weekly intervals thereafter; bonds with less than one year remaining to maturity are excluded. By comparing these computed inflation-indexed bond prices, which form the P_t^r in equation (6), with our function $H1(\mathbf{x}_t)$, we are able to estimate the latent factors. We assume that the standard deviation of the bond price measurement error is 4 basis points. This is motivated by market liaison which suggests that, excluding periods of market volatility, the bid-ask spread has stayed relatively constant over the period considered, at around 8 basis points. Some descriptive statistics for nominal and inflation-indexed bonds are given in table 1.

Note that inflation-indexed bonds are relatively illiquid, especially in comparison to nominal bonds.[5] Therefore, inflation-indexed bond yields potentially incorporate liquidity premia, which could bias our results. As discussed, we use inflation forecasts as a measure of inflation expectations. These forecasts serve to tie down inflation expectations, and as such we implicitly assume that liquidity premia are included in our measure of risk premia. We also assume that the existence of liquidity premia causes a level shift in estimated risk premia but does not greatly bias the estimated *changes* in risk premia.[6]

---

[4] Available from table F16 at www.rba.gov.au/statistics/tables/index.html.

[5] Average yearly turnover between 2003–04 and 2007–08 was roughly $340 billion for nominal government bonds and $15 billion for inflation-indexed bonds, which equates to a turnover ratio of around 7 for nominal bonds and 2½ for inflation-indexed bonds (see Australian Financial Markets Association 2008).

[6] Inflation swaps are now more liquid than inflation-indexed bonds and may provide alternative data for use in estimating inflation expectations at some point in the future. Currently, however, there is not a sufficiently long time series of inflation swap data to use for this purpose.

Table 1. Descriptive Statistics of Bond Price Data

Statistic		Time Period			
		1992–1995	**1996–2000**	**2001–2005**	**2006–2010**
Number of Bonds:	Nominal	12–19	12–19	8–12	8–14
	Inflation Indexed	3–5	4–5	3–4	2–4
Maximum Tenor:	Nominal	11–13	11–13	11–13	11–14
	Inflation Indexed	13–21	19–24	15–20	11–20
Average Outstanding:	Nominal	49.5	70.2	50.1	69.5
	Inflation Indexed	2.1	5.0	6.5	7.1

Note: Tenor in years; outstandings in billions; only bonds with at least one year to maturity are included.

The inflation forecasts are taken from Consensus Economics. We use three types of forecast:

(i) monthly forecasts of the percentage change in CPI over the current and the next calendar year

(ii) quarterly forecasts of the year-on-year percentage change in the CPI for seven or eight quarters in the future

(iii) biannual forecasts of the year-on-year percentage change in the CPI for each of the next five years, as well as from five years in the future to ten years in the future

We use the function $H2(\mathbf{x}_t)$ to relate these inflation forecasts to the latent factors, and use the past forecasting performance of the inflation forecasts relative to realized inflation to calibrate the standard deviation of the measurement errors.

Historical inflation enters the model in the form of $I_{t,\Delta}$ from section 2.2, but otherwise is not used in estimation. This is because the fundamental variable being modeled is the *current instantaneous* inflation rate. Given the inflation law of motion (implicitly defined by equations (2)–(4)), inflation expectations and inflation-indexed bond prices are affected by current inflation and so can inform our estimation. By contrast, the published inflation rate is always "old news" from the perspective of our model and so has nothing direct to say about current instantaneous inflation.[7]

3.2 The Kalman Filter and Maximum-Likelihood Estimation

We use the Kalman filter to estimate the three latent factors, using data on bond prices and inflation forecasts. The Kalman filter can estimate the state of a dynamic system from noisy observations. It does this by using information about how the state evolves over time, as summarized by the state equation, and relating the state to noisy observations using the measurement equation. In our case the latent factors constitute the state of the system and our bond prices and forecast data constitute the noisy observations. From the latent factors we are able to make inferences about inflation expectations and inflation risk premia.

The standard Kalman filter was developed for a linear system. Although our state equation (given by equation (14)) is linear, our measurement equations, using $H1(\mathbf{x}_t)$ and $H2(\mathbf{x}_t)$ as derived in sections 2.2 and 2.3, are not. This is because we work with coupon-bearing bond prices instead of zero-coupon yields. We overcome this problem by using a central difference Kalman filter, which is a type of non-linear Kalman filter.[8]

The approximate log-likelihood is evaluated using the forecast errors of the Kalman filter. If we denote the Kalman filter's forecast of the data at time t by $\hat{\mathbf{y}}_t(\zeta, \mathbf{x}_t(\zeta, \mathbf{y}_{t-1}))$—which depends on the parameters (ζ) and the latent factors $(\mathbf{x}_t(\zeta, \mathbf{y}_{t-1}))$, which in turn depend on the parameters and the data observed up to time $t-1$ $(\mathbf{y}_{t-1})$—then the approximate log-likelihood is given by

$$\mathcal{L}(\zeta) = -\sum_{t=1}^{T} \left(\log |P_{\mathbf{y}_t}| + (\mathbf{y}_t - \hat{\mathbf{y}}_t) P_{\mathbf{y}_t}^{-1} (\mathbf{y}_t - \hat{\mathbf{y}}_t)' \right).$$

[7]Note that our model is set in continuous time; data are sampled discretely, but all quantities—for example, the inflation law of motion as well as inflation yields and expectations—evolve continuously. π_t^i from equation (2) is the current instantaneous inflation rate, not a one-month or one-quarter rate.

[8]See appendix 3 for more detail on the central difference Kalman filter.

Here the estimated covariance matrix of the forecast data is denoted by $P_{\mathbf{y}_t}$.[9] In the model the parameters are given by $\zeta = (K, \boldsymbol{\lambda}_0, \Lambda, \rho_0, \Sigma)$.

We numerically optimize the log-likelihood function to obtain parameter estimates. From the parameter estimates we use the Kalman filter to obtain estimates of the latent factors.

3.3 Calculation of Model Estimates

For a given set of model parameters and latent factors, we can calculate inflation forward rates, expected future inflation rates, and inflation risk premia.

In appendix 2 we show that the expected future inflation rate at time t for time $t+\tau$ can be expressed as

$$\mathbb{E}_t\left(\pi^i_{t+\tau}\right) = \rho_0 + \mathbf{1}' \cdot e^{-K\tau}\mathbf{x}_t.$$

The inflation forward rate at time t for time $t+\tau$, $f^i_{t,\tau}$, is the rate of inflation at time $t+\tau$ implied by market prices of nominal and inflation-indexed bonds trading at time t. It is related to the inflation yield via $y^i_{t,\tau} = \int_t^{t+\tau} f^i_{t,s}\,\mathrm{d}s$.[10] As bond prices incorporate inflation risk, so does the inflation forward rate. In our model the inflation forward rate is given by

$$\begin{aligned} f^i_{t,\tau} &= \rho_0 + \mathbf{1}' \cdot (e^{-K^*\tau}\mathbf{x}_t + (I - e^{-K^*\tau})\boldsymbol{\mu}^*) \\ &\quad - \frac{1}{2}(\mathbf{1}'(I - e^{-K^*\tau})K^{*-1}\Sigma)(\mathbf{1}'(I - e^{-K^*\tau})K^{*-1}\Sigma)'. \end{aligned}$$

See appendix 2 for details on the above and definitions of K^* and $\boldsymbol{\mu}^*$.

