Stock Liquidity Requirements and the Insurance Aspect of the Lender of Last Resort*

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This paper considers a model of information-based bank runs where a central bank sets its lender of last resort (LOLR) policy in order to maximize welfare. To mitigate the risks associated with overinvestment by the banking sector, the central bank sets prudential liquidity requirements for the banking sector in the form of a ratio of liquid assets to deposits. Liquidity requirements then provide a buffer against early deposit withdrawals, but they also allow the central bank to manufacture a distribution of costs to LOLR funding with an expected value equal to 0. It is shown that liquidity requirements, along with an appropriate LOLR policy, become welfare improving if the banking sector is characterized by high-profit opportunities, low leverage, and a relatively volatile deposit base. Otherwise, forgone productive investment due to liquidity restrictions may result in a disproportional cost to the banking sector relative to the insurance value of LOLR.

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1. Introduction

The lender of last resort (LOLR) function of a central bank insures the banking system against a run on liquidity, although no premia

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are explicitly charged against it. Instead, to forestall LOLR exposures from running at unsustainable levels, both in terms of public money at stake and political capital at risk, the official sector can influence the likelihood and extent of LOLR intervention by means of regulatory restrictions on banks. Liquidity regulation, however, may entail welfare losses due to forgone productive investments, implying a trade-off between the welfare benefits from LOLR insurance and the welfare losses from regulatory restrictions. This paper examines this trade-off in a setting in which the official sector aims to maximize welfare conditional on the expected cost of LOLR intervention being equal to 0.

The interaction between LOLR policy and liquidity requirements is discussed in the context of a three-period model with three active risk-neutral agents—namely, a bank, a continuum of depositors, and a central bank that represents the official sector. The bank is the only agent in the economy that has access to a production loan technology and finances its investment through equity and wholesale deposits. In addition, the bank voluntarily holds an amount of riskless assets that pay no return so as to target an optimal probability of solvency. The loan investment is illiquid, and the bank needs to wait for two periods before returns are realized. The probability that the loan *succeeds*, paying a nonzero return, depends on the realization of a risk factor, which we refer to as a *systemic shock*.

When depositors have full information about the systemic shock, liquidity problems do not arise and there is no role for LOLR policy. Relaxing the assumption of full information, the systemic shock is observed with noise by depositors who may then decide either to leave their money with the bank or to run, causing a liquidity shock to the bank. Strategic interaction among depositors, which may give rise to this liquidity shock, is modeled using a global-games methodology as in Morris and Shin (2002) and Rochet and Vives (2004). If the bank has a sufficient cushion of liquid assets in place, it may be able to sustain the liquidity shock. Otherwise, the central bank may have to intervene and extend LOLR provision.

The LOLR policy is defined here as a commitment by the central bank to extend emergency funding to the banking sector. But such a commitment is not unconditional; if a bank runs into liquidity difficulties due to a solvency problem that is severe enough to result in a major hit on the central bank's balance sheet in case of

LOLR intervention, then the central bank may refrain from providing emergency liquidity funding. Otherwise, the LOLR policy would be expected to involve unsustainable losses. To keep things simple, we assume that LOLR intervention takes the form of a capital injection, which may result in a loss to the central bank if bank illiquidity is due to a (moderate) solvency problem; otherwise, the central bank earns a positive return from its LOLR activity.

The central bank is assumed to be able to influence the expected cost of extending LOLR provision by requiring the banking sector to maintain a prescribed ratio of liquid (riskless) assets to deposits. This regulation restricts credit extension by the banking sector, limiting its exposure to risks and mitigating the impact on the central bank's balance sheet should those risks crystallize and LOLR provision has to be extended. The central bank chooses the required ratio with the objective of maximizing welfare conditional on there being zero expected cost of LOLR provision.

While LOLR insurance is welfare enhancing, the associated liquidity regulation may be costly for banks. Absent liquidity requirements, a profit-maximizing bank may have the incentive to free-ride on LOLR insurance and hold a lower stock of liquidity than would be consistent with a zero expected cost of LOLR provision. This is because funding is scarce in this model, and liquid assets are non–interest bearing. The scarcity of funds is modeled along the lines of Dewatripont and Tirole (1994), where an ex post moral hazard problem from the bank's side limits the amount of deposits the bank can raise.

The welfare benefit of LOLR insurance is modeled along the lines of Holmström and Tirole (1998). They examine the impact of liquidity shocks prior to realization of a production process and describe how an irrevocable line of credit—liquidity insurance—can improve welfare. However, Holmström and Tirole (henceforth H&T) assume an exogenously given liquidity shock, which in our model is derived endogenously on the basis of an information-induced bank run. That allows us to consider how the official safety net (LOLR policy complemented with liquidity regulation) influences the magnitude of the liquidity shock and vice versa. Another departure from the H&T framework is that credit lines in their model are extended free of charge, while here liquidity insurance is conditional on liquidity regulation.

By examining the trade-off between the welfare benefit of LOLR insurance and the cost of liquidity regulation, we follow the spirit of Buser, Chen, and Kane (1981), who consider banking regulations by the Federal Deposit Insurance Corporation (FDIC) as a condition for banks receiving deposit insurance and interpret the deadweight cost of regulatory rules as implicit insurance premia. But Buser, Chen, and Kane simply sketch a model of interaction between regulation and the official safety net, stopping short of analyzing the optimal FDIC response. Thus, they offer no insights into welfare implications, which is a key objective of our analysis.

We find that, in line with Posner (1971), who introduced the idea of taxation by regulation, whatever raises the desirability of LOLR activity also makes liquidity regulation more or less desirable ex ante. All things equal, we find that LOLR activity is more desirable the more capital a bank holds. That is because capital provides a cushion to the bank against losses, also offering the central bank some leeway to extend the LOLR insurance without violating its budget constraint. Another way to consider the welfare-enhancing role of LOLR, along with liquidity regulation, is in terms of the bank's leverage ratio. It turns out that LOLR insurance is more valuable when the ratio of bank deposits to bank equity is low. That is because, under low leverage, the bank has a weak incentive to undertake positive-net-present-value (positive-NPV) investment. By increasing the marginal expected return on investment, LOLR insurance increases the bank's marginal propensity to invest, which means that liquidity regulation is more likely to be welfare improving when the bank's leverage ratio is low.

Given the bank's capital ratio, LOLR insurance tends to become more valuable the riskier the liquidity shock that may hit the bank due to a run on deposits. In fact, the liquidity shock becomes riskier, and the LOLR insurance more valuable, the higher the extent of asymmetric information among deposits and the more volatile the bank's deposit base. For example, if the bank relies extensively on interbank funding, then the value of LOLR insurance could be higher than if the bank was raising funds through core deposits, which are generally perceived as more stable over time. In addition, a higher potential return per unit of loan investment increases the value of LOLR insurance and, as a result, the desirability of quid pro quo liquidity restrictions.

