Monetary Policy under Uncertainty about the Nature of Asset-Price Shocks*

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The effects of an asset-price movement on inflation and output depend on whether that movement is fundamental or not. However, central banks cannot observe this. This paper examines the issue of how central banks should respond to asset prices given this constraint. Using a modified version of the Gruen, Plumb, and Stone (2005) model, this paper finds it is better to adopt a three-standard-deviation threshold rule for deciding whether to include asset prices in output-gap and inflation forecasts and monetary policy than to ignore asset prices altogether.

JEL Codes: E32, E52, E60.

1. Introduction

There is almost universal agreement in the monetary policy and asset-prices literature that central banks should take account of asset prices (house and equity prices), at least to the extent that they have implications for inflation. A key question that remains, though, is *how* a central bank should incorporate asset-price information in its inflation and output-gap forecasts and monetary policy decisions. One of the judgments central banks face in using asset-price information is deciding whether an asset-price movement is fundamental or not, i.e., justified by a change in the discounted risk-adjusted future stream of returns, or a misalignment or bubble, as the type

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of movement has different implications for output and inflation and therefore policy.

If there is a fundamental movement, the central bank does not need to adjust policy in response to the asset-price change, because there is no change in the output gap. For example, if asset prices are changing due to a technology shock that causes earnings streams to change, then potential output is changing. In this case, there will be no change in the output gap and no change in inflationary pressure in response to the asset-price change, because any demand increase via channels such as wealth effects or the financial accelerator is matched by an increase in the economy's potential supply.

On the other hand, if the asset-price change is nonfundamental, then potential output will remain unchanged, but demand will shift. This might occur, for example, due to destabilizing speculation and herding in a world of heterogeneous agents (see Delong et al. 1990). This type of movement will result in a change in the output gap and a change in inflationary pressures, which the bank is directly targeting. In this case, it will be necessary for the central bank to adjust policy if it wants to meet its inflation target.

A problem for central banks, though, is that they cannot observe whether an asset-price movement is fundamental or not, and views about the difficulty of judging this directly underpin opinions on the appropriate monetary policy response to asset prices. Bernanke and Gertler (1999) contend it is easier to estimate an output gap than to determine whether an asset-price movement is a bubble. They argue that central banks should ignore stock-price movements that do not appear to be generating inflationary or deflationary pressures. However, Cecchetti, Genberg, and Wadhwani (2002), while acknowledging that it is very difficult to identify small misalignments, argue that central banks can identify very large misalignments—for example, the Japanese stock and property market boom in the late 1980s and the U.S. stock boom in the 1990s—and should take account of these in policy.

One approach to this problem is for the central bank to use a rule to differentiate between asset-price movements and to adjust its forecasts and policy accordingly, rather than treating all asset-price changes in the same manner in policymaking decisions. The central bank never knows with certainty whether an asset-price shift is fundamental, but it does acquire information over time that can assist it in making a decision about whether the change is most likely fundamental or not. One way to use this information is to set a threshold for asset-price changes over which they will be regarded as most likely nonfundamental.

The issues to be examined in this paper include what the appropriate threshold should be and how sensitive this threshold rule is to the assumptions made about the asset-price bubble. To do this, I add a threshold rule to the Gruen, Plumb, and Stone (2005) model and run simulations. The Gruen, Plumb, and Stone model is chosen because it investigates in-depth some of the practical issues in using asset-price information in monetary policymaking. They assume that the central bank can recognize a bubble when it occurs, and they demonstrate that optimal policy depends on what the bank assumes about the nature of the bubble. In this paper I extend their work by assuming that the central bank cannot observe whether a price change is a bubble or not, and I illustrate how the bank's assumption about the nature of the bubble will also affect a threshold rule that the central bank uses to determine what part of an asset-price movement is most likely nonfundamental. The optimal threshold rule is defined as the one that results in the lowest welfare loss in terms of deviations of inflation from the central bank's target and output from potential.

The main findings of the paper include that the optimal threshold is sensitive to the assumptions about the asset-price bubble—in particular, the probability of the bubble bursting and the amount the bubble grows each period. In practice, it is difficult for a central bank to determine these parameter values, but the simulations show that welfare can be improved across almost the entire parameter range by using a threshold of three standard deviations of the asset price rather than by ignoring asset prices. Average welfare across the bubble parameter values can be improved further as the threshold is lowered, but this comes at the expense of a greater range of bubble parameter values over which the chosen threshold rule will lead to a greater loss compared with ignoring asset prices. This suggests that the more risk averse the central bank is, the higher the threshold it should use.

The next section discusses the model used in the simulations and the threshold rule in more detail. Section 3 then discusses the main results and is followed by the conclusion.

2. The Model

The model used in the simulations in this paper is a modified version of the Gruen, Plumb, and Stone (2005) model (GPS model). The GPS model is a two-equation (output gap and inflation), closed-economy model[1] (as per Ball 1999), with the addition of a stochastic asset-price bubble.