The inflation risk premium is given by the difference between the inflation forward rate, which incorporates risk aversion, and

[9] In actual estimation we exclude the first six months of data from the likelihood calculation to allow "burn-in" time for the Kalman filter.

[10] Note that at time t the inflation *forward* rate at time $s > t$, $f^i_{t,s}$, is known, as it is determined by known inflation yields. The inflation rate, π^i_s, that will prevail at s is unknown, however, and in our model is a random variable. π^i_s is related to the known inflation yield by $\exp(-y^i_{t,\tau}) = \mathbb{E}_t(\exp(-\int_t^{t+\tau} \pi^{i*}_s\,\mathrm{d}s))$ so that $y^i_{t,\tau} = -\log(\mathbb{E}_t(\exp(-\int_t^{t+\tau} \pi^{i*}_s\,\mathrm{d}s)))$, where π^{i*}_s is the so-called risk-neutral version of π^i_s (see appendix 2 for details).

the expected future inflation rate, which is free of risk aversion. The inflation risk premium at time t for time $t + \tau$ is given by $f^i_{t,\tau} - \mathbb{E}_t(\pi^i_{t+\tau})$.

4. Results

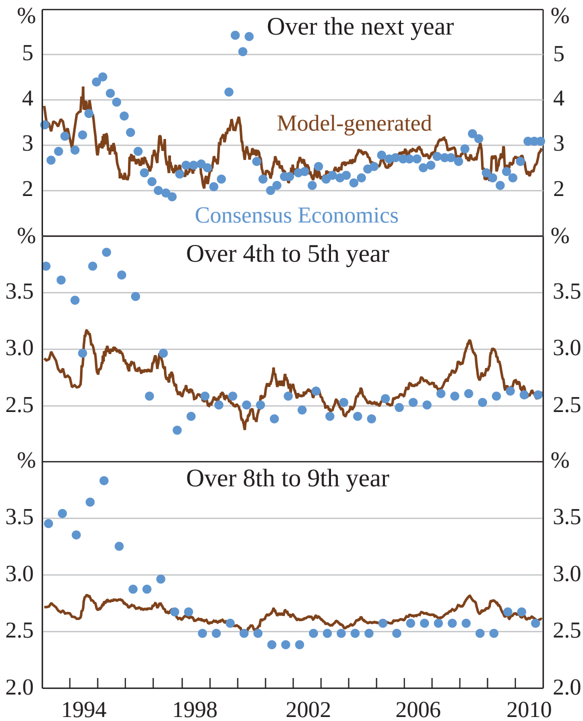
4.1 Model Parameters and Fit to Data

We estimate the model over the period July 31, 1992 to December 15, 2010 using a number of different specifications. First we estimate both two- and three-factor versions of our model. Using a likelihood-ratio test, we reject the hypothesis that there is no improvement of model fit between the two-factor model and three-factor model and so use the three-factor model. (Three factors are usually considered sufficient in the literature, with for example the overwhelming majority of variation in yields captured by the first three principal components.)

We also consider three-factor models with and without forecast data. Both models are able to fit the inflation yield data well, with a mean absolute error between ten-year inflation yields as estimated from the models and ten-year break-even inflation calculated directly from bond prices of around 5 basis points.[11] The model without forecast data gives unrealistic estimates of inflation expectations and inflation risk premia, however: ten-year-ahead inflation expectations are implausibly volatile and can be as high as 8 percent and as low as −1 percent, which is not consistent with economists' forecasts. These findings are consistent with those of Kim and Orphanides (2005), where the use of forecast data is advocated as a means of separating expectations from risk premia. Note, however, that estimates from the model with forecast data are not solely determined by the forecasts; the model estimates of expected future inflation only roughly match the forecast data and on occasion deviate significantly from them, as seen in figure 1.

[11] The divergence between model yields and those measured directly from bond data is mainly due to the different types of yields not being directly comparable—model estimates are zero-coupon yields that take into account indexation lag, while the direct measure is estimated from coupon-bearing bonds which reflect a certain amount of historical inflation.

Figure 1. Forecast Change in CPI



Source: Consensus Economics; authors' calculations.

Our preferred model is thus the three-factor model estimated using forecast data. Likelihood-ratio tests indicate that two parameters of that model (Λ_{11} and Λ_{21}) are statistically insignificant and so they are excluded. Our final preferred model has twenty freely estimated parameters, which are given in table 2. We note that the estimate of ρ_0, the steady-state inflation rate in our model, is 2.6 percent, which is within the inflation target range. The persistence of inflation is essentially determined by the diagonal entries of the K matrix, which drives the inflation law of motion as defined by equations (2)–(4). The first diagonal entry of K is 0.19, which in a single-factor model would imply a half-life of the first latent factor (being the time taken for the latent factor, and so inflation, to revert halfway back to its mean value after experiencing a shock) of around

Table 2. Parameter Estimates for Final Model (Model Estimated 1992–2010)

Parameter	Index Number (i)					
	1		2		3	
ρ_0	2.64	(0.26)	—		—	
$(K)_{1i}$	0.19	(0.02)	0		0	
$(K)_{2i}$	−2.88	(0.05)	1.75	(0.05)	0	
$(K)_{3i}$	1.11	(0.05)	1.74	(0.05)	0.80	(0.01)
$(\Sigma)_{ii}$	0.11	(0.02)	1.51	(0.10)	0.96	(0.02)
$\lambda_{0,i}$	0.12	(0.01)	0.10	(0.01)	−0.01	(0.00)
$(\Lambda)_{1i}$	0		55.44	(0.32)	15.31	(0.06)
$(\Lambda)_{2i}$	0		−107.80	(0.26)	−8.91	(0.06)
$(\Lambda)_{3i}$	−12.38	(0.08)	−144.22	(0.45)	−73.07	(0.20)

Note: ρ_0 and $(\Sigma)_{ii}$ are given in percentage points; standard errors are shown in parentheses.

3½ years. The half-lives of the other two latent factors would be five and ten months.

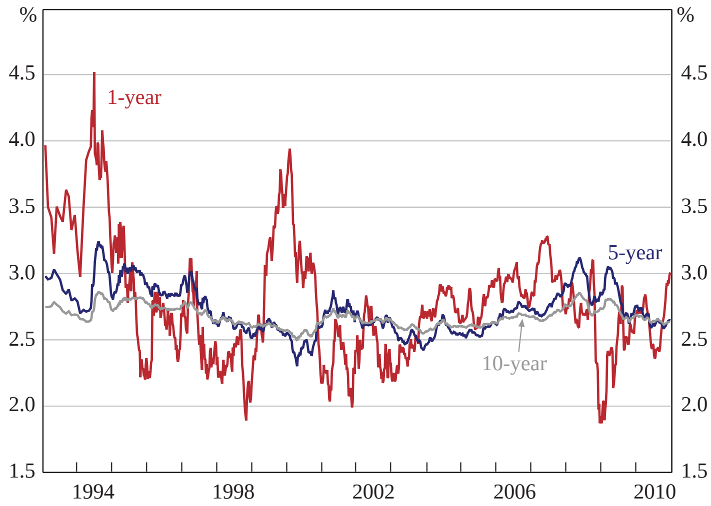
4.2 Qualitative Discussion of Results

4.2.1 Inflation Expectations

Our estimated expected future inflation rates at horizons of one, five, and ten years are shown in figure 2. Two points stand out immediately: one-year-ahead inflation expectations are much more volatile than five- and ten-year-ahead expectations and, as may be expected, are strongly influenced by current inflation (not shown); longer-term inflation expectations appear to be well anchored within the 2 to 3 percent target range.