The rest of the paper is organized as follows. Section 2 describes the three-period model. Section 3 presents the benchmark case in which depositors have full information and there is no LOLR intervention or regulation. Section 4 introduces asymmetric information and characterizes the liquidity shock to the bank, and section 5 solves for the regulatory contract. Section 6 considers welfare implications of prudential liquidity regulation and LOLR insurance, and section 7 concludes. Proofs are included in the appendix.

2. Basic Environment

Consider an economy with three risk-neutral classes of agents: a bank, bank depositors, and a central bank that acts both as liquidity regulator to the bank and as an LOLR. The bank operates for three periods: $t = 0, 1, 2$. At $t = 0$, it holds capital A and raises a volume of uninsured wholesale deposits D. After deposits have been raised, the bank makes a risky loan investment I with constant returns to scale, and any unused funds l are placed in liquid assets that pay no interest.

The loan investment is risky because its payoff at $t = 2$ is binary, paying a gross return $R > 1$ per unit of investment if it succeeds and 0 otherwise. The probability of investment success is characterized by a random variable $\widetilde{\phi}$ with a uniform prior $U(1 - \bar{\phi}, \bar{\phi})$ that is interpreted as a systemic shock affecting the quality of the bank's assets. The bank also acts as a delegated monitor of its loan investment, undertaking an unobservable decision to manage it prudently or to engage in excess risk taking. Excess risk taking implies that the bank's residual claimholders receive a private benefit B per unit of investment if the bank does not fail, but the probability of loan success is scaled down by $\beta > 1$ to $\frac{\widetilde{\phi}}{\beta}$. Loan investment by the bank is assumed to have positive net present value, which implies the following parameter restriction.[1]

[1] Let ϕ_c be the minimum value of ϕ above which the bank is not liquidated and operates for two periods, and let $F(\phi)$ be the cumulative distribution of ϕ. Then, the expected return per unit of investment is $E(\widetilde{R}) = [1 - F(\phi_c)]\frac{\bar{\phi}+\phi_c}{2}R$. If $R \leq \frac{2(2\bar{\phi}-1)}{\bar{\phi}^2}$, then $E(\widetilde{R}) = 1 - \frac{\phi_c}{\bar{\phi}} < 1$, implying a negative-net-present-value investment.

ASSUMPTION 1. $R > \frac{2(2\bar{\phi}-1)}{\bar{\phi}^2}$.

Also, the risky investment is assumed to have a higher net present value when the bank does not engage in excess risk taking, which implies that $R - \frac{B}{\beta-1} > 0$.[2] In addition, we assume that due to moral hazard considerations à la Dewatripont and Tirole (1994), the bank is unable to raise an unlimited amount of deposits and may also need capital to finance its chosen level of investment. That is equivalent to assuming that $R - \frac{B}{\beta-1} < 1$.[3] Together, these inequalities lead to the following parameter restriction.

ASSUMPTION 2. $0 < R - \frac{B}{\beta-1} < 1$.

The bank makes its optimal choice over the amount of loan investment given the actions of two other classes of agents in the economy—namely, depositors and the central bank. A continuum of deposit managers—*depositors* for short—of total measure D are assumed to make no profits and to be able to monitor costlessly the realization of ϕ and manage a unit of wholesale deposits à la Rochet and Vives (2004). Depositors may decide to withdraw prematurely at $t = 1$ on the basis of private signals about ϕ and given their state-contingent payoffs. If a high proportion of depositors receive bad signals, then early withdrawals may occur on a large scale and the bank may face a *liquidity shortfall.* That is a situation where the bank does not have a sufficient amount of liquid assets to meet the withdrawal of deposits. But whether a signal is interpreted as bad or good will eventually depend not only on the realization of ϕ but also on what official measures are in place, as well as on depositors' beliefs about other depositors' actions.

[2]In the absence of excess risk taking and for a given continuation threshold ϕ_c, the expected return per unit of loan investment is given by $E(\widetilde{R}|\text{no excess risk taking}) = [1 - F(\phi_c)]\frac{\bar{\phi}+\phi_c}{2}R$. If the bank engages in excess risk taking, then the expected return per unit of investment becomes $E(\widetilde{R}|\text{excess risk taking}) = [1 - F(\phi_c)]\frac{\bar{\phi}+\phi_c}{2\beta}(R + B)$. It is easy then to show that $E(\widetilde{R}|\text{no excess risk taking}) > E(\widetilde{R}|\text{excess risk taking})$ if and only if $R - \frac{B}{\beta-1} > 0$.

[3]This inequality follows from $D < I$ and the incentive compatibility condition (3b) for bank deposits that we introduce in section 3.1.

Under the possibility of a bank run, the central bank sets its LOLR policy in a way that satisfies a twofold objective: first, to maximize social welfare, which in this model is equivalent to the expected surplus from bank lending, and second, to maintain a zero expected cost of LOLR intervention, i.e., to avoid losses from occurring on a systematic basis. In addition, the central bank is able to control the probability and the extent of a bank run by regulating the amount of cash that the bank holds in relation to deposits. LOLR policy is characterized by a level ϕ^{**} of the systemic shock ϕ such that the bank is bailed out if and only if $\phi \geq \phi^{**}$. In that sense, ambiguity about LOLR intervention does not emanate from strategic behavior by the central bank (i.e., it is not *constructive*); rather, it stems from uncertainty about the systemic shock.

LOLR intervention is assumed to occur via a capital injection,[4] whereby deposit withdrawals are met in full and the central bank acquires control of the bank, repaying all remaining deposits in the final period. Should the proceeds from realized loan returns be sufficient to cover all outstanding deposits, the central bank may make a profit; otherwise, it incurs a loss that could be financed through distortionary taxation.

The following sections examine the optimal behavior of the bank given the actions of depositors and the central bank. In section 3, depositors have full information about the quality of loans and so do not withdraw deposits prematurely as long as the bank is solvent. Therefore, the central bank does not operate an LOLR policy and, consequently, places no restrictions in the actions of the bank. In sections 4 to 6, we relax the assumption of full information and we introduce central bank intervention.

3. Full-Information Benchmark

We envisage a situation in which depositors have full information about the realization of ϕ and, as a result, the bank faces no

[4]In reality, central banks may require preferential terms under LOLR intervention, such as enhanced seniority and extra collateral. Such terms may enhance central banks' willingness to bail out ailing institutions but may also negate the catalyzing impact of the LOLR safety net on private creditors' incentives to roll over, crowding out private credit.

liquidity shortfall as long as it is solvent. Moreover, the bank faces no regulatory restrictions on the amount of liquid assets it can hold, and there is no LOLR insurance. We first solve for a critical level ϕ_0 of the systemic shock above which the bank is solvent and, as a result, it is not liquidated at $t = 1$. Then we solve for the bank's optimal investment and we characterize the bank's voluntary holdings of liquid assets.