2.1 Asset Prices

The market price of an asset, am_t, is assumed to have several components:

$$am_t = af_t + a_t \tag{1}$$

$$= \overline{am_t} + a_t^* + a_t, \tag{2}$$

where af_t is the fundamental asset-price component of the market price. This fundamental component is the sum of $\overline{am_t}$, the long-run average of the market asset price, and a_t^*, the short-run deviation in the fundamental asset price from the long-run average market asset price. The final component of the market asset price is a_t, the deviation of the market asset price from the fundamental asset price, i.e., the bubble or nonfundamental component of the asset price.

The long-run average market asset price is observed by the central bank. The long-run average market price is the anchor for asset prices in the model and is set equal to 0 for convenience. The central bank can also observe the deviation of the market asset price from its long-run average, ap_t, but not its two components:

$$ap_t = am_t - \overline{am_t} \tag{3}$$

$$= a_t^* + a_t, \tag{4}$$

where a_t^* and a_t are the fundamental and nonfundamental parts of the deviation in the market asset price from its long-run average.

[1] The basic conclusions of the simulations will still apply to an open economy. In the open-economy case there would be an extra transmission channel from interest rates to output and inflation via the exchange rate. However, this would not affect the appropriate threshold, which depends on the bubble characteristics rather than the transmission channel for monetary policy.

a_t^* follows an AR(1) process:

$$a_t^* = \tau a_{t-1}^* + \varepsilon_{3t}, \tag{5}$$

where ε_{3t} is a white-noise error term with mean zero and variance σ_3^2. The autoregressive parameter, τ, could be very high so that fundamental deviations from the long-run asset price could be quite persistent.

The bubble, a_t, evolves according to the stochastic process

$$a_t = v_t(a_{t-1} + \gamma), \tag{6}$$

where v_t is a Markov chain and γ is the change in the asset-price level each period. From an asset-price perspective, there are two states of the world: no bubble ($v_t = 0$) and bubble ($v_t = 1$). The bubble can start in any period and can re-form after it has burst. Once the bubble forms, it has a probability p of bursting and probability $1 - p$ of continuing to grow. When $v_t = 0$ the bubble will form with the probability q and not form with probability $1 - q$.

If $v_{t-1} = 1$,

$$\begin{aligned} v_t &= 1 \text{ with probability } 1 - p \\ &= 0 \text{ with probability } p. \end{aligned}$$

Otherwise, if $v_{t-1} = 0$,

$$\begin{aligned} v_t &= 1 \text{ with probability } q \\ &= 0 \text{ with probability } 1 - q. \end{aligned}$$

The unconditional probabilities of each state are given by

$$\begin{aligned} pr(v_t = 1) &= \frac{1 - (1 - q)}{2 - (1 - q) - (1 - p)} \\ &= \frac{q}{q + p} \end{aligned} \tag{7}$$

and

$$pr(v_t = 0) = \frac{p}{p + q}. \tag{8}$$

2.2 Output and Inflation

The output gap is determined by

$$y_t = -\beta r_{t-1} + \lambda y_{t-1} + \iota a_t + \varepsilon_{1t}, \tag{9}$$

where y_t is the output gap, r_t is the deviation of the real interest rate from neutral,[2] ι is the fraction of wealth spent in the current period, a_t measures the size of the asset-price bubble, and ε_{1t} is a white-noise error term.

A key difference between this model and the GPS model is that ιa_t ($\iota < 1$) is added to (9) rather than adding Δa_t to (9). A fraction of the level of a_t rather than Δa_t is added to introduce different properties for the connection between asset prices and the output gap.

The GPS approach of adding Δa_t has a number of implications. First, consumption smoothing occurs entirely via the lag in y_t in (9), as the full amount of the change in the asset price flows through into the output gap. Second, if the bubble were to stop growing, but not collapse, then the output gap would return to zero even though the bubble was still in existence. Third, the economy will operate with a large amount of excess supply when the bubble bursts, as the full value of the bubble will be subtracted from the output gap. Effectively, the economy adjusts its spending in full and immediately to the bursting of the bubble.

By contrast, the model in this paper, which adds a fraction of the level of a_t, introduces an extra layer of consumption smoothing on top of the effect of the lag of y_t in (9), at least as the bubble forms. Adding the level also means the bubble could continue to have an effect on the output gap even if it stopped growing but did not burst. If the bubble bursts in the main simulations, then the positive effect of asset prices on the output gap will cease and the positive output gap will eventually reduce to zero through the term

[2] A simplification used here is that the central bank sets the real interest rate. In reality, it sets the nominal interest rate and, unlike the situation in this model, cannot always set the real interest rate as far below neutral as it wants, because it faces a zero lower bound for nominal interest rates. Robinson and Stone (2005) extend the Gruen, Plumb, and Stone (2005) model by incorporating a nominal interest rate and a zero lower bound. They find that this does impose a further constraint on policymaking, but it does not materially alter the basic conclusions of the Gruen, Plumb, and Stone model.