We see that there is a general decline in inflation expectations from the beginning of the sample until around 1999, the year before the introduction of the Goods and Services Tax (GST). The estimates suggest that the introduction of the GST on July 1, 2000 resulted in a large one-off increase in short-term inflation expectations. This is reflected in the run-up in one-year-ahead inflation expectations over calendar year 1999, although the peak in

Figure 2. Expected Inflation Rates



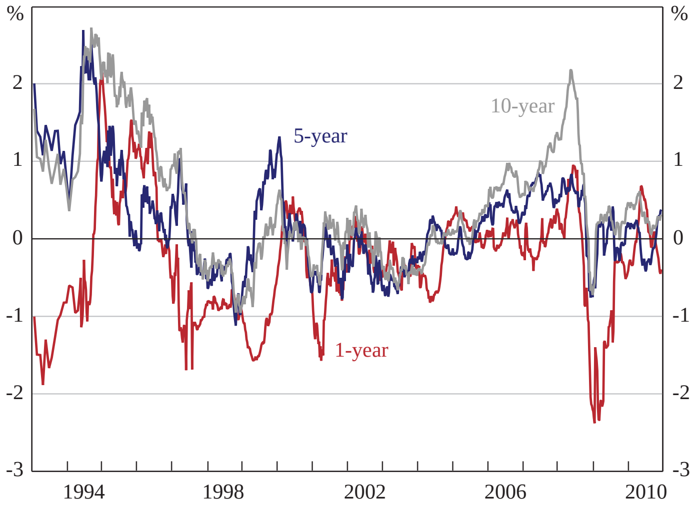
the estimated expectations is below the actual peak in year-end CPI growth of 6.1 percent.[12] Of particular interest, however, is the non-responsiveness of five- and ten-year-ahead expectations, which should be the case if the inflation target is seen as credible.

Long-term expectations increased somewhat between mid-2000 and mid-2001, perhaps prompted by easier monetary conditions globally as well as relatively high inflation in Australia. Interestingly, there appears to have been a sustained general rise in inflation expectations between 2004 and 2008 at all horizons. Again this was a time of rising domestic inflation, strong world growth, a boom in the terms of trade, and rising asset prices.

In late 2008 the inflation outlook changed and short-term inflation expectations fell dramatically, likely in response to forecasts of very weak global demand caused by the financial crisis. Longer-term expectations also fell before rising over the early part of 2009 as authorities responded to the crisis. The subsequent moderation of longer-term expectations, as well as the relative stabilization of short-term expectations over 2010 suggests that financial market

[12] The legislation introducing the GST was passed through Parliament in June 1999.

Figure 3. Inflation Risk Premia



participants considered the economic outlook and Australian authorities' response to the crisis sufficient to maintain inflation within the target range.

The latest data, corresponding to December 2010, shows one-year-ahead inflation expectations exceeding 3 percent, close to the Reserve Bank forecast for inflation of 2.75 percent over the year to December 2011 given in the November 2010 *Statement of Monetary Policy*. Longer-term model-implied inflation expectations as of December 2010 are for inflation close to the middle of the 2 to 3 percent inflation target range.

4.2.2 Inflation Risk Premia

Although more volatile than our long-term inflation expectation estimates, long-term inflation risk premia broadly followed the same pattern—declining over the first third of the sample, gradually increasing between 2004 and 2008 before falling sharply with the onset of the global financial crisis, and then rising again as markets reassessed the likelihood of a severe downturn in Australia (figure 3). The main qualitative point of difference between the two series is in their reaction to the GST. As discussed earlier, the estimates of long-term inflation expectations remained well anchored during the GST period, whereas (as we can see from figure 3) the estimates of long-term risk premia rose sharply. As the terminology suggests, inflation expectations represent investors' central forecast for inflation, while risk premia can be thought of as representing second-order information—essentially how uncertain investors are about their central forecasts and how much they dislike this uncertainty. So while longer-dated expectations of inflation did not change around the introduction of the GST, the rise in risk premia indicates a more variable and uncertain inflation outlook.

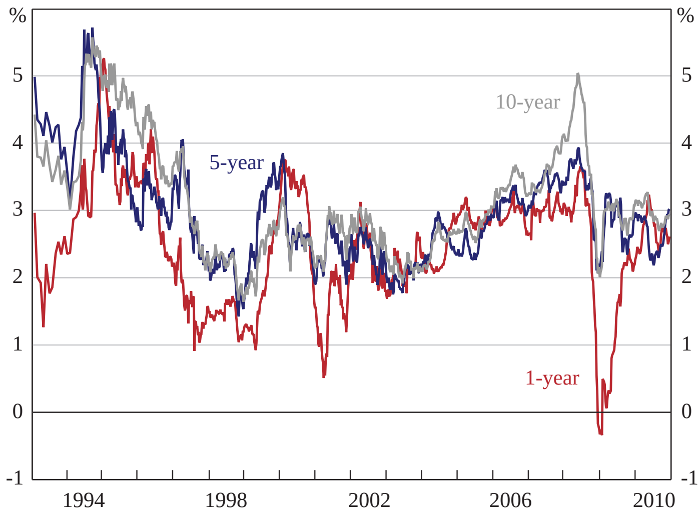
Although our estimates show periods of negative inflation risk premia, indicating that investors were happy to be exposed to inflation risk, this is probably not the case in reality. In our model, inflation risk premia are given by forward rates of inflation (as implied by the inflation yield curve), less inflation expectations. The inflation yield curve is given as the difference between nominal and real yields. Hence if real yields contain a liquidity premium, they will be higher, shifting the inflation yield curve down and reducing the estimated inflation risk premia to below their true level. The inflation-indexed bond market is known to be relatively illiquid in comparison with the nominal bond market, and this provides a plausible explanation for our negative estimates.

If the illiquidity in the inflation-indexed bond market is constant through time, then the level of our estimated risk premium will be biased but *changes* in the risk premium should be accurately estimated. Market liaison suggests that an assumption of relatively constant liquidity, at least during normal times, is not an unreasonable one; as noted earlier, for example, bid-ask spreads have stayed relatively constant over most of the period under consideration. The trough in inflation risk premia around late 2008 and early 2009 may be one exception to this, however, with the liquidity premia for inflation-indexed bonds relative to nominal bonds possibly increasing (in line with increases in liquidity premia for most assets relative to highly rated and highly liquid government securities at this time).