Suppose that I_0 is the bank's chosen level of investment, l_0 is the amount of voluntary liquid holdings, and D_0 is the amount of deposits that the bank would be able to raise at $t = 0$. The solvency threshold ϕ_0 satisfies the breakeven condition

$$\phi_0 R I_0 = D_0 - l_0, \tag{1}$$

where the left-hand side of (1) is the expected return on investment, conditional on $\phi = \phi_0$, and the right-hand side is the amount of deposits that are not covered by liquid assets. Given that $I_0 + l_0 = A + D_0$, from equation (1) we find that

$$\phi_0 = \frac{I_0 - A}{R I_0}. \tag{2}$$

3.1 Voluntary Holdings of Liquid Assets

Let $m(I_0)$ be the marginal net expected return per unit of investment at a total level of investment I_0. From equation (2) it follows that $m(I_0)$ is decreasing in I_0 since the riskier the investment that the bank undertakes, the more vulnerable the bank becomes to a systemic shock and, as a result, the higher its probability of default. Let also $U(I_0) = m(I_0)I_0$ be the expected surplus of the bank's investment. Given that depositors and the central bank make no profits, the surplus of the bank's investment is equal to the bank's net expected utility from loan investment. Then, the optimal amount of investment I_0 solves the following:

$$\max_{I_0} U(I_0) \tag{3a}$$

subject to

$$\int_{\phi_0}^{\bar{\phi}} \phi(RI_0 - D_0)dF(\phi) \geq \int_{\phi_0}^{\bar{\phi}} \frac{\phi}{\beta}(RI_0 - D_0 + BI_0)dF(\phi) \tag{3b}$$

$$I_0 + l_0 = A + D_0. \tag{3c}$$

Inequality (3b) is the bank's incentive compatibility condition, (3c) is the bank's budget constraint, and ϕ_0 is the solvency threshold given by (2). For a given amount of deposits D_0, the optimal choice of I_0 is such that the bank has no incentive to take excessive risks for the private benefit B. Thus, the incentive compatibility condition (3b) can be written as follows:

$$D_0 \leq \left(R - \frac{B}{\beta - 1}\right) I_0. \tag{4}$$

By assumption 1, the bank's investment has positive net present value, and the cost of borrowing from depositors (normalized to 0) is less than the expected net payoff per unit of investment. Consequently, given risk neutrality, the bank always has the incentive to take on more deposits in order to increase its investment, which implies that the incentive compatibility condition (3b) binds

$$D_0 = \left(R - \frac{B}{\beta - 1}\right) I_0. \tag{5}$$

We may now show that the proportion of deposits that the bank keeps in liquid assets is given by the following result.

PROPOSITION 1. *Under full information and no central bank intervention, the liquidity ratio $\frac{l_0}{D_0}$ that the bank opts to maintain is given by*

$$\frac{l_0}{D_0} = 1 - \frac{1}{\left(R - \frac{B}{\beta-1}\right)} + \frac{R\sqrt{\frac{1}{R^2} - C_1^2}}{\left(R - \frac{B}{\beta-1}\right)}, \tag{6}$$

where $C_1 \equiv \sqrt{\bar{\phi}^2 - \frac{2(2\bar{\phi}-1)}{R}}$.

Proof. See the appendix.

Proposition 1 offers a consistent benchmark for comparison with the regulatory case under asymmetric information that we examine next. Moreover, under full information, any voluntary liquidity holdings can be considered *spare liquidity* in the presence of bank

Table 1. Depositors' Payoff Structure

	No Liquidity Shortfall	Liquidity Shortfall and LOLR	Liquidation
Withdraw	$\pi(1-k)$	$\pi(1-k)$	$\pi(1-k)$
Roll Over	π	$\pi(1-k)$	0

capital A, deposits D_0, and an optimal amount of loan investment I_0. Thus, a stock of liquidity under the full-information benchmark plays solely a residual role, rather than serving a deeper economic purpose, relating to the fact that for a given level of capital there are diminishing expected returns on investment since this increases the probability of default. In the following section, we relax the assumption of perfect observability of ϕ by depositors. We then characterize the liquidity shock that hits the bank at $t = 1$ in terms of depositors' equilibrium strategy following the realization of their private signals.

4. Liquidity Shock under Asymmetric Information

Let us assume that at $t = 1$ depositors observe private signals $s_i = \widetilde{\phi} + \widetilde{\varepsilon}_i$, where $\{\widetilde{\varepsilon}_i\}$ are i.i.d. innovations with uniform priors $U(-\varepsilon, +\varepsilon)$. To keep matters simple, we assume that depositors' payoffs are given exogenously in a way that captures one's incentive to run or stay with the bank if others act in the same manner. In particular, depositors receive a bonus π if they roll over and the bank faces no liquidity shortfall. But they receive a reduced bonus in the case of premature withdrawal of deposits for whatever reason, or if they fail to foresee a liquidity shortfall and shift their deposits elsewhere. Finally, in the case of bank liquidation, depositors who did not run lose their total bonus. See table 1.

Having observed signals $\{s_i\}$, depositors are assumed to follow a trigger strategy s^* that is defined as follows.

Definition 1. *A trigger strategy s^* is a rule of action that maps the realization of a depositor's signal s_i to one of the following actions: to withdraw if signal s_i is less than s^*, or to roll over if s_i is greater than or equal to s^*.*

Given the above structure of payoffs, it is easy to verify that a depositor's incentive to withdraw or to roll over increases with the proportion of other depositors undertaking the same action. That is, payoffs satisfy global strategic complementarities and, as shown by Morris and Shin (2002), if a trigger strategy s^* exists and is unique, then it is the only dominant solvable equilibrium strategy.[5] In the remainder of this section, we aim to describe the liquidity shock to a bank, defined as the proportion of depositors that withdraw given ϕ. A uniquely determined s^* allows us to characterize this shock not only in terms of the realized value of the systemic shock ϕ but also in terms of the general economic environment, such as the extent of prudential liquidity maintained by the bank, the central bank's LOLR policy ϕ^{**}, and the degree of noise ε in depositors' signals. Assuming that all model parameters, the LOLR policy ϕ^{**}, and the prior distribution of ϕ are common knowledge, and that the realized sample distribution of depositors is the common distribution of their signals $\{s_i\}$, we prove the following result.

LEMMA 1. *The critical level of systemic shock ϕ^* below which the bank faces a liquidity shortfall is given by*

$$\phi^* = s^* + \varepsilon\left(1 - 2\frac{l}{D}\right). \tag{7}$$

Proof. See the appendix.

We observe that ϕ^* decreases in the ratio of liquid assets to deposits and increases in the strategy s^* that is followed by depositors. ϕ^* also increases with the extent of asymmetric information ε among depositors, implying that the higher the dispersion of beliefs about ϕ, the more likely it is for the bank to face a liquidity shortfall. With ϕ^* in hand, we may now solve for the depositors' equilibrium trigger strategy.

[5] In other words, s^* is the only strategy that survives the iterated deletion of strictly dominated strategies. That typically requires non-empty upper and lower dominance regions—namely, the existence of a level of fundamentals above (below) which *all* depositors accept (refuse) to roll over.

LEMMA 2. *Depositors' equilibrium trigger strategy* s^* *is given by*

$$s^* = \phi^{**} + \varepsilon \left(1 - 2\frac{k}{1-k}\frac{l}{D}\right), \tag{8}$$

where ϕ^{**} *characterizes the central bank's LOLR policy and* $\frac{l}{D}$ *is the ratio of liquid assets to deposits that is maintained by the bank.*

Proof. See the appendix.