λy_{t-1}. However, it is possible that the bursting of a bubble could have more severe negative consequences for the real economy. In particular, after bursting, the bubble may overshoot its fundamental value, leading to a negative bubble, which pushes output below equilibrium. To explore this possibility and test the robustness of the main findings of the paper to the possibility of negative bubbles, further simulations are conducted in section 3.

Excess demand is expected to affect inflation with a lag. Inflation is determined by

$$\pi_t = \pi_{t-1} + \alpha y_{t-1} + \varepsilon_{2t}, \tag{10}$$

where π_t is the deviation of inflation from target and ε_{2t} is a white-noise error term.

The model is parameterized by Ball (1999) for the United States. The parameter values are $\alpha = 0.4$, $\beta = 1$, and $\lambda = 0.8$. The parameter values imply that each period in the model is a year in length. ι is set equal to 0.03, which represents the consensus view of the marginal propensity to consume out of wealth in the United States (Poterba 2000).

2.3 Monetary Policy

The central bank is assumed to be a flexible inflation targeter; i.e., its objective is to stabilize output around potential and inflation around its target. The central bank sets the interest rate to minimize the following loss function:

$$L_t = \sum_{t=0}^{\infty} \delta^t \left[E_t\left(y_{t+1}^2\right) + \mu E_t\left(\pi_{t+1}^2\right)\right], \tag{11}$$

where the central bank discounts future welfare by δ and μ is the weight on future deviations in inflation from the bank's target rate. The target inflation rate is assumed to be zero for convenience.

To attempt to achieve this objective, the central bank must have information on the output gap and inflation. As it can never observe the actual output gap or the bubble, it estimates the output gap and inflation according to

$$yf_t = -\beta r_{t-1} + \lambda yf_{t-1} + \kappa_t \iota ap_t \tag{12}$$

$$\pi f_t = \pi_{t-1} + \alpha yf_{t-1}, \tag{13}$$

where yf_t and πf_t are the central bank's estimates of the output gap and inflation, respectively. κ_t is an indicator variable:

$$\begin{aligned}\kappa_t &= 1(ap_t > \psi\sigma_{ap_t})\\ &= 0 \text{ otherwise,}\end{aligned}$$

where σ_{ap_t} is the standard deviation of the difference between the long-run and market asset prices and ψ is the threshold number of standard deviations chosen by the central bank. The central bank uses σ_{ap_t} in its threshold rule because, unlike the bubble, it can observe this quantity.

The central bank takes account of asset prices in its output-gap forecasts in a nonlinear way. If the deviation of the asset price from the long-run asset price is greater than ψ standard deviations, κ_t is set to unity, and the central bank judges that the asset-price shift is most likely a bubble and relevant to the output-gap forecast. It then includes a fraction of the deviation from the long-run asset price, ap_t, in its forecast of the output gap.[3]

The optimal interest rate rule for a central bank that can identify bubbles and takes account of them, and where the bubble is independent of monetary policy, is a function of the variables in the system and the expected effect of asset prices on the output gap in the next period, given the state of the bubble in this period.

[3]Rather than using a threshold rule, the central bank could also make inferences about the true output gap and whether there is an asset-price bubble by comparing its forecast of inflation from equation (13) with the inflation outturn. If the bank repeatedly observed that actual inflation was above forecast, this would suggest that the actual output gap was higher than estimated, indicating the presence of a bubble, as a repeated error would be unlikely to arise from shocks to inflation, ε_{2_t}. Simulations were run comparing the threshold rule with an inflation-errors rule where the central bank would consider the asset-price movement a bubble when actual inflation exceeded forecast inflation for five years. The results showed that the three-standard-deviation threshold rule generated lower welfare losses across a much wider parameter range for the bubble than an inflation-errors rule. The parameter values where the welfare loss was lower using the threshold rule were identical to those where the threshold rule had lower welfare losses ignoring asset prices, except where $p = 0.1$, where the inflation rule generated lower welfare losses. This result is perhaps not surprising because the inflation-errors test involves making a more indirect inference about whether there is an asset-price bubble than the inference made from observing total asset-price changes.

A standard result for linear-quadratic[4] dynamic programming problems such as finding the optimal monetary policy in this model is that the optimal interest rate will be a linear function of the variables in the system and any known series that affects the system, in this case the expected effect of a_t on the output gap (see Chow 1973, Bertsekas 2000, and appendix 1). An assumption of this solution method is that asset prices are exogenous in the model (there is no feedback from the economy or interest rates to asset prices) and so they can be treated as predetermined in the dynamic programming problem. Gruen, Plumb, and Stone (2005) also show that the difference between the standard optimal interest rate rule used by a central bank that ignores asset-price movements and the one used by a central bank that takes account of them is the expected effect of the asset-price bubble on output in the next period.