4.2.3 Inflation Forward Rates

The inflation forward rate reflects the relative prices of traded nominal and inflation-indexed bonds and is given by the sum of inflation

Figure 4. Inflation Forward Rates



expectations and inflation risk premia. As estimates of longer-term inflation expectations are relatively stable, movements in the five- and ten-year inflation forward rates tend to be driven by changes in estimated risk premia. The inflation forward rate, as shown in figure 4, generally falls during the first third of the sample, rises around the time of the GST, and rises between 2004 and 2008 before falling sharply with the onset of the financial crisis and then rising again.[13]

One notable feature of figure 4 is the negative inflation forward rates recorded in late 2008. This phenomenon is essentially due to very low break-even inflation rates embodied in the bond price data (two-year-ahead nominal less real yields were only around 90 basis points at this time), together with high realized inflation over 2008; as break-even inflation rates reflect around five months of historical inflation, a low two-year break-even inflation rate and high historical inflation necessarily implies a very low or even negative inflation

[13]Note that studies using U.S. and UK data essentially start with the inflation forward rate, which they decompose into inflation expectations and inflation risk premia. Due to a lack of data, we cannot do this, and instead we estimate inflation forward rates as part of our model.

forward rate in the near future. The low break-even inflation rates in turn are due to the yields on inflation-indexed bonds rising relative to the yields on nominal bonds.

5. Discussion and Conclusion

The model just described is designed to give policymakers accurate and timely information on market-implied inflation expectations. It has a number of advantages over existing sources for such data, which primarily constitute either break-even inflation derived from bond prices or inflation forecasts sourced from market economists.

As argued, break-even inflation as derived directly from bond prices has a number of drawbacks as a measure of inflation expectations: such a measure gives average inflation over the tenor of the bond, not inflation at a certain date in the future; government bonds in Australia are coupon bearing, which means that yields of similar-maturity nominal and inflation-indexed bonds are not strictly comparable; there are very few inflation-indexed bonds on issue in Australia, which means that break-even inflation can only be calculated at a limited number of tenors; inflation-indexed bonds are indexed with a lag, which means that their yields reflect historical inflation, not just future expected inflation; and finally, bond yields incorporate risk premia so that the level of, and even changes in, break-even inflation need not give an accurate read on inflation expectations. Our model addresses each of these issues: we model inflation-indexed bonds as consisting of a stream of payments where the value of each payment is determined by nominal interest rates, historical inflation, future inflation expectations, and inflation risk premia. This means we are able to produce estimates of expected future inflation at any time and for any tenor which are free of risk premia and are not affected by historical inflation.

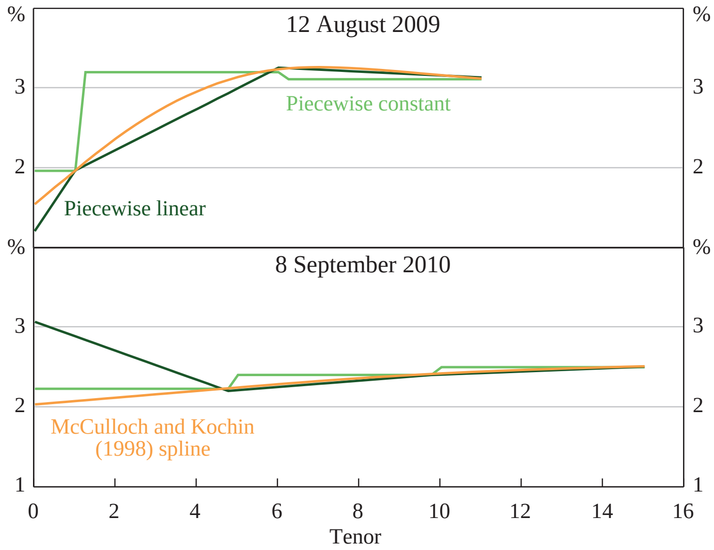
Model-derived inflation expectations also have a number of advantages over expectations from market economists: unlike survey-based expectations, they are again available at any time and for any tenor, and they reflect the agglomerated knowledge of all market participants, not just the views of a small number of economists. By contrast, the main drawback of our model is its complexity—break-even inflation and inflation forecasts have their faults but are transparent and simple to measure, whereas our model, while addressing a number of faults, is by comparison complex and difficult to estimate.

Standard affine term structure models, which take as inputs zero-coupon yield curves and give as outputs expectations and risk premia, have existed in the literature for some time. Our main contribution to this literature, apart from the estimation of inflation expectations and inflation risk premia for Australia, is our reformulation of the model in terms of coupon-bearing bond prices instead of zero-coupon yields. In practice, zero-coupon yields are not directly available but must be estimated, so by fitting the affine term structure model directly to prices, we avoid inserting a second arbitrary yield-curve model between the data and our final model. When many bond prices are available, this is only a small advantage, as accurate zero-coupon yields can be recovered from the well-specified coupon-bearing yield curve. When only a small number of bond prices are available, our method provides a major advantage: one can fit a zero-coupon yield curve to only two or three far-spaced coupon-bearing yields, and indeed McCulloch and Kochin (1998) provide a procedure for doing this, but there are limitless such curves that can be fitted with no a priori correct criteria to choose between them.

The inability to pin down the yield curve is highlighted in figure 5, which shows three yield curves—one piecewise constant, one piecewise linear and starting from the current six-month annualized inflation rate, and one following the method of McCulloch and Kochin (1998)—all fitted to inflation-indexed bond yields on two different dates. All curves fit the bond data perfectly, as would any number of other curves, so there is nothing in the underlying data to motivate a particular choice, yet different curves can differ by as much as 1 percentage point. Our technique provides a method for removing this intermediate curve-fitting step and estimating directly with the underlying data instead of the output of an arbitrary yield-curve model. The fact that we price bonds directly in terms of the underlying inflation process also allows for direct modeling of the lag involved in inflation indexation and the impact that historically observed inflation has on current yields, a second major advantage.

In sum, the affine term structure model used in this paper addresses a number of problems inherent in alternative approaches to measuring inflation expectations, and produces plausible measures of inflation expectations over the inflation targeting era. Given the complexity of the model and the limited number of inflation-indexed

Figure 5. Zero-Coupon Real Yield Curves



bonds on issue, some caution should be applied in interpreting the results. A key finding of the model is that long-term inflation expectations appear to have been well anchored within the inflation target over most of the sample. Conversely, one-year-ahead inflation expectations appear to be closely tied to CPI inflation and are more variable than longer-term expectations. Given the relative stability of our estimates of long-term inflation expectations, changes in five- and ten-year inflation forward rates, and so in break-even inflation rates, are by implication driven by changes in inflation risk premia. As such, our measure has some benefits over break-even inflation rates in measuring inflation expectations.

Appendix 1. Yields and Stochastic Discount Factors

The results of this paper revolve around the idea that inflation expectations are an important determinant of the inflation yield. In this section we make clear the relationships between real, nominal, and inflation yields; inflation expectations; and inflation risk premia. We also link these quantities to standard asset pricing models as discussed, for example, in Cochrane (2005).

Real and Nominal Yields and SDFs

Let M^r_t be the real SDF or pricing kernel, defined such that

$$P_{t,\tau} = \mathbb{E}_t\left(\frac{M^r_{t+\tau}}{M^r_t} x_{t+\tau}\right) \tag{8}$$

holds for any asset, where $P_{t,\tau}$ is the price of the asset at time t which has (a possibly random) payoff $x_{t+\tau}$ occurring at time $t+\tau$. A zero-coupon inflation-indexed bond maturing at time $t+\tau$, $P^r_{t,\tau}$, is an asset that pays one real dollar, or equivalently one unit of consumption, for certain. That is, it is an asset with payoff $x_{t+\tau} \equiv 1$. If we define the real yield by $y^r_{t,\tau} = -\log(P^r_{t,\tau})$, we can use equation (8) with $x_{t+\tau} = 1$ to write

$$y^r_{t,\tau} = -\log\left(P^r_{t,\tau}\right) = -\log\left(\mathbb{E}_t\left(\frac{M^r_{t+\tau}}{M^r_t}\right)\right). \tag{9}$$

This defines the relationship between real yields and the continuous-time real SDF.