Equilibrium strategy s^* essentially captures depositors' reaction to news about $\widetilde{\phi}$. The lower the s^*, the less sensitive depositors are to bad news and vice versa. Equation (8) implies that the higher the extent of asymmetric information about $\widetilde{\phi}$, the less willing depositors are to roll over. Moreover, depositors' willingness to roll over increases with the bank's liquidity ratio $\frac{l}{D}$ and the extent of the central bank's readiness to extend the LOLR provision, as captured by ϕ^{**}.

There are also levels ϕ_U and ϕ_L of the systemic shock such that all depositors roll over if $\widetilde{\phi} \geq \phi_U$, or withdraw if $\widetilde{\phi} \leq \phi_L$. The range $[\phi_U, \bar{\phi}]$ is referred to as the *upper dominance region*, corresponding to realizations of fundamentals that are high enough to prevent any information-induced deposit withdrawals. Similarly, the range $[1 - \bar{\phi}, \phi_L]$ is the *lower dominance region*, where fundamentals are very weak and there is a massive outflow of deposits from the bank. It is easy to show that ϕ_U and ϕ_L are given by

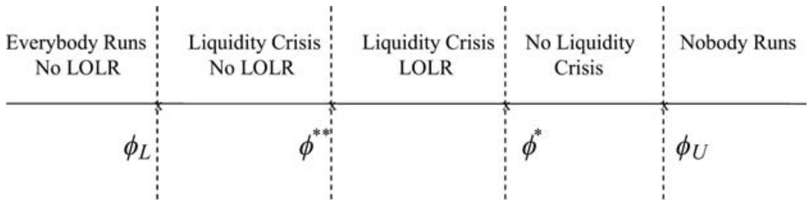
$$\phi_U = \phi^{**} + 2\varepsilon \left(1 - \frac{k}{1-k}\frac{l}{D}\right) \tag{9}$$

$$\phi_L = \phi^{**} - 2\varepsilon \frac{k}{1-k}\frac{l}{D}. \tag{10}$$

Conditional on the realization ϕ of the systemic shock, depositors' signals are drawn independently from a uniform distribution with support $[\phi - \varepsilon, \phi + \varepsilon]$. Then, for small values of $\varepsilon > 0$,[6] the standard global-game argument applies where depositors roll over their funds

[6]For a uniform prior in the real line, size limits for ε are dictated by the bounded support of $\widetilde{\phi}$ in the interval $[1 - \bar{\phi}, \bar{\phi}]$. So, we assume that $2\varepsilon < \min\{\phi_L - (1 - \bar{\phi}), \bar{\phi} - \phi_U\}$, where ϕ_U and ϕ_L are given by equations (9) and (10). We can easily show that such a condition holds if $2\varepsilon < \frac{1}{R} min\{R - \frac{B}{\beta - 1}, \bar{\phi}R - 1\}$.

Figure 1. Event Line Depending on Realization of $\widetilde{\phi}$



if and only if $s_i \geq s^*$, and the event line of the model looks as shown in figure 1.

For values of ϕ within the interval $[\phi_L, \phi_U]$, lemmas 1 and 2 imply that the proportion ρ of depositors who withdraw—the liquidity shock—is given by

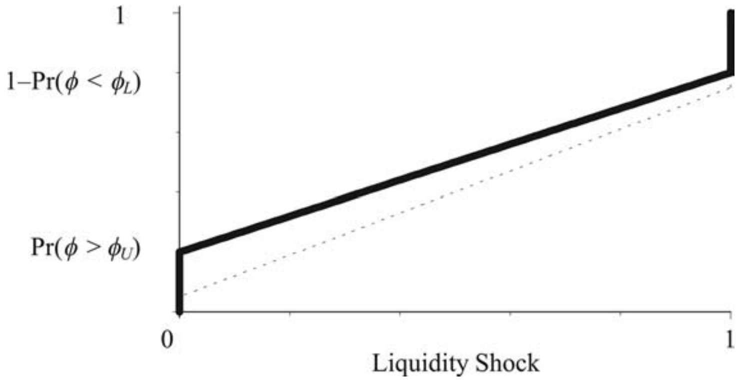
$$\rho(\widetilde{\phi}) = \frac{\phi^{**} + 2\varepsilon\left(1 - \frac{k}{1-k}\frac{l}{D}\right) - \widetilde{\phi}}{2\varepsilon}. \tag{11}$$

We observe that the liquidity shock $\widetilde{\rho}$ increases in ϕ^{**}; i.e., the more partial the LOLR insurance, the higher the proportion of depositors who run. Moreover, $\widetilde{\rho}$ decreases in the ratio of liquid assets to deposits, given that the higher the stock of liquidity, the more confident depositors are that the bank will not face a liquidity shortfall.

Finally, the uncertainty about the size of the liquidity shock $\widetilde{\rho}$ increases with the degree of asymmetric information among depositors, meaning that $\widetilde{\rho}$ becomes *riskier* the higher the ε. To see this, consider the probability distribution of $\widetilde{\rho}$, as implied by equations (9), (10), and (11). That distribution is uniform but with discrete jumps at 0 and 1 of size $\Pr(\phi > \phi_U)$ and $\Pr(\phi < \phi_L)$, respectively, as shown in figure 2 (bold line). In section 5.2 we prove that ϕ^{**} increases with ε, which implies that ϕ_U and ϕ_L also increase with ε. That said, an increase in ε leads to a decrease in $\Pr(\phi > \phi_U)$ and $\Pr(\phi < \phi_L)$ and, as a result, a downward shift in the distribution of $\widetilde{\rho}$ (dotted line in figure 2), meaning that $\widetilde{\rho}$ becomes riskier in a first-order stochastic-dominance sense.

In terms of welfare implications, recognizing that $\widetilde{\rho}$ becomes riskier with increasing information asymmetry is important because the riskiness of $\widetilde{\rho}$ will have an impact on the trade-off between the benefits of LOLR insurance and the costs of liquidity regulation. That is discussed in more detail in section 6.

Figure 2. Probability Distribution of Liquidity Shock ($\tilde{\rho}$)



5. Regulatory Contract

For the purposes of our analysis, the regulatory contract provides for an optimal LOLR policy and a ratio of liquid assets to deposits that needs to be maintained by the bank. As far as the LOLR policy is concerned, we need to solve for an intervention threshold ϕ^{**} that maximizes the expected surplus from bank lending, conditional on a zero expected cost of LOLR intervention. Such a ϕ^{**} corresponds to an optimal amount of loan investment by the bank, which is then implemented by a requirement for the bank to maintain a certain ratio of liquid assets to deposits.