Interest rates affect output with a lag, so to optimally take account of asset prices, the central bank sets interest rates in this period so they will offset the expected effect of asset prices on the output gap in the next period. The expected effect of asset prices on the output gap next period depends on the way in which asset prices are assumed to affect the output gap in (9). If Δa_t is used, as in the GPS model, then the optimal rule is

$$r_t = \phi_1 y_t + \phi_2 \pi_t + \phi_3 (v_t((1-p)\gamma - pa_t) + (1-v_t)(q\gamma)). \qquad (14)$$

Because the output gap is a function of Δa_t, the expected effect of asset prices on the output gap next period, given the bubble state, is equal to $E(\Delta a_{t+1}|v_t)$. If $v_t = 1$ (i.e., there is a bubble), then with probability $(1-p)$ the bubble will still be in existence next period and Δa_{t+1} will equal γ; otherwise, with probability p the bubble will burst and Δa_{t+1} will equal $-a_t$. The expected effect of asset prices on the output gap next period, given a bubble this period, $E(\Delta a_{t+1}|v_t = 1)$, is equal to $(1-p)\gamma - pa_t$.

If $v_t = 0$, then with probability q a bubble will form and Δa_{t+1} will equal γ (the bubble will be γ in size in the first period of its existence), and with probability $1-q$ no bubble will form and Δa_{t+1} will equal 0. The expected effect of asset prices on the output gap

[4]The system of equations is linear and the objective function is quadratic in variables that are in the system.

next period, given no bubble this period, $E(\Delta a_{t+1}|v_t = 0)$, is equal to $q\gamma$.

However, if a_t affects the output gap as in the modified model I use in this paper, then the rule is

$$r_t = \phi_1 y_t + \phi_2 \pi_t + \phi_3(v_t \iota(1-p)(a_t+\gamma) + (1-v_t)((\iota q\gamma)). \quad (15)$$

The output gap is a function of a_t, so the expected effect of asset prices on the output gap next period, given the bubble state, is equal to $E(a_{t+1}|v_t)$. If $v_t = 1$, then with probability $(1-p)$ the bubble will still be in existence next period and a_{t+1} will equal $a_t + \gamma$, and with probability p the bubble will burst and a_{t+1} will equal 0. The expected effect of asset prices on the output gap next period, given a bubble this period, $E(a_{t+1}|v_t = 1)$, is equal to $\iota(1-p)((a_t+\gamma))$ (multiply by the fraction ι because only a fraction of wealth is consumed in each period).

If $v_t = 0$, then with probability q a bubble will form and a_{t+1} will equal γ, and with probability $1-q$ no bubble will form and a_{t+1} will equal 0. Given no bubble this period, the expected effect of asset prices on the output gap next period will be $\iota q\gamma$.

In terms of the monetary policy framework in the paper, ϕ_1, ϕ_2, and ϕ_3 are the weights on the information included in the rule. In the simulation exercise in this paper, these weights are chosen optimally by using the dynamic programming method to solve the linear-quadratic problem for the coefficients in the monetary policy rule (17). Further details are provided in appendix 1.[5]

Which of Δa_t or a_t is added to the output-gap equation will affect the optimal policy as the bubble builds. Adding a_t, as is done in the model in this paper, has the implication that the central bank will increase interest rates above neutral in response to a bubble. In contrast, when adding Δa_t, as is done in the GPS model, there is a point where the expected negative effect of the bubble bursting outweighs its positive effects and the central bank will want to reduce interest rates below neutral. Interest rates will be cut sharply in both models if the bubble bursts.

[5] ϕ_1 and ϕ_2 are obtained from vector G in appendix 1, while ϕ_3 is from vector g in appendix 1. $(v_t((1-p)\gamma - pa_t)+(1-v_t)(q\gamma))$ or $(v_t\iota(1-p)(a_t+\gamma)+(1-v_t)((\iota q\gamma))$ are equivalent to b_t in appendix 1. For the model parameters used here, the optimal weights are $\phi_1 = 1.09$, $\phi_2 = 0.72$, and $\phi_3 = 1$.

Figure 1 shows the results of simulating both the GPS (Δa_t) model and the new model in this paper (a_t) for ten periods, starting in period 1, with bubbles in each model scaled so that the effect of the asset-price bubble shock on the output gap y_t is of a similar magnitude. A bubble commences in period 1 and builds to a peak in period 5 (marked by the vertical dotted lines) and then bursts in period 6. The GPS results are in the left panels. The top left panel shows the deviation of the real interest rate from neutral, r_t, with the black line showing the interest rate set by a central bank that can identify bubbles and which takes account of their expected effect next period according to (14) (the activist). The grey line in the top left panel shows the policy of a central bank that expects no asset-price effect on the output gap next period (the skeptic) and sets their interest rate according to

