Inflation-Forecast-Based Rules and Indeterminacy: A Puzzle and a Resolution*

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We examine an interesting puzzle in monetary economics between what monetary authorities claim (namely, to be forward looking and preemptive) and the poor stabilization properties routinely reported for forecast-based rules. Our resolution is that central banks should be viewed as following "Calvo-type" inflation-forecast-based (IFB) interest rate rules that depend on a discounted sum of current and future rates of inflation. Such rules might be regarded as both within the legal frameworks and potentially mimicking central bankers' practice. We find that Calvo-type IFB interest rate rules are, first, less prone to indeterminacy than standard rules with a finite forward horizon. Second, in difference form, the indeterminacy problem disappears altogether. Third, optimized forms have good stabilization properties as they become more forward looking, a property that sharply contrasts that of standard IFB rules. Fourth, they appear data coherent when incorporated into a well-known estimated dynamic stochastic general equilibrium (DSGE) model of the euro area.

JEL Codes: E52, E37, E58.

*We acknowledge financial support for this research from the Economic and Social Research Council (ESRC), project no. RES-000-23-1126, and from the European Central Bank's Research Visitors Programme for Paul Levine. We thank Gregory Dewalque, Martin Ellison, Keith Kuester, Athanasios Orphanides, Ole Rummel, Frank Smets, Raf Wouters, Peter Welz, two anonymous referees, the editor Michael Woodford, and seminar participants at the University of Surrey and at the 2006 Complexity Conference, Aix en Provence, for helpful comments and suggestions. Views expressed in this paper do not necessarily reflect those of the European Central Bank.

1. Introduction

All modern central banks stress the importance of forward-looking policy, and many stress the notion that interest rates should be based on future inflation expectations. Well-known examples include the central banks of Canada and New Zealand, but this is also true by implication of the practice of other central banks. For example, the monetary policy strategy of the European Central Bank (ECB) states that "price stability is to be maintained over the medium term" (ECB 1999, 47), which precisely suggests a forward-looking non-inflationary strategy. The basis for inflation-forecast-based (IFB) rules is that, by anchoring expectations, they improve the credibility and transparency of monetary policy as well as allow policy to be preemptive.

However, such rules have been criticized on various fronts. One concerns the result that typical forward-looking monetary policy rules (such as Taylor-type rules) tend to lead to real indeterminacy (Woodford 2003, chap. 4). This implies that when a shock displaces the economy from its equilibrium, there are an infinite number of possible paths for the real variables leading back to equilibrium. Such "sunspot equilibria" are of interest because sunspot fluctuations—i.e., persistent movements in inflation and output that materialize even in the absence of shocks to preferences or technology—are typically welfare reducing and potentially quite large. Whether policy rules lead to real indeterminacy depends on whether feedback parameters in the policy rules are insufficiently, or indeed overly, aggressive as well as on the length of the forecast horizon itself (e.g., see Levin, Wieland, and Williams 2003, Batini et al. 2006, and the references therein).

Consequently, there would appear to an interesting puzzle in monetary economics between what policymakers routinely claim (namely, to be forward looking and preemptive) and the poor stabilization properties reported for forecast-based rules. Given the importance of aligning central banks' communication strategies with the modeling frameworks underlying much of their technical analysis, this puzzle should not only be taken seriously but indeed would appear to require an urgent resolution. The purpose of this paper is to suggest such a resolution. We propose viewing central banks as following "Calvo-type" IFB interest rate rules that depend on a discounted sum of current and all future rates of inflation. Such rules, it turns out, are less prone to indeterminacy than standard ones with a finite forward horizon, and, if formulated in difference form, the indeterminacy problem disappears altogether. Indeed, we show that optimized Calvo-type rules have good stabilization properties as they become more forward looking, a property that sharply contrasts that of standard IFB rules. Finally, when taken to the data, they appear to behave at least as well as and sometimes better than more standard monetary-policy reaction functions.

Abstracting from such technical characteristics, moreover, the Calvo-type rules we examine might also be regarded as both within the legal framework of and potentially mimicking central bankers' practice. To illustrate, consider the following tension related to the literal use of forecast-based rules. On the one hand, we know that authorities frequently tailor policy and communication strategies to forward-looking outcomes. On the other, we further know that forward-looking policy rules are susceptible to indeterminacy. Moreover, central banks themselves generate expectations for future-dated outcomes but do so in a chronically uncertain environment. Accordingly, we might conjecture that while policymakers will want to incorporate forecasts into their decision strategies, they may be reluctant to treat them commensurate with realized outcomes.

This "chronically uncertain environment" faced by policymakers takes many forms. Consider a few examples. First, macroeconomic time series tend to be actively revised in the quarters following their publication; thus, rule-based policy prescriptions derived from realized data may depart significantly from their real-time counterparts (e.g., Orphanides 2001). It goes without saying that forecasting in such a "noisy" data environment complicates the policy process considerably; authorities might then take recourse to contemporaneous or backward-looking rules, or else persevere with strategies that explicitly incorporate but potentially downplay (i.e., discount) forward-looking information.

Second, and more fundamentally, central banks often employ strong conditioning assumptions in these very forecasts, such as prescribed projections of financial variables, shock processes, external assumptions, and so on. Potentially, therefore, forecasts (particularly medium-term ones) might be considered more a benchmark for scenario analysis and discussion than a specific expected outturn.

In line with this, forecasts are often wrapped around confidence intervals or "fan charts" whose widths are necessarily increasing in the forecast horizon. Moreover, with every new forecast round, data, assumptions, expert judgment, and risks are updated such that forecasts themselves may be heavily revised over time and differ markedly across institutions. Again, in such circumstances, central banks may wish to incorporate these forecasts in their information set but weigh them accordingly. Similarly, one might consider other germane examples based on the various forms of model and judgmental uncertainty (e.g., Onatski and Stock 2002 and Svensson 2005) and its consequences for attenuated or non-attenuated policymaking. Summing up, we might say that while forward-looking policies require forecasts, the very nature of the policy process—i.e., forecast and judgmental revisions, real-time data problems, model uncertainty, etc.—constrains policymakers to treat such information in a manner different from realized outcomes. Our solution—to think of policy as a Calvo-type IFB rule—though simple, is quite powerful: policymakers target future outcomes (such as future inflation rates) in a geometrically discounted manner. We show that this precludes indeterminacy for a number of cases and, indeed, appears data coherent when appended to an estimated dynamic stochastic general equilibrum (DSGE) model.

The paper proceeds as follows. Section 2 sets out a model chosen for its tractability and summarizes the analytical findings of the literature on IFB rules in their standard form. In the analysis, we focus exclusively on "pure" inflation targeting without a feedback on the output gap. We do this for two reasons: First, there are problems associated with measuring the output gap and therefore implementing rules of the "Taylor" type. Second, since simplicity per se is regarded as a positive aspect of monetary rules, it is of interest to study the stabilizing performance of rules that are indeed as simple and transparent as possible. The new contribution to the IFB literature is in sections 3 and 4. Section 3 introduces and analyzes the general properties of a Calvo-type interest rate rule. Section 4 compares such a rule with more conventional j-period-ahead IFB rules and the benchmark of fully optimal monetary policy. Section 5 illustrates the relative empirical performance of our chosen rule when incorporated into the well-known Smets-Wouters DSGE model of the euro area. Section 6 concludes.

2. The Model and Previous Results for IFB Rules

We adopt a standard and tractable New Keynesian model popularized notably by Clarida, Galí, and Gertler (1999) and Woodford (2003):

$$\pi_t = \beta E_t \pi_{t+1} + \lambda mc_t \tag{1}$$

$$mc_t = -(1+\phi)a_t + \sigma c_t + \phi y_t \tag{2}$$

$$c_t = E_t c_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1}) \tag{3}$$

$$y_t = c_y c_t + g_y g_t \tag{4}$$

$$a_t = \rho_a a_{t-1} + \epsilon_{a,t} \tag{5}$$

$$g_t = \rho_g g_{t-1} + \epsilon_{g,t}. \tag{6}$$

In (1) and (3), π_t is the inflation rate, β is the private sector's discount factor, $E_t(\cdot)$ is the expectations operator, y_t is output, c_t is consumption, and the slope of the Phillips curve λ can be expressed in terms of the average contract length of Calvo-type price contracts. mc_t given by (2) is the marginal cost, where a_t is a technology shock and ϕ is the Frisch parameter. (1) is derived as a linearized form of staggered price setting about a zero-inflation steady state, and (3) is a linearized Euler equation with i_t the nominal interest rate and σ the risk-aversion parameter. (4) is a linearized aggregate equilibrium relation, where g_t is a government-spending shock and c_y and g_y are consumption and government-spending shares, respectively, in the steady state. According to (5) and (6), shocks follow AR(1) processes. All variables are expressed as deviations from the steady state; π_t and i_t as absolute deviations; and c_t, y_t, and g_t as proportional deviations.