The central bank's optimization problem at $t = 0$ can be written as follows:

$$\max_{\phi^{**}} U(\phi^{**}) \tag{12a}$$

subject to

$$\int_{\phi^*}^{\bar{\phi}} \phi(RI - D)dF(\phi) \geq \int_{\phi^*}^{\bar{\phi}} \frac{\phi}{\beta}(RI - D + BI)dF(\phi) \tag{12b}$$

$$\int_{\phi^{**}}^{\phi^*} [\phi RI - (1 - \rho(\phi))D]dF(\phi) \geq \int_{\phi^{**}}^{\phi^*} \left(\rho(\phi) - \frac{l}{D}\right) DdF(\phi) \tag{12c}$$

$$I + l = A + D. \tag{12d}$$

Inequality (12b) is the bank's incentive compatibility condition, (12c) is the central bank's breakeven condition, and (12d) is the budget constraint of the bank. Also, $\rho(\phi)$ is the proportion of depositors who run—the liquidity shock; l is the amount of liquid assets; and A is the amount of capital that the bank holds. In order to determine the central bank's optimal LOLR policy, we first need to establish what is the optimal amount of loan investment $I(\phi^{**})$ for a given LOLR policy ϕ^{**}.

5.1 Optimal Loan Investment

As with the no-regulation benchmark, the amount of deposits D is set before the bank's choice of its optimal investment and is such that the bank has no incentive to engage in excess risk taking for a private benefit B. Thus, the incentive compatibility condition (12b) binds and simplifies as

$$D = \left(R - \frac{B}{\beta - 1}\right) I. \tag{13}$$

We note that the ratio of loans to deposits is fixed and equal to $(R - \frac{B}{\beta-1})^{-1}$ regardless of the choice of investment amount I. Given also positive returns on investment and the fact that the central bank makes no profits, the breakeven condition (12c) of the central bank also binds and can be written as

$$\frac{\phi^* + \phi^{**}}{2} RI = D - l. \tag{14}$$

We are now able to prove the following result.

Lemma 3. *For a given LOLR policy ϕ^{**}, the amount of investment $I(\phi^{**})$ that satisfies simultaneously the bank's incentive compatibility condition (13), the central bank's breakeven condition (14), and the bank's budget constraint (12d) is given by*

$$I(\phi^{**}) = \frac{A\left(\frac{1}{R} - \frac{a\varepsilon}{R - \frac{B}{\beta-1}}\right)}{(C_2 - \phi^{**})}, \tag{15}$$

where $C_2 \equiv \frac{1}{R} - \varepsilon \frac{a + (1-a)\left(R - \frac{B}{\beta-1}\right)}{\left(R - \frac{B}{\beta-1}\right)}$ *and* $a \equiv \frac{1}{1-k}$.

Proof. See the appendix.

Having calculated the bank's optimal loan investment for a given LOLR policy ϕ^{**}, we may now evaluate the optimal LOLR policy ϕ^{**} that maximizes the expected surplus from the bank's loan investment.

5.2 LOLR Policy

Lemma 3 provides the optimal investment $I(\phi^{**})$ by the bank for a given LOLR policy ϕ^{**}. With $I(\phi^{**})$ in hand, the optimal LOLR policy maximizes the expected surplus from the bank's investment and is given by the following result.

PROPOSITION 2. *The central bank's optimal LOLR policy is to bail out the bank if and only if the level of systemic shock $\widetilde{\phi}$ is such that $\widetilde{\phi} \geq \phi^{**}$, where*

$$\phi^{**} = C_2 - \sqrt{C_2^2 - C_1^2}, \tag{16}$$

where $C_1 \equiv \sqrt{\bar{\phi}^2 - \frac{2(2\bar{\phi}-1)}{R}}$ *and* $C_2 \equiv \frac{1}{R} - \varepsilon \frac{a+(1-a)(R-\frac{B}{\beta-1})}{(R-\frac{B}{\beta-1})}$, *with* $a \equiv \frac{1}{1-k}$.

Proof. See the appendix.

An optimal LOLR policy ϕ^{**} could also be considered in terms of the induced probability of a liquidity shortfall at the optimum. In fact, lemmas 1 and 2 imply a one-to-one mapping from the central bank's optimal intervention threshold ϕ^{**} to the critical level of systemic shock ϕ^* below which a liquidity shortfall occurs. Consequently, by choosing an LOLR policy ϕ^{**}, the central bank implicitly induces a certain probability of liquidity shortfall in equilibrium. While this model is stylized, what this optimal policy captures is a balancing act whereby the central bank has to trade off welfare benefits of the official safety net against the intervention cost to the central bank. That is consistent with the principle of *proportionality*, which is widely used in the political debate about the extent and intensity of actions by the official sector.[7]

[7] Under the proportionality principle, the content and form of actions by the official sector shall not exceed what is necessary for the official sector to achieve its

5.3 Prudential Liquidity Ratio

Having established what is the optimal LOLR response to a liquidity shortfall by the bank, the corresponding optimal loan investment $I(\phi^{**})$ can be implemented by means of prudential liquidity regulation, i.e., by requiring the bank at $t = 0$ to maintain a ratio of liquid assets to deposits that is given by the following result.

PROPOSITION 3. *Under the optimal regulatory contract, the ratio of liquid assets to deposits that implements the optimal loan investment by the bank is given by the following expression:*

$$\frac{l}{D} = 1 - \frac{1}{\left(R - \frac{B}{\beta-1}\right)} + \frac{R\sqrt{C_2^2 - C_1^2}}{\left(R - \frac{B}{\beta-1} - aR\varepsilon\right)}, \tag{17}$$

where $C_1 \equiv \sqrt{\bar{\phi}^2 - \frac{2(2\bar{\phi}-1)}{R}}$ *and* $C_2 \equiv \frac{1}{R} - \varepsilon\frac{a+(1-a)(R-\frac{B}{\beta-1})}{(R-\frac{B}{\beta-1})}$, *with* $a \equiv \frac{1}{1-k}$. *Such a ratio is higher than what the bank would voluntarily maintain with no official-sector involvement.*

Proof. See the appendix.

Consequently, in the presence of bank funding constraints, prudential liquidity requirements imply an opportunity cost of funds, which is basically the price to be paid for quid pro quo LOLR insurance. We may also notice that the liquidity requirements increase with the extent of asymmetric information among depositors, which is not surprising given that the riskiness of the liquidity shock, as discussed in section 4, also increases. The question then becomes, Under what circumstances is it worth bearing such a cost? Or, to put it differently, Under what conditions is combining prudential liquidity regulation with LOLR insurance, if at all, welfare improving of the laissez-faire regime? The following section reflects on those questions by identifying a set of conditions under which prudential liquidity is warranted from a welfare perspective.

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policy objective. In our case, such an official-sector objective is the maximization of the surplus from financial intermediation, conditional on avoiding systematic losses under the LOLR facility.