A zero-coupon nominal bond maturing at time $t+\tau$ is an asset that pays one nominal dollar for certain. If we define Q_t to be the price index, then the payoff of this bond is given by $x_{t+\tau} = Q_t/Q_{t+\tau}$ units of consumption. Taking $x_{t+\tau} = Q_t/Q_{t+\tau}$ in equation (8), we can relate the nominal yield $y^n_{t,\tau}$ to the nominal bond price $P^n_{t,\tau}$ and the continuous-time real SDF by

$$y^n_{t,\tau} = -\log\left(P^n_{t,\tau}\right) = -\log\left(\mathbb{E}_t\left(\frac{M^r_{t+\tau}}{M^r_t}\frac{Q_t}{Q_{t+\tau}}\right)\right).$$

Motivated by this result, we define the continuous-time *nominal* SDF by $M^n_{t+\tau} = M^r_{t+\tau}/Q_{t+\tau}$, so that

$$y^n_{t,\tau} = -\log\left(P^n_{t,\tau}\right) = -\log\left(\mathbb{E}_t\left(\frac{M^n_{t+\tau}}{M^n_t}\right)\right). \tag{10}$$

Inflation Yields and the Inflation SDF

The inflation yield is defined to be the difference in yields between zero-coupon nominal and inflation-indexed bonds of the same maturity,

$$y^i_{t,\tau} = y^n_{t,\tau} - y^r_{t,\tau}. \tag{11}$$

As in Beechey (2008), we define the continuous-time *inflation* SDF, $M^i_{t+\tau}$, such that the pricing equation for inflation yields holds—that is, such that

$$y^i_{t,\tau} = -\log\left(\mathbb{E}_t\left(\frac{M^i_{t+\tau}}{M^i_t}\right)\right). \tag{12}$$

All formulations of $M^i_{t+\tau}$ which ensure that equations (9), (10), and (11) are consistent with equation (12) are equivalent from the perspective of our model. One such formulation is to define the inflation SDF as

$$M^i_{t+\tau} = \frac{M^n_{t+\tau}}{\mathbb{E}_t(M^r_{t+\tau})}. \tag{13}$$

We can then obtain equation (12) by substituting equations (9) and (10) into equation (11) and using the definition of the inflation SDF given in equation (13). In this case we have

$$\begin{aligned} y^i_{t,\tau} = y^n_{t,\tau} - y^r_{t,\tau} &= -\log\left(\mathbb{E}_t\left(\frac{M^n_{t+\tau}}{M^n_t}\right)\right) + \log\left(\mathbb{E}_t\left(\frac{M^r_{t+\tau}}{M^r_t}\right)\right) \\ &= -\log\left(\frac{M^r_t}{M^n_t}\mathbb{E}_t\left(\frac{M^n_{t+\tau}}{\mathbb{E}_t\left(M^r_{t+\tau}\right)}\right)\right) = -\log\left(\mathbb{E}_t\left(\frac{M^i_{t+\tau}}{M^i_t}\right)\right), \end{aligned}$$

as desired. If one assumed that $M^r_{t+\tau}$ and $Q_{t+\tau}$ were uncorrelated, a simpler formulation would be to take $M^i_{t+\tau} = 1/Q_{t+\tau}$. Since $M^n_{t+\tau} = M^r_{t+\tau}/Q_{t+\tau}$, in this case we would have $\mathbb{E}_t(M^n_{t+\tau}/M^n_t) = \mathbb{E}_t(M^r_{t+\tau}/M^r_t)\mathbb{E}_t(Q_t/Q_{t+\tau})$, so that $y^n_{t,\tau} = -\log(\mathbb{E}_t(M^r_{t+\tau}/M^r_t)) - \log(\mathbb{E}_t(Q_t/Q_{t+\tau}))$ and $y^i_{t,\tau} = y^n_{t,\tau} - y^r_{t,\tau} = -\log(\mathbb{E}_t(Q_t/Q_{t+\tau})) = -\log(\mathbb{E}_t(M^i_{t+\tau}/M^i_t))$, as desired.

Interpretation of Other SDFs in Our Model

We model M^i_t directly as $\mathrm{d}M^i_t/M^i_t = -\pi^i_t\mathrm{d}t - \boldsymbol{\lambda}^{i\prime}_t\mathrm{d}\mathbf{B}_t$, where we take π^i_t as the instantaneous inflation rate and $\boldsymbol{\lambda}^i_t$ as the market price of inflation risk. Although very flexible, this setup means that in our model the relationship between different stochastic discount factors in the economy is not fixed.

In models such as ours there are essentially three quantities of interest, any two of which determine the other: the real SDF, the nominal SDF, and the inflation SDF. As we make assumptions about only one of these quantities, we do not tie down the model completely. Note that we *could* make an additional assumption to tie down the model. Such an assumption would not affect the model-implied inflation yields or inflation forecasts, however, which are the only data our model sees and so, in the context of our model, would be arbitrary.

This situation of model ambiguity is not confined to models of inflation compensation such as ours. The extensive literature which fits affine term structure models to nominal yields contains a similar kind of ambiguity. Such models typically take the nominal SDF as driven by $\mathrm{d}M_t^n/M_t^n = -r_t^n \mathrm{d}t - \boldsymbol{\lambda}_t^{n\prime} \mathrm{d}\mathbf{B}_t$, where once again the real SDF and inflation process are not explicitly modeled, so that, similar to our case, the model is not completely tied down.

Inflation Expectations and the Inflation Risk Premium

Finally, we link our inflation yield to inflation expectations and the inflation risk premium. The inflation risk premium arises because people who hold nominal bonds are exposed to inflation, which is uncertain, and so demand compensation for bearing this risk. If we set $m_{t,\tau} = \log(M_{t+\tau}^r/M_t^r)$ and $q_{t,\tau} = \log(Q_{t+\tau}/Q_t)$, which are both assumed normal, and use the identity $\mathbb{E}_t(\exp(X)) = \exp(\mathbb{E}_t(X) + \frac{1}{2}\mathbb{V}_t(X))$ where X is normally distributed and $\mathbb{V}(\cdot)$ is variance, we can work from equation (11) to derive

$$y_{t,\tau}^i = \mathbb{E}_t(q_{t,\tau}) - \frac{1}{2}\mathbb{V}_t(q_{t,\tau}) + \mathbb{C}\mathrm{ov}_t(m_{t,\tau}, q_{t,\tau}).$$

The first term above is the expectations component of the inflation yield, while the last two terms constitute the inflation risk premium (incorporating a “Jensen’s” or “convexity” term).

Appendix 2. The Mathematics of Our Model

We first give some general results regarding affine term structure models, then relate these results to our specific model and its interpretation.