To close the model, we require an interest rate rule. The cornerstone of much of the monetary policy literature is the well-known Taylor (1993) rule, a generalized version of which is

$$\begin{aligned} \rho \in [0,1) : i_t &= \rho i_{t-1} + (1-\rho)\big[\pi_t^* + \theta_\pi E_t\big(\pi_{t+j} - \pi_{t+j}^*\big) \\ &\quad + \theta_y E_t(y_{t+k} - \hat{y}_{t+k})\big] \\ \rho = 1 : i_t &= i_{t-1} + \big[\Theta_\pi E_t\big(\pi_{t+j} - \pi_t^*\big) + \Theta_y E_t(y_{t+k} - \hat{y}_{t+k})\big], \end{aligned} \tag{7}$$

where i_t is the nominal interest rate, π_t is inflation over the interval $[t-1,t]$, $\hat{y}$ is potential output, and y_t is actual output, so that $y-\hat{y}_t$ is the output gap. Variables i_t, $\hat{y}_t$, and y_t are measured in deviation form about a zero-inflation steady state and π_t^* is the inflation target. Integers j,k are the policymaker's forecast horizons, which are a feedback on single-period inflation over the interval $[t+j-1,t+j]$ and a feedback on the output gap over the period $t+k$. Thus, this specification of an interest rate rule accommodates not only outcome-based rules (with $j,k \leq 0$) but also forecast-based ones (with $j,k > 0$). Finally, $\theta_\pi, \theta_y > 0$ and $\Theta_\pi, \Theta_y > 0$ are feedback parameters: the larger the values of these parameters, the faster the pace at which the central bank acts to eliminate the gap between expected inflation and the expected output gap and their target values.

The parameter $\rho \in [0,1]$ measures the degree of interest rate smoothing. If $\rho = 1$, we have an *integral (or difference) rule* that is equivalent to the interest rate responding to a *price-level target.*[1] For $\rho < 1$, (7) can be written as $\Delta i_t = \frac{1-\rho}{\rho}[\theta_\pi E_t(\pi_{t+j} - \pi^*_{t+j}) + \theta_y E_t(y_{t+k} - \hat{y}_{t+k}) - i_t]$, which is a partial adjustment to a static IFB rule, $i_t = \theta_\pi E_t(\pi_{t+j} - \pi^*_{t+j}) + \theta_y E_t(y_{t+k} - \hat{y}_{t+k})$.

For reasons already discussed, our analysis focuses on standard IFB rules without an output-gap target ($\theta_y = 0$) and with a zero-inflation target $\pi_t^* = 0$.[2] Then, writing $\theta_\pi \equiv \theta$ and $\Theta_\pi \equiv \Theta$, (7) becomes

$$\begin{aligned} i_t &= \rho i_{t-1} + \theta(1-\rho)E_t\pi_{t+j}; \rho \in [0,1), \theta > 0 \\ &= i_{t-1} + \Theta E_t \pi_{t+j}; \rho = 1, \Theta > 0. \end{aligned} \tag{8}$$

Stability and indeterminacy of a dynamic system are associated with the roots of the system's characteristic equation or,

[1] Unlike its non-integral counterpart, an integral rule responding to inflation does not require observations of the steady-state (natural) rate of interest, about which i_t is expressed, to implement. The merits of price-level versus inflation targeting are examined in Vestin (2006).

[2] Another form of IFB rule found in the literature targets *average* inflation over a specified time horizon, as investigated by Batini and Pearlman (2002). This is represented as $i_t = \rho i_{t-1} + \theta(1-\rho)\frac{E_t \sum_{r=0}^{j} \pi_{t+r}}{1+j}$. As indicated after result 4 below, these rules have roughly the same determinacy properties as a standard IFB rule with half the horizon j.

equivalently, the eigenvalues of its state-space setup. If the number of unstable roots (outside the unit circle) exactly matches the number of nonpredetermined variables, there is a unique solution path. Too few unstable roots then leads to indeterminacy, while too many leads to instability.

The mechanism through which indeterminacy arises can be illustrated in the context of a simplified version of our model. On the demand side, we replace the Keynes-Ramsey condition (3) with an ad hoc IS curve $y_t = -\alpha(i_t - E_t\pi_{t+1})$ and assume $y_t = c_t$ and $\beta = 1$. We also remove the productivity shock a_t. Moreover, suppose that the central bank employs a non-integral rule without interest rate smoothing ($\rho = 0$) so that (7) becomes $i_t = \theta E_t \pi_{t+1}$. Substituting out for y_t and i_t, we arrive at the following process for inflation:

$$E_t(\pi_{t+1}) = \frac{1}{1 - \lambda(\sigma + \phi)\alpha(\theta - 1)}\pi_t. \tag{9}$$

Consider the case in which private-sector expectations are driven by a nonfundamental-shock process and anticipate that inflation next period will be equal to 1. This will lead to an increase in real interest rates, with a consequent reduction in demand of $\alpha(\theta - 1)$. Given (9), price-setting behavior will thus imply a current inflation rate of $1 - \lambda(\sigma + \phi)\alpha(\theta - 1)$, which we define as π_0.

Now assume that θ is chosen so that $0 < \pi_0 < 1$, which is the case if $1 + 1/(\lambda(\sigma + \phi)\alpha) > \theta > 1$. If we then lead equation (9) forward in time and take expectations, consistency requires that the sequence of successive inflationary expectations is given by 1, $1/\pi_0, 1/\pi_0^2, 1/\pi_0^3, \ldots$. However, these inflation expectations tend to infinity—a solution that clashes with private-sector expectations. Thus, the unique possible solution is $\pi_t = y_t = i_t = 0$ for all $t > 0$. On the other hand, suppose that the central bank is *not aggressive*, and $\theta < 1$. In this case, $\pi_0 > 1$, and hence the sequence of inflationary expectations tends to zero—a solution that fulfills private-sector expectations, making these "self-fulfilling." Now suppose the central bank is *overaggressive* such that $\pi_0 < -1$. This happens when $\theta > 1 + \frac{2}{\lambda(\sigma + \phi)\alpha}$. In this case, the economy experiences cycles of positive and negative inflation, but again the sequence of inflationary expectations tends to zero and fulfills private-sector expectations.

"Self-fulfilling expectations" implies that any initial private-sector expectation leads to an acceptable path for inflation—hence indeterminacy. Furthermore, if these (nonfundamental) shocks to private-sector expectations follow a stochastic process, then sunspot equilibria are generated. These are typically welfare reducing because they induce increased volatility in the system.

So far, research on monetary policy strategy has identified a series of circumstances under which forward-looking optimal and simple one-period-ahead IFB rules might result in multiple equilibria or instability. One of the earliest contributions on indeterminacy under inflation-targeting forward-looking rules is Bernanke and Woodford (1997). Assuming that agents form their expectations rationally, they showed that the equilibrium associated with forward-looking optimal inflation-targeting rules under commitment may not be unique when the central bank targets current (exogenously determined) private-sector forecasts of inflation, either those made explicitly by professional forecasters or those implicit in asset prices. In this sense, their finding squares with the more general one in Sargent and Wallace (1975), who showed that any policy rule responding uniquely to exogenous factors may induce multiple rational-expectations equilibria.

Subsequent work by Svensson and Woodford (2005), again assuming rational expectations and commitment on the side of the central bank, revealed, however, that forward-looking optimal inflation targeting based on endogenously determined forecasts as opposed to exogenous, private-sector forecasts might not necessarily lead to superior results. As their work emphasizes, the purely forward-looking procedure, often assumed in discussions of inflation-forecast targeting, prevents the target variables from depending on past conditions. In other words, the target variables are not "history dependent."[3] This feature makes the rules suboptimal, perhaps seriously so (Currie and Levine 1993), and can lead to indeterminacy of the equilibrium (Woodford 1999).

Perhaps the best-known theoretical result in the literature on IFB rules is that to avoid indeterminacy, the monetary authority must respond aggressively—i.e., with a coefficient above unity,

[3] As we shall see in section 5, this is a property of the optimal-commitment rule.

but not excessively large, to expected inflation in the closed-economy context (see, among others, Clarida, Galí, and Gertler 2000 and, in the small-open-economy context, see De Fiore and Liu 2002). Bullard and Mitra (2002) reaffirmed this result, and in a further paper, Bullard and Mitra (2007) show how monetary policy inertia (a high ρ in (7)) can help alleviate problems of inertia. These same authors follow the pioneering work of Evans and Honkapohja (2001) and examine conditions for the stability of one-period-ahead IFB rules under least-squares learning ("e-stability"). A possible resolution of our "puzzle" is suggested by Evans and Honkapohja (2003) in that forward-looking rules with least-squares learning can lead to stability. However, we address the question of stability and determinacy within a strict rational-expectations framework.

This literature on the determinacy of IFB rules applies to one-period-ahead IFB rules. In a series of papers, Batini and Pearlman (2002), Batini, Levine, and Pearlman (2004), and Batini et al. (2006) extend this literature to any forward horizon for both the closed and open economy. The main results for the former can be summarized as follows.