6. Welfare Analysis

From section 5, the expected surplus U of the bank's investment under the optimal regulatory contract is given by

$$U = \frac{A}{2(2\bar{\phi} - 1)} \frac{\left(C_1^2 - \phi^{**2}\right)}{\left(C_2 - \phi^{**}\right)} \left(1 - \frac{aR}{R - \frac{B}{\beta - 1}} \varepsilon\right). \tag{18}$$

By substituting the optimal LOLR policy ϕ^{**} from proposition 2 into equation (18), we get

$$U = \frac{A}{2\bar{\phi} - 1} \left(C_2 - \sqrt{C_2^2 - C_1^2}\right) \left(1 - \frac{aR}{R - \frac{B}{\beta - 1}} \varepsilon\right). \tag{19}$$

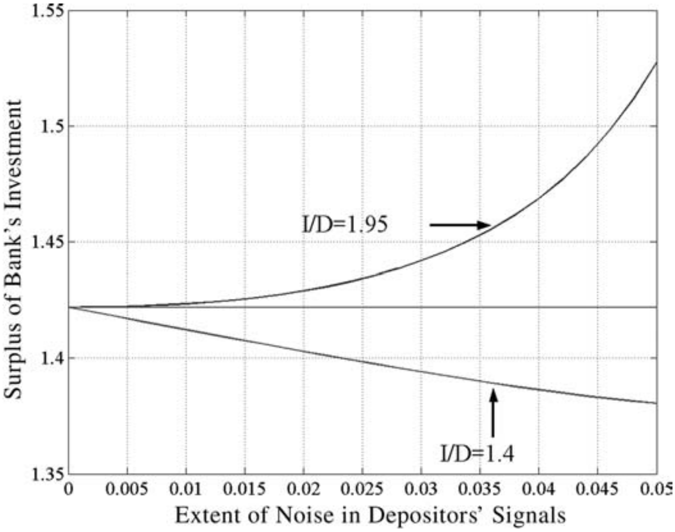
Similarly, from section 3, the expected surplus U_0 of the bank's investment, absent any official-sector involvement, is given by

$$U_0 = \frac{A}{2\bar{\phi} - 1} \left[\frac{1}{R} - \sqrt{\frac{1}{R^2} - C_1^2}\right]. \tag{20}$$

From equations (19) and (20) it follows that, as asymmetric information among depositors dissipates, the expected surplus of the bank's investment under the optimal regulatory contract tends toward its level under the no-regulation benchmark. However, for $\varepsilon > 0$, liquidity regulation may or may not lead to a welfare improvement of the no-regulation case. As we show next, that depends on the extent of the bank's funding constraints as measured by the ratio of loan investments to deposits. In figure 3, for example, we plot the expected surplus of loan investments against different values of ε and for a ratio of loans to deposits equal to 1.4 and 1.95, while the horizontal line corresponds to the expected surplus under the no-regulation benchmark.

As figure 3 shows, prudential liquidity regulation may become socially desirable the more debt constrained the banking sector is, i.e., the higher the ratio of loans to deposits. Otherwise, a requirement for the bank to hoard liquidity may be too costly, even after taking into account the social value of the LOLR safety net. Next,

Figure 3. Welfare Implications of Liquidity Regulation



we derive a threshold for the ratio of loans to deposits above which liquidity requirements, quid pro quo for LOLR insurance, lead to a welfare improvement of the no-regulation case.

Proposition 4. *For small values of* ε*, liquidity regulation improves welfare, relative to the no-regulation benchmark, if and only if the bank's loans-to-deposits ratio is such that*

$$\frac{I}{D} > \frac{a-1}{a\left(1-\sqrt{1-R^2C_1^2}\right)}, \tag{21}$$

where $C_1 \equiv \sqrt{\bar{\phi}^2 - \frac{2(2\bar{\phi}-1)}{R}}$ *and* $a \equiv \frac{1}{1-k}$.

Proof. See the appendix.

Proposition 4 results from a trade-off between the benefit of LOLR insurance and the cost of hoarding liquid assets for regulatory purposes. Whatever raises the desirability of LOLR activity, it also makes liquidity regulation more or less desirable ex ante. As in Rothschild and Stiglitz (1970, 1971), liquidity insurance becomes more valuable the riskier the liquidity shock from premature withdrawal of deposits. The higher the extent of asymmetric information ε among depositors, the riskier the liquidity shock and, as a result, the more valuable the LOLR insurance. Also, the value of LOLR insurance increases with the prima facie incentive of depositors to withdraw, which is captured by parameter k in depositors' payoffs. The lower the value of k, the less stable the deposit base and the easier it is for (21) to be satisfied. By the same token, the higher the potential return R on investment, the easier it is for (21) to hold and the higher the benefit of LOLR insurance relative to the cost of liquidity regulation.

We also stated in section 5.1 that the ratio of loans to deposits is fixed and depends on parameters B and β, which determine the bank's moral hazard problem. The stronger the moral hazard incentive of the bank—i.e., the higher the B and the lower the β—the lower the amount of deposits that the bank can raise relative to assets, which makes it easier for (21) to hold and for liquidity regulation to be welfare improving. To put it differently, when moral hazard incentives are strong, the ratio of external to internal funding of the bank is low. Proposition 4 can then be interpreted in terms of the bank's *leverage ratio*—namely, the ratio of deposits to bank's equity. By substituting equations (12d) and (17) into (21), it turns out that prudential liquidity restrictions, accompanied by an optimal LOLR policy, are welfare improving if the leverage ratio is sufficiently low.

The intuition that underlies this point is subtle: Given that the marginal expected return on investment decreases with the total amount of loan investment since this increases the probability of default, low leverage implies weak incentives to undertake positive-NPV investments. But LOLR insurance increases the marginal expected return on investment and, as a result, the bank's marginal propensity to invest in positive-NPV projects. That makes liquidity regulation more likely to be welfare improving exactly when the leverage ratio of the banking sector is low. Lower bank leverage also implies a higher level of bank capital relative to investment. But the higher the capitalization of the banking sector, the higher the cushion against losses, which also offers some leeway to the central bank to extend the LOLR insurance without violating its budget constraint. That said, the lower the leverage ratio—or the higher the capital cushion—the lower the liquidity requirements and, as a result, the more likely it is for central bank intervention to be warranted.

7. Conclusion

We presented a model of information-based bank runs where prudential liquidity requirements of the banking sector allow the central bank to run a balanced budget under its LOLR facility. In the presence of bank funding constraints and diverse beliefs among depositors about the quality of bank loans, it was shown that optimal liquidity regulation implies a ratio of liquid assets to deposits that is higher than voluntary liquidity holdings, absent any official-sector involvement. Such a higher liquidity ratio is to enable the banking sector to meet liquidity shocks from its own resources and up to a certain level of confidence. In other words, we viewed stock liquidity requirements as serving as a first line of defense against deposit withdrawals, allowing the central bank to maintain a zero expected cost of LOLR intervention, while counteracting any excess risk taking by the bank due to the existence of LOLR insurance. It was shown that prudential liquidity is warranted, from a welfare perspective, if the banking sector is characterized by high-profit opportunities, low leverage, and a relatively volatile deposit base (e.g., interbank funding, as opposed to funding through core deposits). Otherwise, stock liquidity becomes too costly, even after accounting for the social value of LOLR insurance.