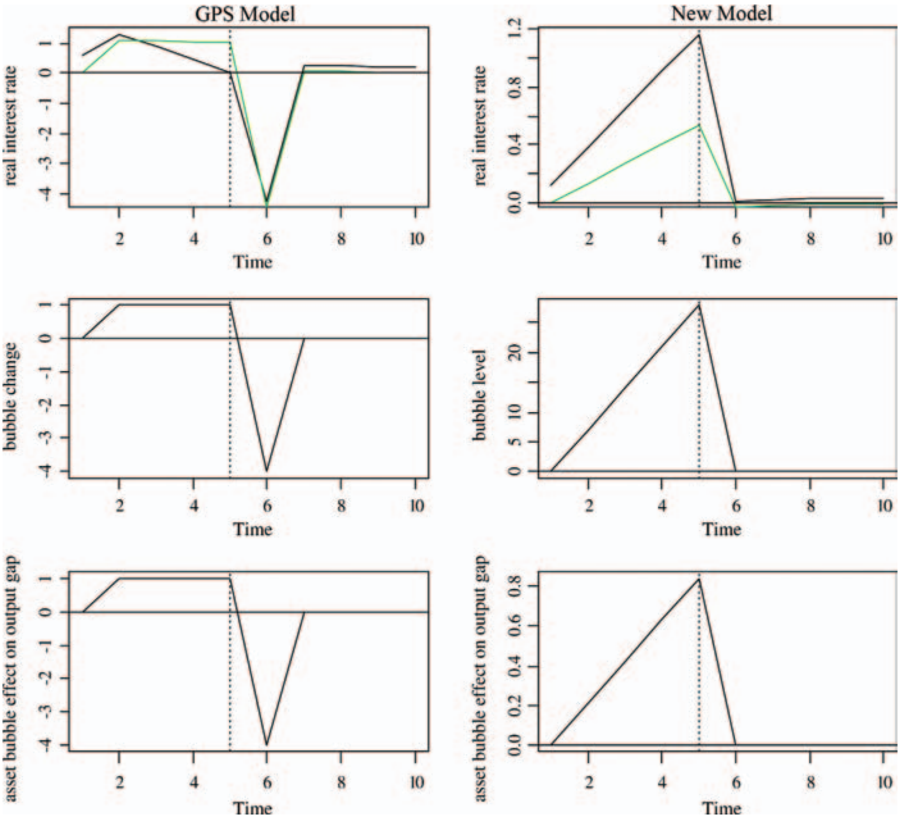
$$r_t = \phi_1 y_t + \phi_2 \pi_t. \tag{16}$$

The policy recommendations implemented in the economy are those of the skeptic central bank. The top right panel shows the equivalent results for the new model in this paper, where a_t is added to the output-gap equation, with activist policy being set according to (15). The middle left panel shows Δa_t, which is added to the output-gap equation in the GPS model, while the middle right panel shows a_t. The bottom panels show the effect of the asset-price bubble on the output gap in each model.[6]

In the GPS model, the activist central bank initially raises interest rates above neutral in response to the bubble, but then begins to moderate this tightening from period 2 onward as the expected negative effect of the bubble bursting begins to counter its ongoing positive effects. When the bubble bursts in period 6, interest rates are cut sharply. In the level (a_t) model in the right panel, the activist continues to tighten interest rates as the positive effect on the output gap from the bubble continues to grow. When the bubble bursts, the activist central bank also cuts interest rates sharply.

[6]In the GPS model, Δa_t is equivalent to the real shock to the economy from asset prices, while in the new model a_t is a component of asset prices, which has an effect on the real economy via wealth effects and depends on the fraction of wealth spent in each period, ι. To make the two shocks to the real economy similar in each simulation, γ_t is set equal to 1 in the GPS model and 7 in the new model.

Figure 1. Simulating the Effect of a Bubble on Monetary Policy



The optimal rules can be used by the central bank if it can identify bubbles. However, it is assumed in the simulations (except the base one) that this is not the case. Instead, the central bank uses the information that it does know, viz. the entire deviation from the long-run asset price, ap_t, and adopts the rule given by

$$rf_t = \phi_1 yf_t + \phi_2 \pi f_t + \kappa_t \phi_3 \iota (1 - pf_t)(ap_t + \gamma f_t), \qquad (17)$$

where pf_t is the bank's estimate of p and γf_t is the bank's estimate of γ. As described above, if the market asset-price deviation from the long-run asset price is more than a certain threshold number of standard deviations, then the central bank will regard it as most likely a bubble, i.e., $\kappa_t = 1$. In this case, the bank will be concerned about the effects on inflation and will want to take account of its estimate of the expected effect of the asset-price bubble on the output gap next period, $(1 - pf_t)\iota ap_t + \gamma f_t$. The bank estimates the probability of the bubble bursting, pf_t, from the average length of periods it has regarded as bubbles in the last 100 years:[7]

$$pf_t = \frac{1}{\overline{bl}_t}, \tag{18}$$

where $\overline{bl}_t$ is the average bubble length in the last 100 years.

The bank estimates the size of the change in the bubble from the last change in ap_t:

$$\gamma f_t = ap_t - ap_{t-1}. \tag{19}$$

The bank will only use this estimate when it considers a bubble to be in existence and most of the movement in ap_t to be due to the bubble.

2.4 The Threshold Rule

The central bank uses a threshold rule for deciding whether an asset price is most likely a bubble because it cannot observe whether an asset-price movement is a bubble or not. The threshold rule is used on the grounds that if the asset-price shift is sufficiently distant from the long-run asset price, it is unlikely to represent a fundamental shift and therefore is most likely a bubble.