Result 1. *For an integral rule feeding back on current inflation* ($j = 0$), $\Theta > 0$ *is a necessary and sufficient condition for determinacy.*[4] *For higher feedback horizons* ($j \geq 1$), $\Theta > 0$ *is a necessary but not sufficient condition for stability and determinacy.*

Result 2. *For* j*-period-ahead integral IFB rules,* $j \geq 1$, *there exists a range* $\Theta \in [0, \bar{\Theta}(j)]$ *with* $\bar{\Theta}(j) > 0$ *such that the model is stable and determinate.*

Result 3. *For a non-integral rule feeding back on current inflation* ($j = 0$), $\theta > 1$ *is a necessary and sufficient condition for determinacy. For higher feedback horizons* ($j \geq 1$), $\theta > 1$ *is a necessary but not sufficient condition for stability and determinacy.*

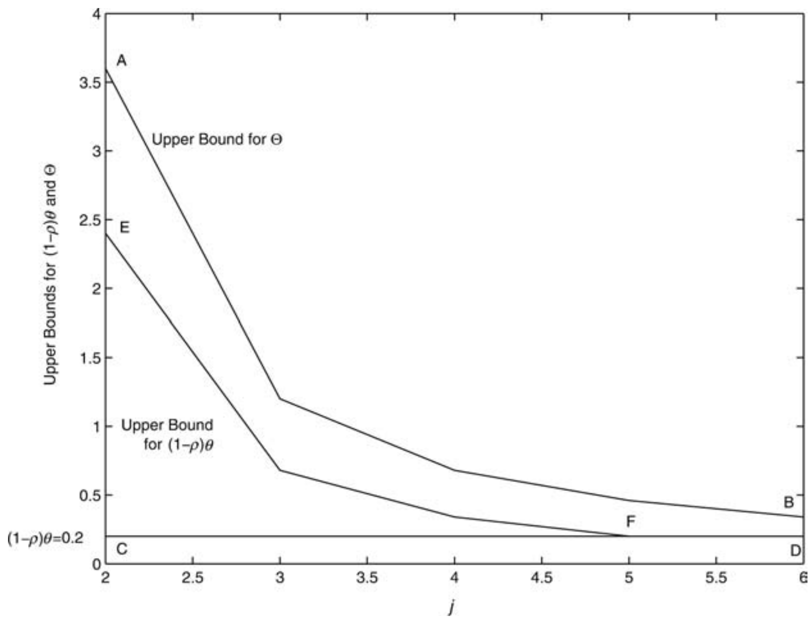
[4]Note that determinacy implies saddle-path stability; however, a model can be stable with non-explosive behavior but without a unique solution (i.e., indeterminate).

RESULT 4. *For j-period-ahead non-integral IFB rules, $j \geq 1$, there exists some lead J such that for $j > J$, there is indeterminacy for all values of θ.*[5] *J is given by*

$$J = \frac{1}{1-\rho} + \frac{(1-\beta)\sigma}{\lambda(\sigma+\phi)}. \tag{10}$$

To get a feel for these results, we now provide numerical values for threshold values $\bar{\theta}$ for non-integral rules and $\bar{\Theta}$ for integral rules. In figure 1, based on table 1, parameter estimates are taken from Batini et al. (2006).[6] For non-integral rules, we set $\rho = 0.8$.

Figure 1. Critical Upper Bounds for $(1-\rho)\theta$ and Θ



[5] Strictly, there are some mild conditions on the parameters that a plausible calibration easily satisfies for this result to hold—see Batini and Pearlman (2002). For the average-inflation rule of the type set out in the previous footnote, the corresponding lead $\hat{J}$ is given by $\hat{J} = 2J - 1$.

[6] Parameter values are $\lambda = 0.27$, $\beta = 0.99$, $\sigma = 3.91$, $\phi = 2.16$, $\rho_a = \rho_g = 0.9$, $sd(\epsilon_g) = 2.75$, and $sd(\epsilon_a) = 0.59$, found using Bayesian methods and U.S. data.

Table 1. Critical Upper Bounds for $\bar{\theta}(j)$ and $\bar{\Theta}(j)$

Threshold	ρ	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
$\bar{\theta}(j)$	0.8	102	12.0	3.4	1.70	1.00	Indeterminacy
$\bar{\Theta}(j)$	1.0	23	3.6	1.2	0.68	0.46	0.34

These numerical results corroborate the analytical results summarized above.[7] The indeterminacy problem becomes more acute as the horizon j increases, imposing a tighter constraint on the range of IFB rules available. For non-integral rules with $\rho = 0.8$, the maximum horizon J is just over five quarters. In accordance with result 2, for integral rules, as j increases, there is always some feedback coefficient on expected inflation $0 < \Theta < \bar{\Theta}$ such that the IFB rule yields stability and determinacy. For non-integral rules, the area of determinacy in $(j, (1-\rho)\theta)$ space is EFC. For integral rules, the corresponding space in (j, Θ) space is ABDC.[8]

3. Calvo-Type Interest Rate Rules

We now turn to the main focus of this paper, which is an alternative way of thinking about IFB rules, referred to in the introduction as *Calvo-type interest rate rules.*[9] To formulate this, first define the discounted sum of future expected inflation rates as

$$\Theta_t = (1-\varphi)E_t(\pi_t + \varphi\pi_{t+1} + \varphi^2\pi_{t+2} + \cdots); \varphi \in (0,1). \qquad (11)$$

[7] In fact, qualitatively similar results are found in a more developed New Keynesian model with consumption habit and price indexing in Batini et al. (2006).

[8] Further insight into these results can be provided by writing the expected value of future inflation approximately as $E_t\pi_{t+j} = (\lambda^{max})^j\pi_t$, where λ^{max} is the largest stable eigenvalue of the system under control. Then as j increases, $(\lambda^{max})^j$ decreases, so that the feedback effect becomes negligible, and the system exhibits indeterminacy similar to the $\theta < 1$ type.

[9] We use this terminology since they have the same structure as Calvo-type price or wage contracts (Calvo 1983). One can think of the rule as a feedback from expected future inflation that continues in any one period with probability φ and is switched off with probability $1-\varphi$. The probability of the rule lasting for just j periods is then $(1-\varphi)\varphi^j$, and the mean-lead horizon is therefore $(1-\varphi)\sum_{j=1}^{\infty} j\varphi^j = \frac{\varphi}{1-\varphi}$.

Then

$$\varphi E_t \Theta_{t+1} - \Theta_t = -(1-\varphi)\pi_t. \tag{12}$$

With this definition, a rule of the form

$$\begin{aligned} i_t &= \rho i_{t-1} + \theta(1-\rho)\Theta_t; \rho \in [0,1), \theta > 0 \\ &= i_{t-1} + \Xi\Theta_t; \rho = 1, \Xi > 0 \end{aligned} \tag{13}$$

emerges that describes feedback on forward-looking inflation with mean-lead horizon $\frac{\varphi}{1-\varphi}$. Thus, with $\varphi = 0.5$, e.g., we have a Calvo-type rule that compares with (7) with a horizon $j = 1$.[10]

Consider first *non-integral* rules. With a Calvo-type rule—writing (1), (3), (12), and (13) in matrix form—the characteristic equation of the system can be shown to be

$$\begin{aligned} (1-\varphi z)(z-\rho)((\beta z - 1)(z-1) - \frac{\lambda(\sigma+\phi)}{\sigma} z) \\ + \theta(1-\varphi)(1-\rho)\frac{\lambda(\sigma+\phi)}{\sigma} z = 0, \end{aligned} \tag{14}$$

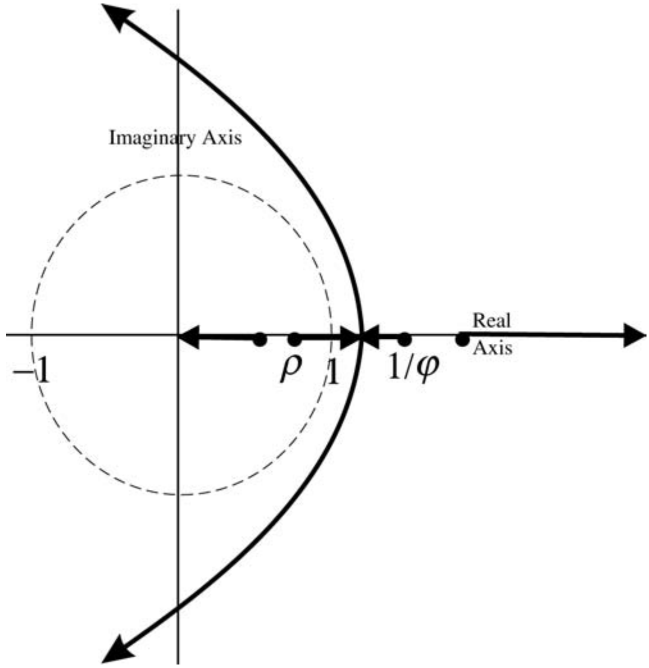
where z is the forward operator (i.e., $zx_t \equiv x_{t+1}$). Noting that the system (1), (3), and (13) has only one lag term, the condition for stability and indeterminacy of the system is that exactly one root of (14) must lie within the unit circle. Accordingly, we investigate (14) using the *root-locus method*. This is a standard method for analyzing the stability of dynamic linear systems found in the engineering literature. (See Evans 1954 and Aoki 1981 for an early

[10] It is of interest to note that for $\rho \in [0,1)$, this rule can also be expressed as

$$i_t = (1+\varphi\rho)^{-1}[\rho i_{t-1} + \varphi E_t i_{t+1} + \theta(1-\rho)\pi_t].$$

Whether the rule is expressed in this way or as (13), it is evident from the model (1)–(6) that current variables and one-step-ahead forecasts are sufficient statistics for the decisions of private agents. Under least-squares learning of the rule, the private sector does not need to look further than one period, but with fully rational expectations, assumed in this paper, the rule expressed in terms of $E_t i_{t+1}$ or (13) must be solved forward, getting us back to the original infinite-horizon formulation (11).