Appendix

Proof of Proposition 1. Equation (2) implies that the net expected return $m(I_0)$ per unit of investment is

$$m(I_0) = \frac{R}{2(2\bar{\phi} - 1)} \left[C_1^2 - \left(\frac{I_0 - A}{RI_0} \right)^2 \right], \tag{22}$$

where $C_1 \equiv \sqrt{\bar{\phi}^2 - \frac{2(2\bar{\phi}-1)}{R}}$. Then, the bank's expected utility U_0 from investment I_0 is given by

$$U_0 = m(I_0)I_0. \tag{23}$$

Given (22), (23) can be written as

$$U_0 = \frac{R}{2(2\bar{\phi} - 1)} \left[C_1^2 - \left(\frac{I_0 - A}{RI_0} \right)^2 \right] I_0, \tag{24}$$

where the amount of investment I_0 that maximizes U_0 in (24) solves

$$I_0 = \frac{A}{R\sqrt{\frac{1}{R^2} - C_1^2}}. \tag{25}$$

Given that $I_0 + l_0 = A + D_0$, (3b) and (25) imply that the amount of liquid assets l_0 that the bank would hold voluntarily is given by

$$l_0 = \left[1 - \frac{1}{\left(R - \frac{B}{\beta - 1} \right)} + \frac{R\sqrt{\frac{1}{R^2} - C_1^2}}{\left(R - \frac{B}{\beta - 1} \right)} \right] \left(R - \frac{B}{\beta - 1} \right) I_0. \tag{26}$$

Finally, from (3b) and (26), the ratio of liquid assets to deposits under no official-sector involvement is

$$\frac{l_0}{D_0} = 1 - \frac{1}{R - \frac{B}{\beta - 1}} + \frac{R\sqrt{\frac{1}{R^2} - C_1^2}}{\left(R - \frac{B}{\beta - 1} \right)}. \tag{27}$$

Proof of Lemma 1. Conditional on systemic shock ϕ, depositors' signals $\{s_i\}$ are i.i.d. uniform with support on $[\phi - \varepsilon, \phi + \varepsilon]$. Given ϕ and depositors' equilibrium strategies s^*, the proportion of withdrawn deposits is equal to the probability that a signal s_i is lower than s^*:

$$\Pr(s_i \leq s^* | \phi) = \frac{s^* - \phi + \varepsilon}{2\varepsilon}. \tag{28}$$

From (28), the critical level of shock ϕ^* solves

$$\phi^* = s^* + \varepsilon \left(1 - 2\frac{l}{D} \right). \tag{29}$$

Proof of Lemma 2. Conditional on signal $s_i = s^*$, let P_{00} be the probabilities of no liquidity shortfall and P_{10} be the probability of liquidity shortfall and LOLR intervention. P_{00} and P_{10} are given by

$$P_{00} = \frac{s^* + \varepsilon - \phi^*}{2\varepsilon}. \tag{30}$$

By substituting ϕ^* from lemma 1 into equation (30), we get

$$P_{00} = \frac{l}{D}, \tag{31}$$

where l is the amount of liquid assets held by the bank and D is the total amount of deposits. Similarly, given ϕ^* and ϕ^{**}, P_{10} is given by

$$P_{10} = \frac{\phi^* - \phi^{**}}{2\varepsilon}. \tag{32}$$

With P_{00} and P_{10} in hand, depositors' equilibrium in trigger strategies s^* is such that a depositor who observes a signal $s_i = s^*$ is indifferent between withdrawing and rolling over; i.e., s^* solves

$$\pi P_{00} + \pi(1-k)P_{10} = \pi(1-k). \tag{33}$$

The left-hand side of (33) is the expected payoff from rolling over, conditional on s^*, while the right-hand side is the (certain) payoff from withdrawing. By substituting P_{00} and P_{10} from (31) and (32) into (33), we get

$$\frac{l}{D}\pi + \frac{\phi^* - \phi^{**}}{2\varepsilon}\pi(1-k) = \pi(1-k). \tag{34}$$

Then, by substituting ϕ^* from lemma 1 into equation (34), we derive s^*:

$$s^* = \phi^{**} + \varepsilon\left(1 - 2\frac{k}{1-k}\frac{l}{D}\right). \tag{35}$$

Proof of Lemma 3. By substituting (12d) into (14), we get

$$\frac{\phi^* + \phi^{**}}{2}RI = I - A. \tag{36}$$

From lemmas 1 and 2, the term $\frac{\phi^*+\phi^{**}}{2}$ on the left-hand side of (36) can be written as

$$\frac{\phi^* + \phi^{**}}{2} = \phi^{**} + \varepsilon\left(1 - \frac{1}{1-k}\frac{l}{D}\right). \tag{37}$$

By substituting (37) into (36) and setting $a \equiv \frac{1}{1-k}$, the central bank's breakeven condition becomes

$$\left[\phi^{**} + \varepsilon\left(1 - a\frac{l}{D}\right)\right] RI = I - A. \tag{38}$$

From the incentive compatibility condition (13) and budget constraint (12d), (38) becomes

$$(C_2 - \phi^{**})I = A\left[\frac{1}{R} - \frac{a\varepsilon}{R - \frac{B}{\beta-1}}\right], \tag{39}$$

where $C_2 \equiv \frac{1}{R} - \varepsilon\frac{a+(1-a)(R-\frac{B}{\beta-1})}{(R-\frac{B}{\beta-1})}$ and $a \equiv \frac{1}{1-k}$. Then, equation (39) implies that for a given ϕ^{**}, the amount of investment $I(\phi^{**})$ that satisfies simultaneously constraints (12b), (12c), and (12d) solves

$$I(\phi^{**}) = \frac{A\left(\frac{1}{R} - \frac{a\varepsilon}{R-\frac{B}{\beta-1}}\right)}{(C_2 - \phi^{**})}. \tag{40}$$

Proof of Proposition 2. For a given LOLR policy ϕ^{**}, the bank's net expected return per unit of investment is given by

$$m(\phi^{**}) = [1 - F(\phi^{**})]\frac{\bar{\phi} + \phi^{**}}{2}R - 1, \tag{41}$$

which can be restated as

$$m(\phi^{**}) = \frac{R}{2(2\bar{\phi} - 1)}\left(C_1^2 - \phi^{**2}\right), \tag{42}$$

where parameter $C_1 \equiv \sqrt{\bar{\phi}^2 - \frac{2(2\bar{\phi}-1)}{R}}$. Given (42), the central bank's optimization problem (12a) is equivalent to the following unconstrained problem:

$$\max_{\phi^{**}} U(\phi^{**}), \tag{43}$$

where $U(\phi^{**}) = m(\phi^{**})I(\phi^{**})$, $I(\phi^{**})$ is given by lemma 3, and $m(\phi^{**})$ is the net expected return per unit of investment. Given (42), (43) can be expressed as

$$\max_{\phi^{**}} \frac{A}{2(2\bar{\phi}-1)} \frac{\left(C_1^2 - \phi^{**2}\right)}{(C_2 - \phi^{**})} \left(1 - \frac{a\varepsilon R}{R - \frac{B}{\beta-1}}\right), \tag{44}$$

where $C_1 \equiv \sqrt{\bar{\phi}^2 - \frac{2(2\bar{\phi}-1)}{R}}$, $C_2 \equiv \frac{1}{R} - \varepsilon \frac{a+(1-a)(R-\frac{B}{\beta-1})}{(R-\frac{B}{\beta-1})}$, and $a \equiv \frac{1}{1-k}$. However, maximizing the expression in (44) is equivalent to maximizing the following expression of ϕ^{**}:

$$f(\phi^{**}) = C_2 + \phi^{**} - \frac{C_1^2 - C_2^2}{\phi^{**} - C_2}. \tag{45}$$