The deviation of the market price from the long-run asset price, ap_t, is the sum of a_t^*, the short-run deviation in the fundamental asset price from the long run, and a_t, the bubble, so the variance of ap_t used in the threshold rule is a function of the variance of both a_t^* and a_t.

[7] Results from simulations where the central bank uses only the last twenty-five years of asset-price data to calculate the probability of the bubble bursting and the variance of ap_t (see further on in the main text) were similar to the full simulation results. In particular, the main result that using the three-standard-deviation rule performs better than ignoring asset prices across most of the parameter range for variables that affect the variance of a_t relative to ap_t remains.

The mean and variance of ap_t, a_t^*, and a_t are given by the following (for details see appendix 2):

$$E(ap_t) = E\left(a_t^*\right) + E(a_t), \tag{20}$$

where

$$E\left(a_t^*\right) = 0 \tag{21}$$

$$E(a_t) = \frac{q\gamma}{(q+p)p} \tag{22}$$

$$var(ap_t) = var\left(a_t^*\right) + var(a_t)\left(a_t^* \text{ is independent of } a_t\right), \tag{23}$$

where

$$var\left(a_t^*\right) = \sigma^2/(1-\tau^2) \tag{24}$$

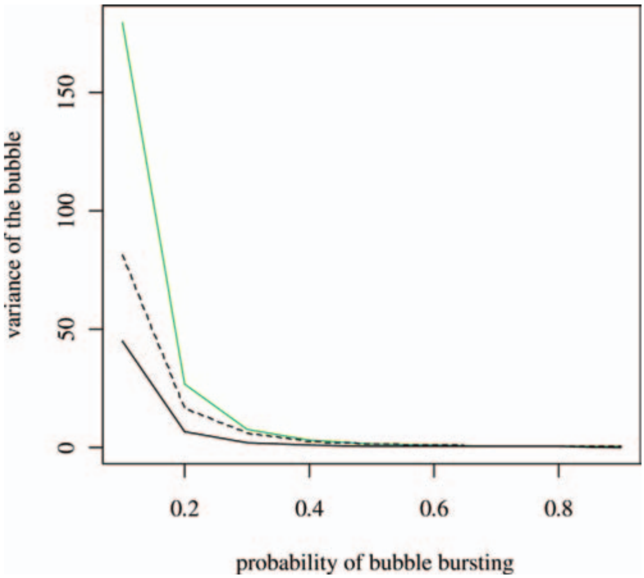
$$var(a_t) = \frac{q}{q+p}\left[\frac{\gamma^2(2-p)-\gamma^2}{p^2}\right]. \tag{25}$$

The variance of a_t is dependent on the probability of the bubble bursting, p, in a nonlinear way. An increase in p will reduce the variance of a_t and vice versa. In some ranges for p, a relatively small change in p will have a large effect on the variance of a_t. An increase in γ leads to a proportional shift up in the variance at every p. In figure 2 the solid black line is drawn for $q = 0.1$ and $\gamma = 1$. An increase in γ to 2 shifts the variance up to the grey line, where the variance at every p is $\gamma^2 = 4$ times larger than when $\gamma = 1$.

In contrast to the increase in γ, an increase in the probability of the bubble forming, q, has less effect on the variance of a_t. In figure 2, a large increase in q from 0.1 to 0.9, holding γ equal to 1, shifts the variance curve up from the solid line to the dashed line. The increase is proportionally larger the lower p is. For example, at $p = 0.9$ and $\gamma = 1$, the increase in q from 0.1 to 0.9 increases the relatively small variance by a factor of 5, while if $p = 0.1$, the variance only increases by a factor of 1.8.

Any change in a_t will be reflected in a change in ap_t; however, the bank does not know it is a_t that is changing. The bank chooses a volatility threshold beyond which it considers the movement in ap_t to be nonfundamental. This threshold is a number of standard deviations of ap_t. Up until this threshold, the bank ignores the changes

Figure 2. The Variance of the Bubble, a_t, versus the Probability of the Bubble Bursting, p



Note: $q = 0.1, \gamma = 1$ (solid black line); $q = 0.9, \gamma = 1$ (dashed line); $q = 0.1, \gamma = 2$ (grey line).

in ap_t, considering them most likely to be changes in fundamental values or noise. Beyond this threshold, the asset-price movement is regarded as nonfundamental. In this case, the bank takes account of ap_t in its estimates of y_t, i.e., in yf_t and in its optimal interest rate rule (i.e., $\kappa_t = 1$).

In choosing a threshold number of standard deviations, there is a trade-off between catching the bubble early and mistaking a fundamental asset-price movement for a nonfundamental one and acting when it is not necessary to do so. If a relatively low threshold is adopted, then the central bank will be more likely to catch bubbles early—before they send the economy a long way from equilibrium—but it is also more likely to make mistakes about asset-price bubbles. If the threshold is high, the bank will not catch the bubble as early but is less likely to make a mistake confusing a nonfundamental and fundamental price movement.