Figure 2. Position of Roots as θ Changes from 0 to ∞



application to economics.)[11] It was used for the first time to study the indeterminacy of forward-looking interest rate rules by Batini and Pearlman (2002). The method provides an elegant way of locating the position in the complex plane of all the roots of the characteristic equation as one of the parameters changes. In our application, the parameter in question is the feedback parameter from future inflation, θ.

The root-locus diagram for (14) as θ changes is shown in figure 2, which depicts the complex plane, and is a generic shape for all parameter values of the system. The root locus starts out at the roots of (14) for $\theta = 0$; these roots are denoted on the diagram by •. Note that one root, $z = 1/\varphi$, is outside the unit circle, while another, $z = \rho$, is inside the unit circle, and it is easy to show that $(\beta z - 1)(z - 1) - \frac{\lambda(\sigma+\phi)}{\sigma} z = 0$ has one root outside and one root inside

[11]See appendix 1 for a brief guide to the root-locus method.

the unit circle. The arrows then show how the four roots change as θ changes. Note in particular that the smallest root has a branch from it leading to $z = 0$, while the largest has a branch leading to $z = \infty$. Of the other two roots, one of them has a branch passing through $z = 1$, and where their branches meet, they both branch into the complex plane and head to infinity at an angle of $60°$ asymptotically to the real line.[12]

There are several things to note about this root-locus diagram. First, when $\theta = 1$, then $z = 1$ as well; this is immediate from (14).[13] Second, the diagram therefore implies that the system has a single stable root for all values of $\theta > 1$, no matter what the values are of the other parameters. However, this apparently general result needs some explanation and, indeed, some slight qualification. We summarize the main results as follows and provide a proof in appendix 2.

Result 5. *A sufficient condition for the system (1)–(3) with the Calvo-type interest rate rule (13) to be determinate for all* $\theta > 1$ *is that* $\rho > \varphi$.

Estimated interest rate rules (including our estimates in section 5) suggest substantial smoothing, with typically $\rho > 0.95$. The condition in this last result is therefore that $\varphi < 0.95$, or, in other words, the mean lead must be less than nineteen quarters. A final observation is that as the interest rate smoothing increases, at the limit where we have an integral rule, result 5 *always* holds. Thus the result for integral (price-level) rules is an immediate corollary of result 5.

Result 6. *The system (1)–(3) with the Calvo-type integral (i.e., price-level) interest rate rule is determinate for all* $\Xi > 0$.

The proof follows once one has replaced $(1 - \rho)\theta$ with Ξ, so that the same argument follows as for the proof of the previous result (with $z = 1$ when $\Xi = 0$) and, in addition, $1 > \varphi$.

[12] The root-locus diagram would look qualitatively the same if $1/\varphi$ were the largest real root, or ρ the smallest real root, for $\theta = 0$.

[13] Note that the Taylor principle—that interest rate should react by more than one-to-one to expected or current or past inflation—means that $\theta > 1$ for non-integral rules.

Thus, according to result 6, the indeterminacy problem *disappears altogether* (in the context of our simple model) if the authorities target a weighted average of present and future price levels with geometrically declining weights.

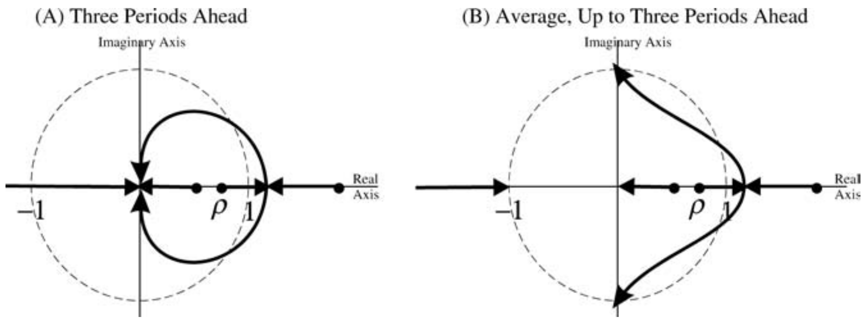
The Calvo-type IFB rule (13) is not completely forward looking, as it includes a reaction to current inflation with weight unity. Do the improved determinacy properties of this rule compared with standard IFB rules crucially depend on the presence of current inflation? We can see this is not the case because average-inflation rules, which also react to current inflation over a finite time horizon j referred to in footnotes 2 and 3, have similar indeterminacy properties to single-period IFB rules with horizon $\frac{j+1}{2}$ if j is odd and $\frac{j}{2}$ if j is even. Further suppose that the Calvo-type rule involves only forward-looking inflation, so that (12) contains the term $E_t\pi_{t+1}$ instead of π_t, so that the characteristic equation (14) becomes

$$(1-\varphi z)(z-\rho)((\beta z-1)(z-1)-\frac{\lambda(\sigma+\phi)}{\sigma}z) + \theta(1-\varphi)(1-\rho)\frac{\lambda(\sigma+\phi)}{\sigma}z^2=0. \tag{15}$$

Using the root-locus technique, one can show that, provided $\rho>\varphi$, there is indeterminacy only for values of θ beyond that value at which $z=-1$. Thus the critical value of θ for indeterminacy is that which satisfies (15) at $z=-1$. It is easy to see that this critical value is given by $f(\rho)(1+\varphi)/(1-\varphi)$, where $f(\rho)$ is a function of ρ (and the other parameters), which is an increasing function of φ. Thus for the Calvo-type rule, as φ (and therefore the expected horizon) increases, the proneness to indeterminacy actually *falls*. Furthermore, the function $f(\rho)\to\infty$ as $\rho\to 1$, so with Calvo-type *integral* rules that are purely forward looking, the critical value for θ above which there is indeterminacy becomes infinite and result 6 holds.

We end this section by noting the contrast between result 5 and result 3, which can be illustrated by the root-locus diagrams of figure 3 for IFB rules. These diagrams depict the cases for interest rates depending on either single-period inflation, three periods ahead (panel A), or average inflation over the current period and up

Figure 3. Position of Roots for (A) Single-Period-Inflation Forward-Looking Rules and (B) Average-Inflation Forward-Looking Rules as θ Changes from 0 to ∞



to three periods ahead (panel B). Both diagrams demonstrate that there may be a range of $\theta > 1$ for which there is determinacy (exactly one stable root), but for values of θ that are too large, there is indeterminacy.

4. Optimal Monetary Policy

4.1 Utility-Based Welfare

In the simple model of this paper, a quadratic approximation to the utility of the household that underlies the model takes the form

$$\Omega_0 = E_0 \left[\frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left[(y_t - \hat{y}_t)^2 + w_\pi \pi_t^2 + w_i i_t^2 \right] \right], \tag{16}$$

where $\hat{y}_t$ is potential output achieved when prices are flexible ($mc_t = 0$ in (1)) and

$$w_\pi = \frac{\zeta}{\lambda(\sigma + \phi)} = \frac{\zeta \xi}{(1 - \xi)(1 - \beta \xi)(\sigma + \phi)}. \tag{17}$$

In (17), $1 - \xi$ is probability of a price optimization for each firm, σ is the risk-aversion parameter, $1 + \phi$ is the elasticity of disutility with respect to hours worked, and ζ is the elasticity of substitution of differentiated goods making up aggregate output.[14] For estimated or calibrated parameter values (see footnote 6 above) reported in Batini et al. (2006), this gives $w_\pi = 1.826$.[15]

Although there is no cost-push ("markup") shock in our model, the existence of a penalty on the variability of the interest rate, driven by concerns for the zero-lower-bound constraint, leads to an inflation/output-gap trade-off. Whereas without such a constraint optimal policy simply sets the interest rate to keep both inflation and the output gap at zero, with the constraint this is not possible and a nontrivial policy problem emerges.

4.2 Optimal Policy with and without Commitment

We first compute the optimal policies where the policymaker can commit and the optimal discretionary policy where no commitment mechanism is in place.[16] To obtain the weight on interest rate variance, w_i, we first compute the optimal-commitment rule with the interest rate responding only to current inflation (see below). We impose an approximate *zero lower bound* on the nominal interest rate by experimenting with w_i so that it is sufficiently high so as to ensure $i_t > 0$ with almost unit probability of 0.99 percent; i.e., (assuming a normal distribution), $sd(i_t) < \frac{i}{2.33}$, where $i = (\frac{1}{\beta} - 1) \times 100$ is the natural rate of interest. A weight $w_i = 0.5$ was necessary to achieve this condition.

4.3 Optimized IFB Rules

We now turn to optimized IFB rules. The general form of the rule that covers both integral and non-integral rules is given by

$$i_t = \rho i_{t-1} + \Xi E_t \pi_{t+j}; \rho \in [0, 1], \Xi, j \geq 0. \tag{18}$$

[14] See Woodford (2003, chap. 6).

[15] Based on an annual inflation rate, this is equivalent to $w_\pi = \frac{1.826}{16} = 0.11$, which is at the lower end of commonly used weights.

[16] Full details of the solution procedures for optimal commitment and discretion are given in Levine, McAdam, and Pearlman (2007). For the former we show that, in general, optimal commitment can be implemented as a rule that feeds back on both current and past predetermined state variables and is therefore history dependent. See also Woodford (2003, chap. 8) for ways in which optimal commitment can be implemented as a rule.