Some Results Regarding Affine Term Structure Models

Start with the latent factor process,

$$\mathrm{d}\mathbf{x}_t = K(\boldsymbol{\mu} - \mathbf{x}_t)\mathrm{d}t + \Sigma\,\mathrm{d}\mathbf{B}_t.$$

Given $\mathbf{x}_t$, we have, for $s > t$ (see, for example, Duffie 2001, p. 342),

$$\begin{aligned}\mathbf{x}_s &= e^{-K(s-t)}\left(\mathbf{x}_t + \int_t^s e^{K(u-t)}K\boldsymbol{\mu}\,\mathrm{d}u + \int_t^s e^{K(u-t)}\Sigma\,\mathrm{d}B_u\right)\\ &\overset{D}{=} e^{-K(s-t)}\mathbf{x}_t + (I - e^{-K(s-t)})\boldsymbol{\mu} + \boldsymbol{\epsilon}_{t,s},\end{aligned} \tag{14}$$

where '$\overset{D}{=}$' denotes equality in distribution and $\boldsymbol{\epsilon}_{t,s} \sim N(\mathbf{0}, \Omega_{s-t})$ with

$$\begin{aligned}\Omega_{s-t} &= e^{-K(s-t)}\left(\int_t^s e^{K(u-t)}\Sigma\Sigma' e^{K'(u-t)}\,\mathrm{d}u\right)e^{-K'(s-t)}\\ &= \int_0^{s-t} e^{-Ku}\Sigma\Sigma' e^{-K'u}\,\mathrm{d}u.\end{aligned}$$

Further, if we define

$$\pi_t = \rho_0 + \boldsymbol{\rho}'\mathbf{x}_t,$$

then since $\int_t^{t+\tau}\pi_s\,\mathrm{d}s$ is normally distributed,

$$\begin{aligned}&\mathbb{E}_t\left(\exp\left(-\int_t^{t+\tau}\pi_s\,\mathrm{d}s\right)\right)\\ &\quad= \exp\left(-\mathbb{E}_t\left(\int_t^{t+\tau}\pi_s\,\mathrm{d}s\right) + \frac{1}{2}\mathbb{V}_t\left(\int_t^{t+\tau}\pi_s\,\mathrm{d}s\right)\right)\end{aligned}$$

with

$$\begin{aligned}\int_t^{t+\tau}\pi_s\,\mathrm{d}s &= \int_t^{t+\tau}\rho_0 + \boldsymbol{\rho}'\mathbf{x}_s\,\mathrm{d}s\\ &= \int_t^{t+\tau}\rho_0 + \boldsymbol{\rho}'\left(e^{-K(s-t)}\mathbf{x}_t + (I - e^{-K(s-t)})\boldsymbol{\mu}\right.\\ &\qquad\left. + e^{-K(s-t)}\int_t^s e^{K(u-t)}\Sigma\mathrm{d}B_u\right)\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&= \int_t^{t+\tau} \rho_0 + \boldsymbol{\rho}'(e^{-K(s-t)}\mathbf{x}_t + (I - e^{-K(s-t)})\boldsymbol{\mu})\mathrm{d}s \\
&\quad + \int_t^{t+\tau} \boldsymbol{\rho}' \left(\int_u^{t+\tau} e^{-K(s-t)}\mathrm{d}s \right) e^{K(u-t)}\Sigma \, \mathrm{d}B_u, \qquad (15)
\end{aligned}
$$

where we have used a stochastic version of Fubini's theorem to change the order of integration (see, for example, Da Prato and Zabczyk 1993, p. 109). Evaluating the inner integral of line (15), using Itô's isometry (see, for example, Steele 2001, p. 82) and making the change of variable $s = t + \tau - u$, we have

$$
\begin{aligned}
\mathbb{E}_t \left(\int_t^{t+\tau} \pi_s \, \mathrm{d}s \right) &= \int_0^{\tau} \rho_0 + \boldsymbol{\rho}'(e^{-Ks}\mathbf{x}_t + (I - e^{-Ks})\boldsymbol{\mu})\mathrm{d}s \\
\mathbb{V}_t \left(\int_t^{t+\tau} \pi_s \, \mathrm{d}s \right) &= \int_0^{\tau} (\boldsymbol{\rho}'(I - e^{-Ks})K^{-1}\Sigma)^2 \, \mathrm{d}s,
\end{aligned}
$$

where for $\mathbf{x}$, a vector we define $\mathbf{x}^2 = \mathbf{x}'^2$ as the vector dot-product $\mathbf{x}'\mathbf{x}$. Hence

$$
\begin{aligned}
\mathbb{E}_t \left(\exp \left(- \int_t^{t+\tau} \pi_s \, \mathrm{d}s \right) \right) &= \exp \left(- \int_0^{\tau} \boldsymbol{\rho}' e^{-Ks}\mathbf{x}_t \, \mathrm{d}s \right. \\
&\quad \left. - \int_0^{\tau} \rho_0 + \boldsymbol{\rho}'(I - e^{-Ks})\boldsymbol{\mu} - \frac{1}{2}(\boldsymbol{\rho}'(I - e^{-Ks})K^{-1}\Sigma)^2 \mathrm{d}s \right).
\end{aligned}
$$

Now for $M'_{1,\tau} = (I - e^{-K\tau})K^{-1}$ we have

$$
\int_0^{\tau} \boldsymbol{\rho}' e^{-Ks}\mathbf{x}_t \mathrm{d}s = \boldsymbol{\rho}'(I - e^{-Kt})K^{-1}\mathbf{x}_t = \boldsymbol{\rho}' M'_{1,\tau}\mathbf{x}_t,
$$

while

$$
\begin{aligned}
\int_0^{\tau} \boldsymbol{\rho}'(I - e^{-Ks})\boldsymbol{\mu} \, \mathrm{d}s &= \boldsymbol{\rho}'(\tau I + e^{-K\tau}K^{-1} - K^{-1})\boldsymbol{\mu} \\
&= \boldsymbol{\rho}'(\tau I - M'_{1,\tau})\boldsymbol{\mu}
\end{aligned}
$$

and

$$\begin{aligned}
&\int_0^\tau -\frac{1}{2}(\boldsymbol{\rho}'(I-e^{-Ks})K^{-1}\Sigma)^2 \mathrm{d}s \\
&\quad = -\frac{1}{2}\boldsymbol{\rho}'K^{-1}\left(\int_0^\tau (I-e^{-Ks})\Sigma\Sigma'(I-e^{-K's})\mathrm{d}s\right)K^{-1'}\boldsymbol{\rho} \\
&\quad = -\frac{1}{2}\boldsymbol{\rho}'K^{-1}(\tau\Sigma\Sigma' - \Sigma\Sigma' M_{1,\tau} - M'_{1,\tau}\Sigma\Sigma' + M_{2,\tau})K^{-1'}\boldsymbol{\rho},
\end{aligned}$$

where from Kim and Orphanides (2005), for example,

$$\begin{aligned}
M_{2,\tau} &= \int_0^\tau e^{-Ks}\Sigma\Sigma' e^{-K's}\,\mathrm{d}s \\
&= -vec^{-1}(((K\otimes I)+(I\otimes K))^{-1}vec(e^{-K\tau}\Sigma\Sigma' e^{-K^{'}\tau} - \Sigma\Sigma')).
\end{aligned}$$

Putting this together, we have

$$\mathbb{E}_t\left(\exp\left(-\int_t^{t+\tau}\pi_s\,\mathrm{d}s\right)\right) = \exp(-\alpha_\tau - \boldsymbol{\beta}'_\tau \mathbf{x}_t) \tag{16}$$

with

$$\begin{aligned}
\alpha_\tau &= \tau\rho_0 + \boldsymbol{\rho}'(\tau I - M'_{1,\tau})\boldsymbol{\mu} \\
&\qquad - \frac{1}{2}\boldsymbol{\rho}'K^{-1}(\tau\Sigma\Sigma' - \Sigma\Sigma' M_{1,\tau} - M'_{1,\tau}\Sigma\Sigma' + M_{2,\tau})K^{-1'}\boldsymbol{\rho} \qquad (17) \\
\boldsymbol{\beta}_\tau &= M_{1,\tau}\boldsymbol{\rho}. \qquad (18)
\end{aligned}$$

Equivalent formulas are available in Kim and Orphanides (2005).