Given this trade-off, what might the appropriate threshold and its determinants be for the central bank to use when determining whether an asset-price movement is most likely a bubble or not? In the next section, simulation exercises are conducted to investigate these issues.

3. Simulation: Monetary Policy under Uncertainty about Asset-Price Shocks

In this section the model described above is simulated to determine what threshold will be appropriate and how sensitive the threshold choice is to the asset-price bubble parameters. The initial values for the model are $a_0^* = v_0 = y_0 = \pi_0 = a_0 = 0$; i.e., the economy is equilibrium, the output gap is 0, inflation is equal to target, and the market asset price is equal to its long-run average. Inflation and output have equal weight in the loss function; i.e., $\mu = 1$.

The model is simulated across different thresholds with varying parameter values for p, the probability of the bubble bursting, and γ, the amount the bubble grows each period. These two parameters are varied because, relatively, they have the most influence on the variance of a_t and, through this, the optimal threshold. Other parameters that affect the variance of ap_t—including q, the probability of a bubble forming, and τ, the autoregressive coefficient in the short-run fundamental asset-price equation—are fixed.[8] Changing q and τ, which is examined in the extensions subsection below, does not materially alter the main conclusions.

The model is simulated for 11,000 periods, and the first 1,000 values are discarded to remove any influence of initial values. The first step is to generate shocks to a_t^*, y_t, and π_t of 11,000 periods in length. The values of γ and p are fixed, and the model is simulated assuming the central bank cannot observe bubbles and uses monetary policy rules, which vary by the threshold number (one, two, three) of standard deviations of ap_t that the asset price must move before the central bank will consider it a bubble.

In reality, the central bank only has a limited amount of asset-price data to assess the variance of ap_t, so in the simulation the

[8]The assumed parameters are $q = 0.1$ and $\tau = 0.9$.

bank only uses the previous 100 years of ap_t to calculate the variance. This introduces some error into the estimation of $var(ap_t)$. Simulations where the central bank ignores asset-price movements completely in setting interest rates, and a base simulation where the central bank can identify bubbles, are also conducted. The different monetary policy rules are compared by calculating the welfare loss as measured by the bank's objective function (11). The values of γ and p are then changed, and the model is simulated again with the varying thresholds.

The welfare losses from using each monetary policy rule are shown across a variety of parameter values in table 1 and in the following figures.[9] Table 1 expresses the welfare losses as a ratio of the loss from following a particular rule to the loss if the central bank could identify bubbles and reacted optimally to them.

Comparing the three threshold-based monetary policy rules, the results in figure 3 show the threshold number of standard deviations of ap_t that will deliver the minimum welfare loss across a range of parameter values for the bubble. The parameter values vary from 1 to 8 for γ and 0.1 to 0.9 for p.[10] As can be seen from the plot, the threshold that delivers the minimum welfare loss depends on both p and γ. When p is high and γ is low—for example, $p = 0.9$

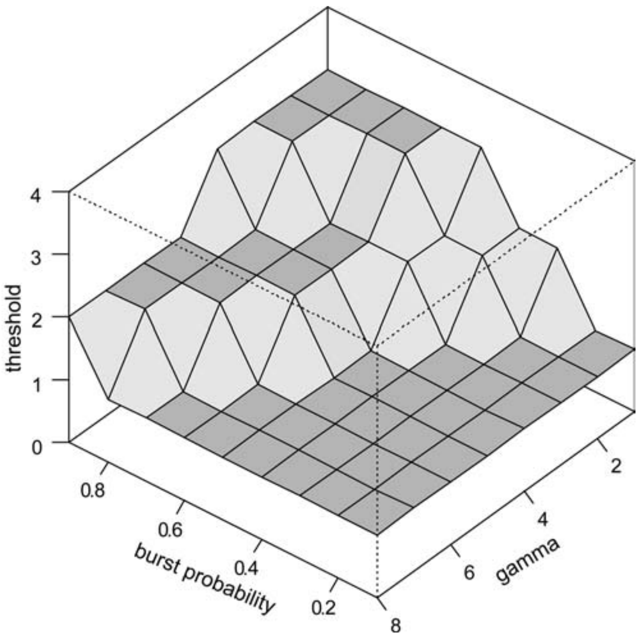
[9] A full set of results across the entire parameter range considered in the simulations is provided in appendix 3.

[10] With this parameter range, the expected effect of bubbles on the output gap ranges from 0.03 percent ($p = 0.9, \gamma = 1$) to 2.4 percent of GDP ($p = 0.1, \gamma = 8$), although actual bubble effects could be much larger in some cases. The larger expected shocks are around the maximum size of the U.S. output gap over the 1987 to 2007 period, when the output gap moved in a range of around ± 2 percent (OECD Economic Outlook database estimates). More importantly, the parameter range chosen for p and γ allows the testing of the various rules across a very wide range for the variance of the bubble relative to the total asset-price movement, as it is this relative variance that is the key determinant of the appropriate standard-deviation threshold for considering asset-price movements. In the simulations, the asset-price bubble variance as a percentage of the total asset-price variance ranges from less than 1 percent of the total asset-price variance ($p = 0.9, \gamma = 1$) to over 99 percent ($p = 0.1, \gamma = 8$). The results in the paper are robust to changing the range of output-gap effects that may be of interest to a central bank. For example, if ι (the marginal propensity to consume out of wealth) was increased, this would increase the size of the expected effect of the asset-price bubbles on the output gap at each γ and p, while not changing the overall results, as the relative variance of the bubble to the total asset price at each γ and p combination would remain the same.