The corresponding Calvo-type rules are given by

$$i_t = \rho i_{t-1} + \Xi E_t \Theta_t; \rho \in [0, 1], \Xi, j \geq 0. \tag{19}$$

Given the estimated variance-covariance matrix of the white-noise disturbances, an optimal combination (Ξ, ρ) can be found for each rule defined by the time horizon $j \geq 0$.

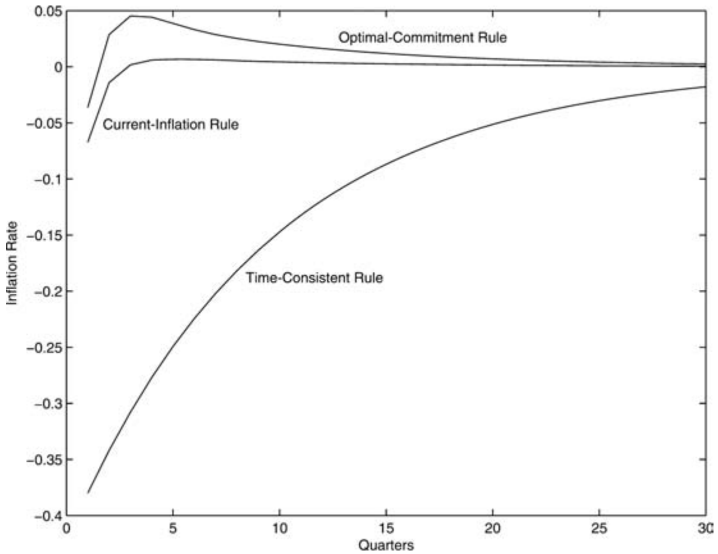
4.4 Numerical Results

We first focus on the optimal-commitment rule, the optimal-discretionary (time-consistent) rule, and an optimized current-inflation rule. Results for the three types of rules are summarized in table 2. Figures 4–11 compare the responses under the three rules following an unanticipated productivity shock ($a_0 = 1$) and an unanticipated government-spending shock ($g_0 = 1$). The output and inflation equivalent welfare differences compared with the optimal-commitment policy are computed as follows. Suppose the welfare

Table 2. Comparision of Welfare-Based Optimal Rules and Optimized IFB Rules

Rule	ρ	Ξ	Loss Function	y_e	π_e
Minimal Feedback on π_t	1	0.001	49.01	0.97	0.72
IFB(0)	1	2.035	2.509	0.11	0.08
IFB(1)	1	12.00	2.676	0.12	0.09
IFB(2)	1	3.570	4.574	0.23	0.17
IFB(3)	1	1.216	32.02	0.78	0.57
IFB(4)	1	0.675	208.1	2.03	1.50
Calvo IFB($\varphi = 0.5$)	1	2.203	2.602	0.12	0.09
Calvo IFB($\varphi = 0.67$)	1	2.351	2.636	0.12	0.09
Calvo IFB($\varphi = 0.75$)	1	2.444	2.653	0.12	0.09
Calvo IFB($\varphi = 0.875$)	1	2.616	2.678	0.13	0.09
Calvo IFB($\varphi = 0.917$)	1	2.683	2.689	0.13	0.09
Optimal Commitment	n.a.	n.a	1.896	0.00	0.00
Optimal Discretion	n.a.	n.a	53.55	1.02	0.75

Figure 4. Inflation following Shock $a_0 = 1$



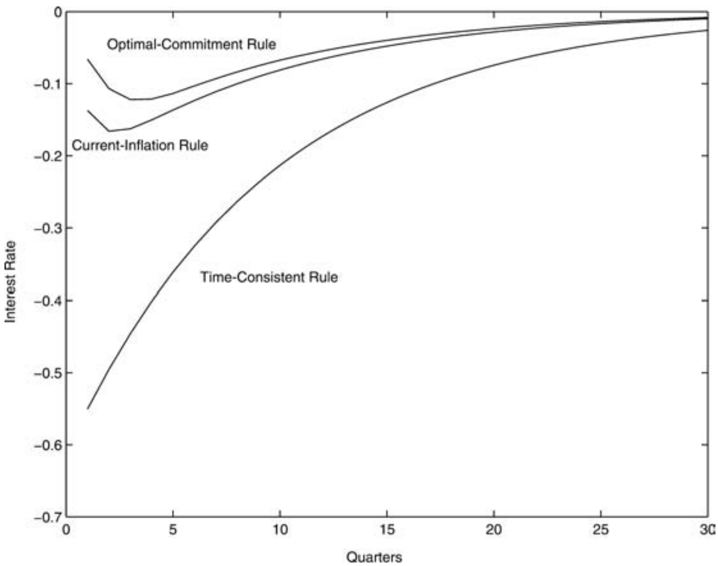
loss difference is X. This is equivalent to a permanent output gap of y_e if $\frac{1}{2(1-\beta)} y_e^2 = X$ and to a permanent inflationary bias of π_e if $\frac{1}{2(1-\beta)} w_\pi \pi_e^2 = X$; i.e.,

$$y_e = \sqrt{2(1-\beta)X} \tag{20}$$

$$\pi_e = \sqrt{\frac{2(1-\beta)X}{w_\pi}}. \tag{21}$$

Comparing the three types of rules, there are two notable results. First, Clarida, Galí, and Gertler (1999) stress the existence of stabilization gains from commitment in New Keynesian models: we show in our simple model that these are substantial, amounting to a permanent output equivalent of 1.02 percent or an inflationary bias of 0.75 percent per quarter, or 3 percent per year. The source of this time-inconsistency problem is from pricing and consumption behavior together. Following a shock that diverts the economy from its steady state, given expectations of inflation, the opportunist

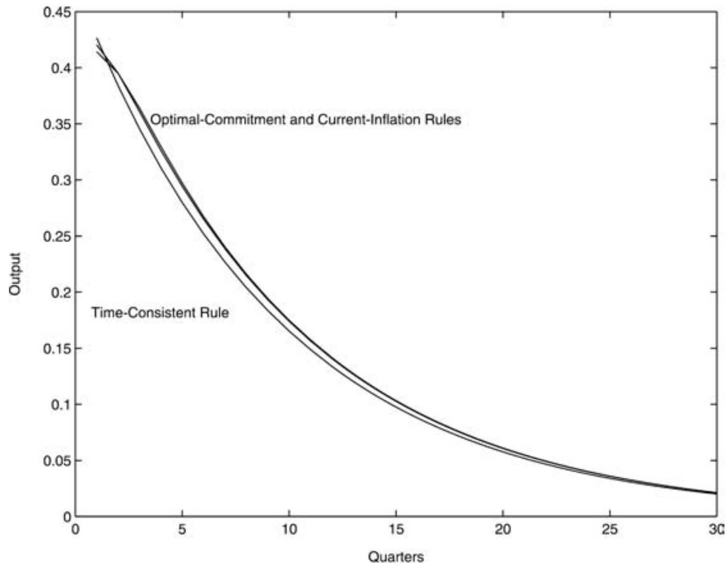
Figure 5. Interest Rate following Shock $a_0 = 1$



policymaker can increase or decrease output by reducing or increasing the interest rate, which increases or decreases inflation. Consider the case where the economy is below its steady-state level of output. A reduction in the interest rate then causes consumption demand to rise. Firms that are locked into price contracts respond to an increase in demand by increasing output and increasing the price according to their indexing rule. Those that can reoptimize only increase their price. These changes are for given inflationary expectations and illustrate the incentive to inflate when the output gap increases. In a noncommitment equilibrium, however, the incentive is anticipated, and the result is greater inflation variability as compared with the commitment case. This contrast between the commitment and discretionary cases is seen clearly in the figures.

The second notable result concerns the optimized current-inflation rule. We find that most of the gains from commitment (in fact, over 80 percent) can be achieved by this very simple optimized rule without an output-gap feedback, and the cost of simplicity is

Figure 6. Output following Shock $a_0 = 1$



only 0.11 percent of output, or an inflationary bias of 0.08 per quarter, or 0.32 percent per year. We find the optimized rule over parameters ρ and Ξ in (18) gives $\rho = 1$ so that the best current-inflation rule is of the integral type. In the figures, we see how the optimized current-inflation rule closely mimics the optimal-commitment rule.

Turning now to IFB rules, we compute the optimized standard rules with future horizon $j = 0, 1, 2, 3, 4$, denoted by IFB(j), and compare these with Calvo-type rules with probability of survival $\varphi = 0.5, 0.67, 0.75, 0.875$, and 0.917 corresponding to an average future horizon of $\frac{\varphi}{1-\varphi} = 1, 2, 4, 7$, and 11 quarters, respectively. Our results first confirm a finding of Batini et al. (2006): that the stabilization performance of standard optimized IFBj rules deteriorates sharply as the horizon j increases. Our new result that follows from the stability analysis of Calvo-type rules and the absence of an "indeterminacy constraint" is that *this sharp deterioration is not a feature of Calvo-type optimized IFB rules.* Even optimized rules with an expected future horizon of three years perform almost as well as the current-inflation rule. Again, we find that integral rules perform the best.