The first derivative of $f(\cdot)$ with respect to ϕ^{**} is

$$\frac{\partial f(\phi^{**})}{\partial \phi^{**}} = 1 + \frac{C_1^2 - C_2^2}{(C_2 - \phi^{**})^2}, \tag{46}$$

while the second derivative is given by

$$\frac{\partial^2 f(\phi^{**})}{\partial \phi^{**2}} = 2\frac{C_1^2 - C_2^2}{(C_2 - \phi^{**})^3}. \tag{47}$$

We consider two cases: (i) $C_2 \leq C_1$ and (ii) $C_2 > C_1$. However, we can easily show that $C_2 \leq C_1$ holds if and only if the bank's loans-to-deposits ratio is such that $\frac{I}{D} \geq \frac{1-(C_1-\varepsilon)R}{2R\varepsilon}$, which for small values of ε implies a high value of $\frac{I}{D}$. Nevertheless, given that the loans-to-deposits ratio under both the regulation and the no-regulation case is equal to $(R - \frac{B}{\beta-1})^{-1}$, the case where $C_2 \leq C_1$ can be easily ruled out by the fact that the ratio of liquid assets to deposits, as given by proposition 1, is non-negative. Thus, the only relevant case to consider is $C_2 > C_1$, under which (47) becomes negative and the optimal LOLR policy ϕ^{**} is given by

$$\phi^{**} = C_2 - \sqrt{C_2^2 - C_1^2}. \tag{48}$$

Proof of Proposition 3. Given $I + l = A + D$, the liquidity ratio $\frac{l}{D}$ can be expressed as

$$\frac{l}{D} = \frac{D + A - I}{D}. \tag{49}$$

By substituting (13) and (40) into (49), the ratio of liquid assets to deposits is given by

$$\frac{l}{D} = 1 - \frac{1}{R - \frac{B}{\beta-1}} + \frac{R(C_2 - \phi^{**})}{\left(R - \frac{B}{\beta-1} - aR\varepsilon\right)}, \tag{50}$$

where $C_2 \equiv \frac{1}{R} - \varepsilon \frac{a+(1-a)(R-\frac{B}{\beta-1})}{(R-\frac{B}{\beta-1})}$ and $a \equiv \frac{1}{1-k}$. Then, by substituting ϕ^{**} from proposition 2 into equation (50), the optimal liquidity ratio is given by

$$\frac{l}{D} = 1 - \frac{1}{\left(R - \frac{B}{\beta-1}\right)} + \frac{R\sqrt{C_2^2 - C_1^2}}{\left(R - \frac{B}{\beta-1} - aR\varepsilon\right)}. \tag{51}$$

Moreover, from proposition 1 and equation (51), the ratio of liquid assets to deposits under the optimal regulatory contract is higher than under no official-sector involvement if and only if

$$\frac{\sqrt{\left[\frac{1}{R} - \varepsilon \frac{a+(1-a)\left(R-\frac{B}{\beta-1}\right)}{\left(R-\frac{B}{\beta-1}\right)}\right]^2 - C_1^2}}{\left(R - \frac{B}{\beta-1} - aR\varepsilon\right)} > \frac{\sqrt{\frac{1}{R^2} - C_1^2}}{\left(R - \frac{B}{\beta-1}\right)}. \tag{52}$$

By recasting (52) we derive the following inequality:

$$\begin{aligned}\left(\varepsilon - \frac{K}{aR}\right)^2 < \frac{1}{(1-L)} \left(\frac{a + (1-a)K}{a}\right)^2 \\ \times \left(\varepsilon - \frac{K}{R[a + (1-a)K]}\right)^2 - \frac{K^2 L}{a^2 R^2 (1-L)},\end{aligned} \tag{53}$$

where $K \equiv R - \frac{B}{\beta-1}$, $L \equiv (RC_1)^2$, and $C_1 \equiv \sqrt{\bar{\phi}^2 - \frac{2(2\bar{\phi}-1)}{R}}$. With respect to ε, the geometric loci defined by the left-hand side and right-hand side of (53) are parabolas. The parabola defined by the left-hand side of (53) has its vertex at the point $[\frac{K}{aR}, 0]$, and its focal parameter is $p_{LHS} = \frac{1}{2}$. Similarly, the parabola defined by the right-hand side of (53) has its vertex at the point $[\frac{K}{R[a+(1-a)K]}, -\frac{K^2 L}{a^2 R^2 (1-L)}]$, and its focal parameter is $p_{RHS} = \frac{(1-L)}{2}\left[\frac{a}{a+(1-a)K}\right]^2$. From analytical geometry we know that if the focal parameter p of a parabola is positive, then the parabola faces upward. That is definitely the case for the left-hand side of (53)—i.e., $\frac{1}{2} > 0$—while for the right-hand side of (53), that is true if and only if $1 - L > 0$, or $C_1 < \frac{1}{R}$. But, given $C_2 > C_1$ and $C_2 < \frac{1}{R}$, it also follows that $C_1 < \frac{1}{R}$. Consequently, the parabola defined by the right-hand side of (53) is also facing upward. Finally, we observe that both parabolas intersect at $[0, (\frac{K}{aR})^2]$ and their vertices lie to the right of their intersection point. Thus, for relatively small values of ε, a sufficient condition for (53) to hold is $0 > -\frac{K^2 L}{a^2 R^2 (1-L)}$, or $1 - LR > 0$. But this has already been shown to be true, implying that the optimal regulatory contract stipulates a higher liquidity ratio than what the bank would voluntarily maintain under no official-sector involvement.

Proof of Proposition 4. Given that $U|_{\varepsilon=0} = U_0$, a necessary and sufficient condition for $U(\varepsilon) > U_0$ for small values of ε is that $\frac{\vartheta U(\varepsilon)}{\vartheta \varepsilon}|_{\varepsilon=0} > 0$. With a bit of algebra, we get

$$\left.\frac{\partial U}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{A}{(2\bar{\phi}-1)} \frac{(1-\sqrt{1-L})}{K\sqrt{1-L}}\left[a(1-\sqrt{1-L}) + (1-a)K\right], \quad (54)$$

where $K \equiv R - \frac{B}{\beta-1}$, $L \equiv R^2 C_1^2$, $C_1 \equiv \sqrt{\bar{\phi}^2 - \frac{2(2\bar{\phi}-1)}{R}}$, and $a \equiv \frac{1}{1-k}$. Then, a necessary and sufficient condition for $\frac{\vartheta U(\varepsilon)}{\vartheta \varepsilon}|_{\varepsilon=0} > 0$ is

$$\frac{I}{D} > \frac{a-1}{a\left(1-\sqrt{1-R^2 C_1^2}\right)}. \quad (55)$$

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