Bond Price Formula

If we model the SDF according to

$$\begin{aligned}
\mathrm{d}M_t/M_t &= -\pi_t \mathrm{d}t - \boldsymbol{\lambda}'_t\,\mathrm{d}\mathbf{B}_t \\
\pi_t &= \rho_0 + \boldsymbol{\rho}'\mathbf{x}_t,\ \ \boldsymbol{\lambda}_t = \boldsymbol{\lambda}_0 + \Lambda\mathbf{x}_t \\
\mathrm{d}\mathbf{x}_t &= K(\boldsymbol{\mu} - \mathbf{x}_t)\mathrm{d}t + \Sigma\,\mathrm{d}\mathbf{B}_t,
\end{aligned} \tag{19}$$

then the price of a zero-coupon bond at t paying one dollar at $t+\tau$ is given by (see, for example, Cochrane 2005)

$$\mathbb{E}_t\left(\frac{M_{t+\tau}}{M_t}\right) = \mathbb{E}_t\left(\exp\left(-\int_t^{t+\tau}\pi_t + \frac{1}{2}\boldsymbol{\lambda}_t'\boldsymbol{\lambda}_t\,\mathrm{d}t - \int_t^{t+\tau}\boldsymbol{\lambda}_t'\mathrm{d}\mathbf{B}_t\right)\right)$$
$$= \mathbb{E}_t\left(\exp\left(-\int_t^{t+\tau}\pi_s^*\,\mathrm{d}s\right)\right), \tag{20}$$

where π_s^* is like π_s in equation (19) above but with

$$\mathrm{d}\mathbf{x}_t = K^*(\boldsymbol{\mu}^* - \mathbf{x}_t)\mathrm{d}t + \Sigma\,\mathrm{d}\mathbf{B}_t,$$

where $K^* = (K + \Sigma\Lambda)$ and $\boldsymbol{\mu}^* = {K^*}^{-1}(K\boldsymbol{\mu} - \Sigma\boldsymbol{\lambda}_0)$. (Here π_s^* is the "risk-neutral" version of π_s.) Hence we can price bonds via equation (16) using K^* and $\boldsymbol{\mu}^*$ in place of K and $\boldsymbol{\mu}$ in equations (17) and (18). We can write equation (20) as

$$\exp(-\alpha_\tau^* - \boldsymbol{\beta}_\tau^{*\prime}\mathbf{x}_t) = \mathbb{E}_t\left(\exp\left(-\int_t^{t+\tau}\pi_s^*\,\mathrm{d}s\right)\right).$$

In terms of the inflation yield from equation (12), this can be written as $y_{t,\tau}^i = \alpha_\tau^* + \boldsymbol{\beta}_\tau^{*\prime}\mathbf{x}_t$.

Inflation Forecast Formula

Inflation expectations are reported in terms of percentage growth in the consumer price index, *not* average inflation (the two differ by a Jensen's inequality term). As such, expectations at time t of how the CPI will grow between time $s > t$ and time $s + \tau$ in the future correspond to a term of the form

$$\mathbb{E}_t\left(\exp\left(\int_s^{s+\tau}\pi_u\,\mathrm{d}u\right)\right) = \mathbb{E}_t\left(\mathbb{E}_s\left(\exp\left(-\int_s^{s+\tau}-\pi_u\,\mathrm{d}u\right)\right)\right)$$
$$= \mathbb{E}_t(\exp(-\bar{\alpha}_\tau - \bar{\boldsymbol{\beta}}_\tau'\mathbf{x}_s))$$
$$= \exp\left(-\bar{\alpha}_\tau - \bar{\boldsymbol{\beta}}_\tau'(e^{-K(s-t)}\mathbf{x}_t + (I - e^{-K(s-t)})\boldsymbol{\mu}) + \frac{1}{2}\bar{\boldsymbol{\beta}}_\tau'\Omega_{s-t}\bar{\boldsymbol{\beta}}_\tau\right),$$

where the last line follows since $\mathbf{x}_s|\mathbf{x}_t \sim N(e^{-K(s-t)}\mathbf{x}_t + (I - e^{-K(s-t)})\boldsymbol{\mu}, \Omega_{s-t})$. Here $\bar{\alpha}_\tau$ and $\bar{\boldsymbol{\beta}}_\tau$ are equivalent to α_τ and $\boldsymbol{\beta}_\tau$ from equations (17) and (18) but with the market price or risk $\boldsymbol{\lambda}_t$ set to zero and using $-\rho_0$ and $-\boldsymbol{\rho}$ in place of ρ_0 and $\boldsymbol{\rho}$. So if the CPI is expected to grow by 3 percent between s and $s+\tau$, for example, we would have

$$\tau \log(1+3\%) = -\bar{a}_\tau - \bar{\boldsymbol{\beta}}'_\tau(e^{-K(s-t)}\mathbf{x}_t + (I - e^{-K(s-t)})\boldsymbol{\mu}) + \frac{1}{2}\bar{\boldsymbol{\beta}}'_\tau \Omega_{s-t} \bar{\boldsymbol{\beta}}_\tau.$$

Appendix 3. Central Difference Kalman Filter

The central difference Kalman filter is a type of sigma-point filter. Sigma-point filters deal with non-linearities in the following manner:

- First, a set of points *around* the forecast of the state is generated. The distribution of these points depends on the variance of the forecast of the state.
- The measurement equations (functions $H1(\mathbf{x}_t)$ and $H2(\mathbf{x}_t)$) are used to calculate a set of forecast observation points. This set of points is used to estimate a mean and variance of the data forecasts.
- The mean and variance of the data forecasts are then used to update the estimates of the state and its variance.

The algorithm we use is that of an additive-noise central difference Kalman filter, the details of which are given below. For more details on sigma-point Kalman filters, see van der Merwe (2004).

- *Step 1.* Initialize the state vector and its covariance matrix to their unconditional expected values,

$$\hat{\mathbf{x}}_0 = [0, 0, 0]'$$
$$P_{\mathbf{x}_0} = \Omega_\infty.$$

- *Step 2.* Loop over $k = 1 : n$, where n is the length of our data set.
- *Step 2a.* Time-update equations:

$$\hat{\mathbf{x}}_k^- = e^{-\mathbf{K}d_k}\hat{\mathbf{x}}_{k-1}$$
$$\mathbf{P}_{\mathbf{x}_k}^- = e^{-\mathbf{K}d_k}\mathbf{P}_{\mathbf{x}_{k-1}}e^{-\mathbf{K}'d_k} + \Omega_{d_k},$$

where d_k is the time in years between data point k and data point $k-1$.