Figure 7. Output Gap following Shock $a_0 = 1$

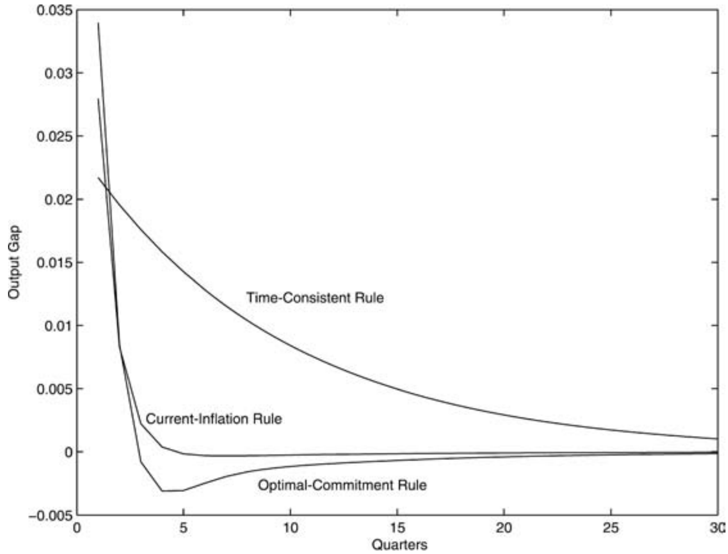


Figure 8. Inflation following Shock $g_0 = 1$

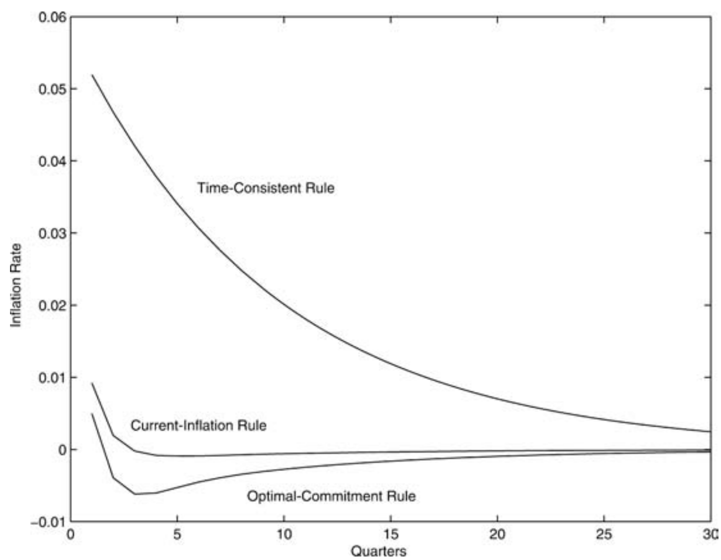


Figure 9. Interest Rate following Shock $g_0 = 1$

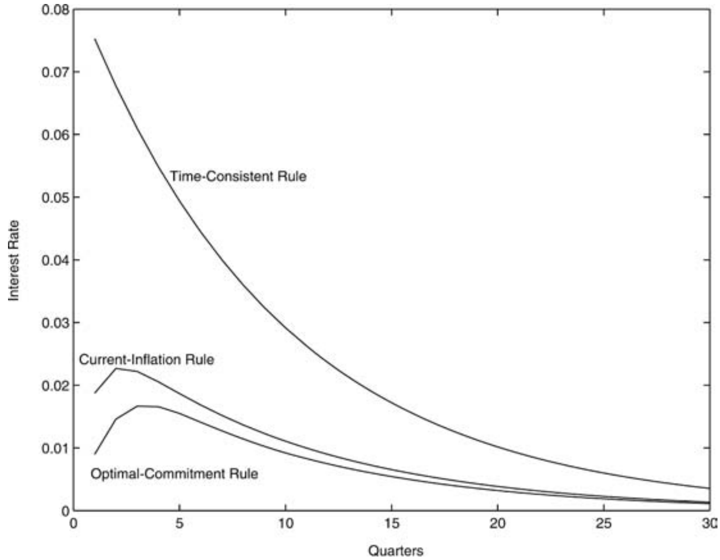


Figure 10. Output following Shock $g_0 = 1$

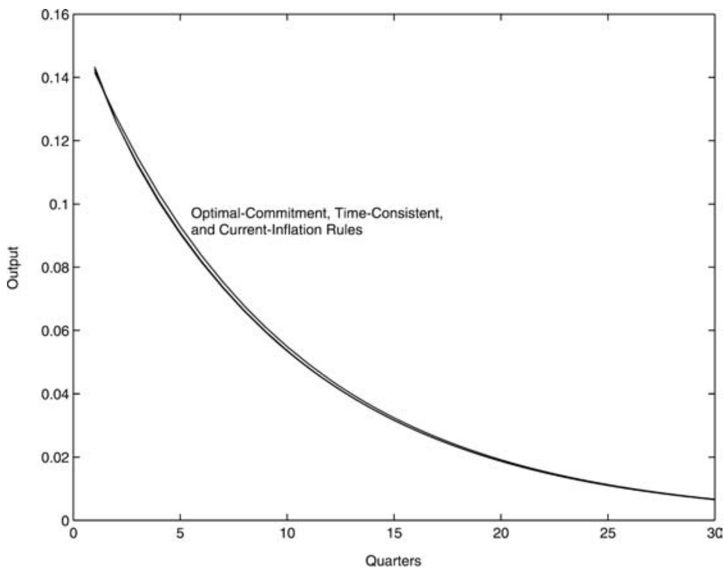
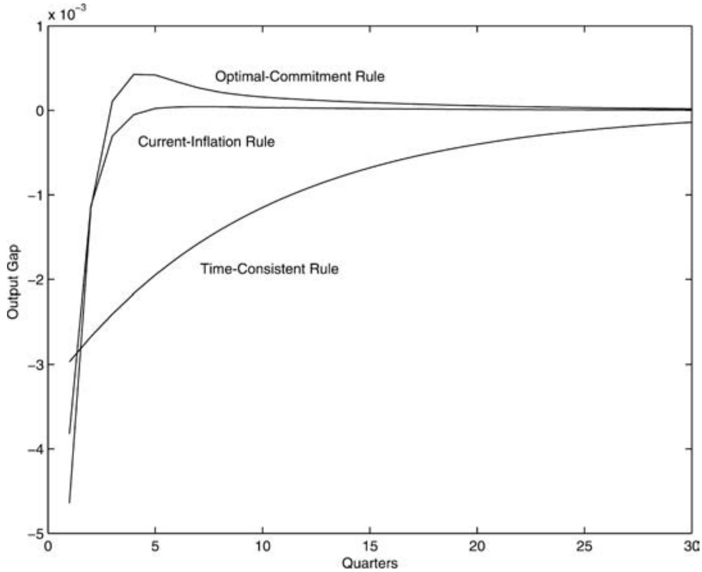


Figure 11. Output Gap following Shock $g_0 = 1$



5. Calvo-Type Interest Rate Rules: A DSGE Model Illustration

In this section, for illustrative purposes, we implement the aforementioned Calvo-type interest rate rules in a benchmark model of the euro area—namely, that of Smets and Wouters (2003) (SW henceforth). A brief description of the model is provided in appendix 3.

5.1 Monetary-Policy Reaction Functions

In line with the empirical approach to monetary rules in SW, we modify the previous monetary-policy reaction functions for the standard and Calvo-type IFB rules as, respectively,

$$\begin{aligned} i_t &= \rho i_{t-1} + (1-\rho)[\bar{\pi}_t + \theta_\pi E_t(\pi_{t+j} - \bar{\pi}_{t+j}) + \theta_y \tilde{y}_t] \\ &\quad + \theta_{\Delta\pi}(\pi_t - \pi_{t-1}) + \theta_{\Delta y}(\tilde{y}_t - \tilde{y}_{t-1}) \\ i_t &= \rho i_{t-1} + (1-\rho)[\bar{\pi}_t + \theta_\pi \Theta_t(\varphi) + \theta_y \tilde{y}_t] \\ &\quad + \theta_{\Delta\pi}(\pi_t - \pi_{t-1}) + \theta_{\Delta y}(\tilde{y}_t - \tilde{y}_{t-1}), \end{aligned} \tag{22}$$

where $\tilde{y}_t$ is the output gap. Then in the SW model $j = -1$ and the interest rate feeds back on lagged inflation. To incorporate this rule as a special case, we also modify (11) to become

$$\Theta_t = (1-\varphi)E_t\big[\pi_{t-1} - \pi^*_{t-1} + \varphi\big(\pi_t - \pi^*_t\big) + \varphi^2\big(\pi_{t+1} - \pi^*_{t+1}\big) + \cdots\big];\ \varphi \in (0,1), \tag{23}$$

so (12) now becomes

$$\varphi E_t\Theta_{t+1} - \Theta_t = -(1-\varphi)\big(\pi_{t-1} - \pi^*_{t-1}\big). \tag{24}$$

This modified Calvo-type rule reduces to the past-inflation-rate rule in SW as a special case by putting $\varphi = 0$. The mean-lead horizon is now given by $\frac{1}{1-\varphi} - 2$.[17]

5.2 Results

Thus, we reestimate by Bayesian methods the SW model with the policy rule replaced by (22) and the model supplemented with (23), where, to repeat, the special case of $\varphi = 0$ retrieves the default, backward-looking SW policy rule.[18] Table 3 reports the parameters of the policy reaction function for each model variant, from IFB(−1) to IFB(4) to the Calvo case.[19] As standard, two sets of parameter results are presented: (i) the estimated posterior mode of the parameters, which is obtained by directly maximizing the log of the posterior distribution with respect to the parameters (and a standard error based on the corresponding Hessian) and (ii) the 5th and 95th percentile of the posterior distribution of the parameters

[17] It is straightforward to show that the results of section 3 still hold with this modification.

[18] We are grateful to Gregory De Walque and Raf Wouters for providing the SW model in Dynare code.

[19] Results for the other parameters (as well as the Dynare files to replicate our results) are available on request from the authors. Notably, the full set of parameter values appeared very well identified and stable across the model variants.

Table 3. Comparison of Calvo-Type and Standard IFB Rules

Rule		IFB(−1)	IFB(0)	IFB(1)	IFB(4)	Calvo-Type IFB
ρ	Mode	0.969	0.969	0.965	0.951	0.958
		(0.012)	(0.013)	(0.015)	(0.025)	(0.021)
	Mean	0.965	0.967	0.958	0.940	0.951
		[0.943:0.984]	[0.951:0.986]	[0.932:0.983]	[0.891:0.977]	[0.918:0.982]
θ_π	Mode	1.700	1.701	1.700	1.700	1.702
		(0.099)	(0.100)	(0.100)	(0.099)	(0.099)
	Mean	1.697	1.698	1.700	1.703	1.707
		[1.535:1.853]	[1.531:1.860]	[1.542:1.873]	[1.541:1.867]	[1.539:1.868]
θ_y	Mode	0.121	0.121	0.117	0.111	0.120
		(0.045)	(0.045)	(0.046)	(0.049)	(0.045)
	Mean	0.117	0.123	0.109	0.107	0.120
		[0.045:0.186]	[0.057:0.193]	[0.034:0.178]	[0.028:0.171]	[0.053:0.186]
$\theta_{\Delta\pi}$	Mode	0.146	0.146	0.111	0.118	0.121
		(0.052)	(0.052)	(0.049)	(0.049)	(0.049)
	Mean	0.155	0.110	0.116	0.119	0.126
		[0.070:0.239]	[0.034:0.195]	[0.034:0.200]	[0.041:0.196]	[0.048:0.212]
$\theta_{\Delta y}$	Mode	0.154	0.154	0.152	0.147	0.151
		(0.023)	(0.023)	(0.022)	(0.022)	(0.021)
	Mean	0.152	0.152	0.146	0.139	0.146
		[0.120:0.191]	[0.115:0.187]	[0.112:0.183]	[0.099:0.178]	[0.109:0.181]
φ	Mode	–	–	–	–	0.8398
		–	–	–	–	(0.103)
	Mean	–	–	–	–	0.797
		–	–	–	–	[0.646:0.956]
Prob.		0.289	0.096	0.158	0.224	0.234

Note: IFBj rule: $i_t = \rho i_{t-1} + (1-\rho)[\pi_t^* + \theta_\pi E_t(\pi_{t+j} - \pi_{t+j}^*) + \theta_y \tilde{y}_t] + \theta_{\Delta\pi}(\pi_t - \pi_{t-1}) + \theta_{\Delta y}(\tilde{y}_t - \tilde{y}_{t-1})$. Calvo IFB ($\varphi$): $i_t = \rho i_{t-1} + (1-\rho)[\pi_t^* + \theta_\pi \Theta_t(\varphi) + \theta_y \tilde{y}_t] + \theta_{\Delta\pi}(\pi_t - \pi_{t-1}) + \theta_{\Delta y}(\tilde{y}_t - \tilde{y}_{t-1})$, where $\varphi E_t \Theta_{t+1} - \Theta_t = -(1-\varphi)(\pi_{-t} - \pi_{t-1}^*)$. Hessian standard errors are in parentheses and 5th and 95th percentiles are in squared brackets. Log marginal likelihood of IFB(−1) = −298.65.

obtained through the Metropolis-Hastings sampling algorithm (using 100,000 draws from the posterior, three parallel chains, and an average acceptance rate of around 0.25) for the various model variants. The models are estimated using the Dynare software (Juillard 2004). Note that, in reestimation, we used identical priors to those used in SW. Moreover, for the additional parameter, φ, we assumed a beta distribution with a prior mean of 0.8 (corresponding to a mean-lead horizon of three quarters), with a standard error of 0.1.

Turning to the results, we see that the Calvo rule yields a φ value centered at an implied mean-lead horizon of three to four quarters in the policy rule. As shown in the last row of table 3, which reports the model odds, this rule beats all contemporaneous and forecast-based rules in marginal-likelihood terms without leading to any deterioration in the parameter values. Comparing the likelihood values of the Calvo rule with the backward-looking rule, IFB(–1), there is a very close data coherence. Indeed, in terms of Bayesian odds ratio (0.234/0.289 = 0.81), we effectively could not discriminate between these two types of rules.[20] Summing up, one might say that while by no means conclusive, these results do suggest that a Calvo-type rule is competitive with more conventional monetary policy rules.

6. Conclusions

The large literature on IFB rules now strongly suggests that rules that target future inflation with a specified time horizon are prone to indeterminacy and have poor stabilization properties. This raises

[20] As discussed in Geweke (1999), the Bayesian approach to estimation allows a formal comparison of different models based on their marginal likelihoods. The marginal likelihood of model M_i is given by

$$p(Y \mid M_i) = \int_{\Xi} p(\xi \mid M_i) p(Y \mid \xi, M_i) d\xi,$$

where $p(\xi \mid M_i)$ is the prior density for model M_i and $p(Y \mid \xi, M_i)$ is the data density for model M_i given the parameter vector ξ and the data vector Y. Then the posterior odds ratio is given by

$$PO_{ij} \equiv \frac{p(M_i|Y)}{p(M_j|Y)} = \frac{p(Y|M_i)p(M_i)}{p(Y|M_j)p(M_j)} = \frac{p(Y|M_i)}{p(Y|M_j)},$$

assuming equal prior model probabilities ($p(M_i) = p(M_j)$). The posterior model probabilities are reported in table 3.

an interesting puzzle of why many central banks insist on forward-looking inflation targets. Part of the answer lies in the fact that these rules assume commitment to a low long-run inflation rate (in fact, zero inflation in our setup). For central banks, forward-looking rules with a low inflation target signals commitment to this low long-run inflation rate. But can IFB rules with a long forward lead be implemented without negative consequences? Our paper proposes a resolution of this puzzle by suggesting that the policy process of central banks may in fact be best modeled in the form of a Calvo-type rule that targets a discounted infinite sum of future expected inflation.

Our main findings are, first, that Calvo-type IFB interest rate rules are less prone to indeterminacy than standard ones with a finite forward horizon. Second, for such rules in integral (i.e., difference) form, the indeterminacy problem disappears altogether. In this case, the Calvo rule takes the form of a weighted-average future price-level target with geometrically declining weights. Third, as a consequence of these results, optimized Calvo-type rules have good stabilization properties as they become more forward looking, which sharply contrasts with the substantial deterioration in the corresponding performance of standard IFB rules. Fourth, in terms of data coherence in the context of the SW model, a Calvo-type rule with a mean forward horizon of just less than one year is perfectly competitive with more conventional monetary policy rules.

A number of possible directions for future research are suggested by this study. First, Calvo-type reaction functions can be estimated directly using generalized method of moments (GMM) estimation methods and can be compared with more standard Taylor-type and IFB rules. Second, in DSGE modeling in general (open economy, closed economy, interacting economies, etc.), it is commonplace to compare the performance of optimized Taylor-type rules with their optimal counterparts. Calvo-type rules could be added to this exercise. Finally, the design of robust rules using, e.g., the Bayesian estimated posterior distribution as in Batini et al. (2006) could be extended to rules of this form.[21]

[21]See Levine et al. (2007).

Appendix 1. A Topological Guide to the Root-Locus Technique

Here we present a brief guide on how to use the root-locus technique. We start with some standard rules as provided in control-theory textbooks and then apply them to the specific example of the paper.

The idea is to track the roots of the polynomial equation $f(z)+\theta g(z) = 0$ as θ moves from 0 to ∞. Clearly for $\theta = 0$, the roots are those of $f(z) = 0$, whereas when $\theta \rightarrow \infty$, the roots are those of $g(z) = 0$. The root locus then connects the first set of roots to the second set by a series of lines and curves. We shall assume without loss of generality that the coefficient of the highest power of f is negative and that of g is positive. Since the roots of a polynomial may be complex, the root locus is plotted in the complex plane.

There are a number of different ways to state the standard rules that underlie the technique. One popular way (see Evans 1954) of sketching the root locus by hand involves just six steps:

(i) a. Define $n(f)$ = number of zeros of $f(z)$, $n(g)$ = number of zeros of $g(z)$. For our case, $n(f) = 4, n(g) = 1$.

b. Loci start at the zeros of $f(z)$ and end at the zeros of $g(z)$ and at ∞ if $n(f) > n(g)$.

(ii) Number of loci must be equal to $\max(n(f), n(g)) = 4$, in our case.

(iii) A point on the real axis is on the root locus if the number of zeros of f and g on the real axis to its left is odd.

(iv) Loci ending at ∞ do so at angles to the positive real axis given by $2k\pi/(n(f) - n(g))$, where the integer k ranges from 0 to $(n(f) - n(g)) - 1$. In our case, these angles are 0, $2\pi/3$, $4\pi/3$.

(v) If all coefficients of f and g are real, then the root locus is symmetric about the real axis.

(vi) Loci leave the real axis where $\partial\theta/\partial z = 0$.