- *Step 2b.* Create the sigma points,

$$
\begin{aligned}
X_k^0 &= \hat{\mathbf{x}}_k^- \\
X_k^i &= \hat{\mathbf{x}}_k^- + \left(h\sqrt{\mathbf{P}_{\mathbf{x}_k}^-}\right)_i \qquad i = 1, \dots, L \\
X_k^i &= \hat{\mathbf{x}}_k^- - \left(h\sqrt{\mathbf{P}_{\mathbf{x}_k}^-}\right)_i \qquad i = L+1, \dots, 2L,
\end{aligned}
$$

where $(\sqrt{P_{\mathbf{x}_k}^-})_i$ is the ith column of the matrix square root of $P_{\mathbf{x}_k}^-$, L is the number of latent factors, and h is the central difference step size, which is set to $\sqrt{3}$.

- *Step 2c.* Propagate the sigma points through the pricing functions $H1(\cdot)$ and $H2(\cdot)$. Let m_k be the number of observed inflation-indexed bond prices in period k. Let n_k be the number of observed inflation forecasts in period k. For each observed price $j = 1, \dots, m_k$ we propagate each sigma point X_k^i, $i = 0, \dots, 2L$ through the pricing function for bond j in period k, $H1_{k,j}(\cdot)$. For each observed forecast $j = m_k + 1, \dots, m_k + n_k$ we propagate each sigma point X_k^i, $i = 0, \dots, 2L$ through the pricing function for forecast j in period k, $H2_{k,j}(\cdot)$. Denote the output by φ_k, which is a matrix of dimension $n_k + m_k$ by $2L + 1$ with elements

$$
(\varphi_k)_{j,i} = \begin{cases} H1_{k,j}(X_i) & i = 0, \dots, 2L, \quad j = 1, \dots, m_k \\ H2_{k,j}(X_i) & i = 0, \dots, 2L, \quad j = m_k{+}1, \dots, m_k{+}n_k. \end{cases}
$$

Denote the ith column of φ_k by $\boldsymbol{\varphi}_k^i$.

- *Step 2d.* Observation-update equations. For weightings of

$$
\begin{aligned}
w_0^{(m)} &= \frac{h^2 - L}{h^2} \quad w_i^{(m)} = \frac{1}{2h^2} \qquad \forall i \geq 1 \\
w_i^{(c_1)} &= \frac{1}{4h^2} \quad w_i^{(c_2)} = \frac{h^2 - 1}{4h^4} \qquad \forall i \geq 1,
\end{aligned}
$$

the estimate of the price vector is given by a weighted average of the $\boldsymbol{\varphi}_k^i$s,

$$\hat{\mathbf{y}}_k = \sum_{i=0}^{2L} w_i^{(m)} \boldsymbol{\varphi}_k^i,$$

and the estimated covariance matrix of $\hat{\mathbf{y}}_k$ is given by

$$P_{\mathbf{y}_k} = \sum_{i=1}^{L} \left[w_i^{(c_1)} \left(\boldsymbol{\varphi}_k^i - \boldsymbol{\varphi}_k^{L+i} \right)^{[2]} + w_i^{(c_2)} \left(\boldsymbol{\varphi}_k^i + \boldsymbol{\varphi}_k^{L+i} - 2\boldsymbol{\varphi}_k^0 \right)^{[2]} \right] + R_k,$$

where R_k is the covariance matrix of the noise present in the observed prices. Here $(\cdot)^{[2]}$ denotes the vector outer product.

Next the estimate of the covariance between the state estimate and the price estimate is given by

$$P_{\mathbf{x}_k \mathbf{y}_k} = \sqrt{w_1^{(c1)} P_{\mathbf{x}_k}^-} \left[\boldsymbol{\varphi}_k^{1:L} - \boldsymbol{\varphi}_k^{L+1:2L} \right]^T.$$

- *Step 2e.* Calculate the Kalman gain matrix G_k,

$$G_k = P_{\mathbf{x}_k \mathbf{y}_k} P_{\mathbf{y}_k}^{-1}.$$

- *Step 2f.* Update the state estimates,

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + G_k(\mathbf{y}_k - \hat{\mathbf{y}}_k)$$
$$P_{\mathbf{x}_k} = P_{\mathbf{x}_k}^- - G_k P_{\mathbf{y}_k} G_k^T,$$

where $\mathbf{y}_k$ is the vector of observed prices.

References

Australian Financial Markets Association. 2008. *Australian Financial Markets Report.*

Beechey, M. 2008. "Lowering the Anchor: How the Bank of England's Inflation-Targeting Policies Have Shaped Inflation Expectations and Perceptions of Inflation Risk." FEDS Working Paper No. 2008-44, Board of Governors of the Federal Reserve System.

Cochrane, J. 2005. *Asset Pricing.* Princeton, NJ: Princeton University Press.

D'Amico, S., D. Kim, and M. Wei. 2008. "Tips from TIPS: The Informational Content of Treasury Inflation-Protected Security Prices." FEDS Working Paper No. 2008-30, Board of Governors of the Federal Reserve System.

Da Prato, G., and J. Zabczyk. 1993. *Stochastic Equations in Infinite Dimensions.* Number 45 in *Encyclopaedia of Mathematics and Its Applications.* Cambridge, UK: Cambridge University Press.

Duffie, D. 2001. *Dynamic Asset Pricing Theory.* Princeton, NJ: Princeton University Press.

Durham, J. 2006. "An Estimate of the Inflation Risk Premium Using a Three-Factor Affine Term Structure Model." FEDS Working Paper No. 2006-42, Board of Governors of the Federal Reserve System.

Finlay, R., and M. Chambers. 2009. "A Term Structure Decomposition of the Australian Yield Curve." *Economic Record* 85 (271): 383–400.

Hördahl, P. 2008. "The Inflation Risk Premium in the Term Structure of Interest Rates." *BIS Quarterly Review* (September): 23–38.

Joyce, M., P. Lildholdt, and S. Sorensen. 2010. "Extracting Inflation Expectations and Inflation Risk Premia from the Term Structure: A Joint Model of the UK Nominal and Real Yield Curves." *Journal of Banking and Finance* 34 (2): 281–94.

Kim, D., and A. Orphanides. 2005. "Term Structure Estimation with Survey Data on Interest Rate Forecasts." FEDS Working Paper No. 2005-48, Board of Governors of the Federal Reserve System.

Kim, D., and J. Wright. 2005. "An Arbitrage-Free Three-Factor Term Structure Model and the Recent Behavior of Long-Term Yields and Distant-Horizon Forward Rates." FEDS Working Paper No. 2005-33, Board of Governors of the Federal Reserve System.

McCulloch, J., and L. Kochin. 1998. "The Inflation Premium Implicit in the U.S. Real and Nominal Term Structures of Interest Rates." Ohio State University Working Paper No. 98-12.

Steele, J. 2001. *Stochastic Calculus and Financial Applications.* New York: Springer.

van der Merwe, R. 2004. *Sigma-Point Kalman Filters for Probabilistic Inference in Dynamic State-Space Models.* Dissertation, Oregon Health and Science University.