Appendix 2. Proof of Result 5

We prove this result in two steps. First, we need to show that the branch point into the complex plane near $z = 1$ is to the right of $z = 1$. Second, we have to show that the branches of the root locus do not cross the unit circle twice (otherwise, there are too many stable roots, and hence indeterminacy, over a certain range of values of θ greater than 1).

STEP 1. The branch point is to the right of $z = 1$, provided that the root locus passes through this point from left to right as θ increases. But this means that we require $\partial z/\partial\theta > 0$ at $z = \theta = 1$. By implicit differentiation of (14), we find that

$$\left[(1-\beta)(1-\rho)(1-\varphi) + \frac{\lambda(\sigma+\phi)}{\sigma}(1-2\varphi+\rho\varphi)\right]\frac{\partial z}{\partial\theta}\bigg|_{\theta=1} \\ = (1-\rho)(1-\varphi).$$

It is easy to see that a sufficient condition for $\partial z/\partial\theta|_{\theta=1} > 0$ is $\rho > \varphi$.

STEP 2. We now investigate those points on the root loci that lie on the unit circle. These are, of course, characterized by $z = e^{i\phi} = cos\phi + isin\phi$. To solve for ϕ, the easiest approach is to substitute $z = e^{i\phi}$ directly and then multiply (14) through by $e^{-i\phi}$. Then the imaginary part of this expression is independent of θ and can be written as

$$\begin{aligned} \varphi\beta sin3\phi - [\beta(1+\varphi\rho) + \varphi(1+\beta+\lambda(\sigma+\phi)/\sigma))]sin2\phi \\ + [(1+\varphi\rho)(1+\beta+\lambda(\sigma+\phi)/\sigma) \\ + \varphi + \rho\beta - \rho]sin\phi = 0. \end{aligned} \tag{25}$$

Using the substitutions $sin2\phi = 2sin\phi\cos\phi$, $sin3\phi = (4cos^2\phi - 1)$ $sin\phi$, it is clear that one solution to (25) is $sin\phi = 0$, which corresponds to $\phi = 0$ $(z = 1)$ and $\phi = \pi$ $(z = -1$, which is technically a solution when $\theta < 0$). It follows that the other solutions are given by

$$4\varphi\beta cos^2\phi - 2[\beta(1+\varphi\rho) + \varphi(1+\beta+\lambda(\sigma+\phi)/\sigma))]cos\phi \\ + (1+\varphi\rho)(1+\beta+\lambda(\sigma+\phi)/\sigma) \\ + \varphi + \rho\beta - \rho - \varphi\beta = 0.$$

Provided that $\rho > \varphi$, it is easy to show that the coefficient of $cos\phi$ is more than twice that of $cos^2\phi$; it follows that at least one of the solutions to $cos\phi$ is greater than 1. But this means that there is no more than one real solution for ϕ, so that there cannot be a double crossing of the unit circle for $\rho > \varphi$.

Appendix 3. The Smets-Wouters Model

The Smets-Wouters (SW) model is an extended version of the standard New Keynesian DSGE closed-economy model with sticky prices and wages. The model features three types of agents: households, firms, and the monetary policy authority. Households maximize a utility function with two arguments (goods and leisure) over an infinite horizon. Consumption appears in the utility function relative to a time-varying external habit-formation variable. Labor is differentiated over households, so that there is some monopoly power over wages, which results in an explicit wage equation and allows for the introduction of sticky nominal Calvo-type wage contracts. Households also rent capital services to firms and decide how much to accumulate given certain capital-adjustment costs. Firms produce differentiated goods, decide on labor and capital inputs, and set Calvo-type price contracts. Wage and price setting is augmented by the assumption that those prices and wages that cannot be freely set are partially indexed to past inflation. Prices are therefore set as a function of current and expected real marginal cost but are also influenced by past inflation. Real marginal cost depends on wages and the rental rate of capital. The short-term nominal interest rate is the instrument of monetary policy. The stochastic behavior of the model is driven by ten exogenous shocks—five shocks arising from technology and preferences, three cost-push shocks, and two monetary policy shocks.

Consistent with the DSGE setup, potential output is defined as the level of output that would prevail under flexible prices and wages in the absence of cost-push shocks.

References

Aoki, M. 1981. *Dynamic Analysis of Open Economies.* New York: Academic Press.

Batini, N., A. Justiniano, P. Levine, and J. Pearlman. 2006. "Robust Inflation-Forecast-Based Rules to Shield against Indeterminacy." *Journal of Economic Dynamics and Control* 30 (9–10): 1491–1526.

Batini, N., P. Levine, and J. Pearlman. 2004. "Indeterminacy with Inflation-Forecast-Based Rules in a Two-Bloc Model." ECB Working Paper No. 340 and FRB International Finance Discussion Paper No. 797. Presented at the International Research Forum on Monetary Policy, Washington, DC, November 14–15, 2003.

Batini, N., and J. Pearlman. 2002. "Too Much Too Soon: Instability and Indeterminacy with Forward-Looking Rules." Bank of England External MPC Discussion Paper No. 8.

Bernanke, B. S., and M. Woodford. 1997. "Inflation Forecasts and Monetary Policy." *Journal of Money, Credit and Banking* 29 (4): 653–84.

Bullard, J., and K. Mitra. 2002. "Learning about Monetary Policy Rules." *Journal of Monetary Economics* 49 (6): 1106–29.

———. 2007. "Determinacy, Learnability, and Monetary Policy Inertia." *Journal of Money, Credit and Banking* 39 (5): 1177–1212.

Calvo, G. A. 1983. "Staggered Prices in a Utility-Maximizing Framework." *Journal of Monetary Economics* 12 (3): 383–98.

Clarida, R., J. Galí, and M. Gertler. 1999. "The Science of Monetary Policy: A New Keynesian Perspective." *Journal of Economic Literature* 37 (4): 1661–1707.

———. 2000. "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory." *Quarterly Journal of Economics* 115 (1): 147–80.

Currie, D., and P. Levine. 1993. *Rules, Reputation and Macroeconomic Policy Coordination.* Cambridge University Press.

De Fiore, F., and Z. Liu. 2002. "Openness and Equilibrium Determinacy under Interest Rate Rules." ECB Working Paper No. 173.

European Central Bank. 1999. "The Stability-Oriented Monetary Policy Strategy of the Eurosystem." *Monthly Bulletin* (January): 43–56.

Evans, G. W., and S. Honkapohja. 2001. *Learning and Expectations in Macroeconomics.* Princeton, NJ: Princeton University Press.

———. 2003. "Expectations and the Stability Problem for Optimal Monetary Policies." *Review of Economic Studies* 70 (4): 807–24.

Evans, W. R. 1954. *Control Systems Dynamics.* McGraw-Hill.

Geweke, J. 1999. "Using Simulation Methods for Bayesian Econometric Models: Inference, Development, and Communication." *Econometric Reviews* 18 (1): 1–126.

Juillard, M. 2004. *Dynare Manual.* Available at www.cepremap.cnrs.fr/dynare/.

Levin, A., V. Wieland, and J. C. Williams. 2003. "The Performance of Forecast-Based Monetary Policy Rules under Model Uncertainty." *American Economic Review* 93 (3): 622–45.

Levine, P., P. McAdam, and J. Pearlman. 2007. "Quantifying and Sustaining Welfare Gains from Monetary Commitment." ECB Working Paper No. 709. Presented at the 12th International Conference on Computing in Economics and Finance, Cyprus, June 2006.

Levine, P., P. McAdam, J. Pearlman, and R. Pierse. 2007. "Robust Monetary Rules in a Model of the Euro Area." Mimeo. Presented at the ECB DSGE Forum, July 2007.

Onatski, A., and J. H. Stock. 2002. "Robust Monetary Policy under Model Uncertainty in a Small Model of the U.S. Economy." *Macroeconomic Dynamics* 6 (1): 85–110.

Orphanides, A. 2001. "Monetary Policy Rules Based on Real-Time Data." *American Economic Review* 91 (4): 964–85.

Sargent, T. J., and N. Wallace. 1975. "'Rational' Expectations, the Optimal Monetary Instrument, and the Optimal Money Supply Rule." *Journal of Political Economy* 83 (2): 241–54.

Smets, F., and R. Wouters. 2003. "An Estimated Dynamic Stochastic General Equilibrium Model of the Euro Area." *Journal of the European Economic Association* 1 (5): 1123–75.

Svensson, L. E. O. 2005. "Monetary Policy with Judgment: Forecast Targeting." *International Journal of Central Banking* 1 (1): 1–55.

Svensson, L. E. O., and M. Woodford. 2005. "Implementing Optimal Policy through Inflation-Forecast Targeting." In *The Inflation-Targeting Debate*, ed. B. S. Bernanke and M. Woodford, 19–83. Chicago: University of Chicago Press.

Taylor, J. B. 1993. "Discretion versus Policy Rules in Practice." *Carnegie-Rochester Conference Series on Public Policy* 39: 195–214.

Vestin, D. 2006. "Price-Level Versus Inflation Targeting." *Journal of Monetary Economics* 53 (7): 1361–76.

Woodford, M. 1999. "Commentary: How Should Monetary Policy Be Conducted in an Era of Price Stability?" In *New Challenges for Monetary Policy*, ed. Federal Reserve Bank of Kansas City. Kansas City, MO: Federal Reserve Bank of Kansas City.

———. 2003. *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton, NJ: Princeton University Press.