Table 1. Welfare-Loss Ratio (Threshold/Optimal)

p	One Standard Deviation	Two Standard Deviations	Three Standard Deviations	Ignore Asset Price
		$\gamma = 1$		
0.1	4.58	11.85	40.80	50.20
0.2	2.02	3.03	5.39	10.43
0.3	1.81	1.62	1.90	2.50
0.4	1.78	1.36	1.43	1.57
0.5	1.80	1.29	1.18	1.20
0.6	1.79	1.28	1.16	1.13
0.7	1.78	1.23	1.10	1.07
0.8	1.77	1.22	1.08	1.05
0.9	1.78	1.23	1.05	1.03
Mean	2.12	2.68	6.12	7.80
		$\gamma = 3$		
0.1	13.91	70.18	194.38	238.16
0.2	3.01	9.78	23.88	46.68
0.3	1.73	3.32	5.68	15.00
0.4	1.53	1.79	2.46	5.91
0.5	1.52	1.38	1.76	3.07
0.6	1.64	1.34	1.50	1.98
0.7	1.66	1.25	1.28	1.46
0.8	1.67	1.23	1.21	1.38
0.9	1.73	1.23	1.15	1.16
Mean	3.15	10.17	25.92	34.98
		$\gamma = 5$		
0.1	25.87	113.37	356.07	405.26
0.2	4.47	15.64	42.17	101.72
0.3	2.09	5.96	14.49	36.86
0.4	1.58	2.62	4.74	13.60
0.5	1.43	1.75	2.59	7.09
0.6	1.49	1.40	1.80	3.41
0.7	1.53	1.30	1.57	2.42
0.8	1.54	1.22	1.33	1.75
0.9	1.60	1.24	1.30	1.45
Mean	4.62	16.06	47.34	63.73

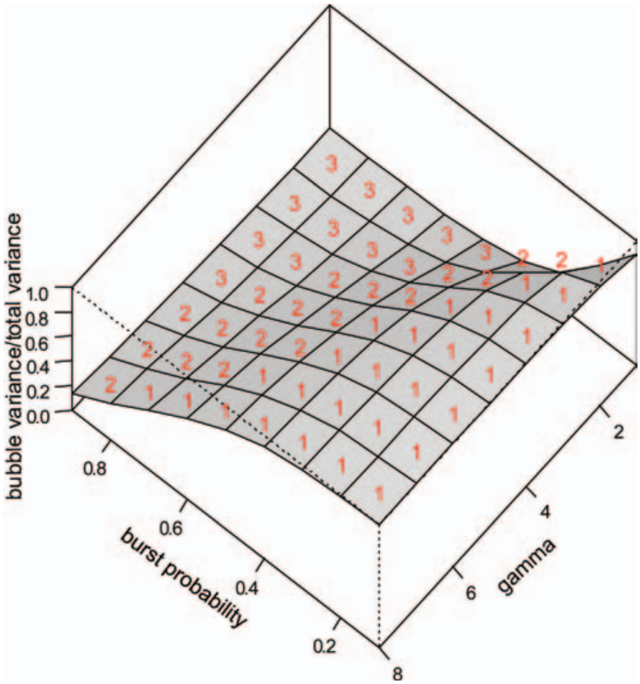
Figure 3. Thresholds Generating the Minimum Welfare Loss across the Parameters p and γ



and $\gamma = 1$—the central bank minimizes welfare losses with a conservative rule, only regarding an asset-price movement as a bubble if the asset price moves more than three standard deviations away from the long-run asset price. As p decreases and γ increases, the welfare-loss-minimizing threshold falls.

The parameters, p and γ, are important determinants of the threshold because they are key influences on the relative variances of the fundamental asset price and the bubble. The parameter γ has a positive nonlinear relationship with the variance of the bubble, while (as shown in figure 2) p has a negative nonlinear relationship with the bubble variance. The surface in figure 4 shows how the ratio of the bubble variance to the total asset-price variance depends on the parameters p and γ. When p is low and γ is high, the bubble has a high variance relative to the total asset-price variance. As p increases and γ decreases, the variance of the bubble relative to the total asset price decreases. Superimposed on the surface of figure 4

Figure 4. Relative Variance of the Bubble (Bubble Variance/Total Asset-Price Variance)



Note: Optimal thresholds are superimposed on the surface.

is the welfare-loss-minimizing threshold number of standard deviations. As the variance of the bubble relative to the total asset-price variance falls, the optimal threshold rises.

The relative variance of the bubble to the total asset price is an important determinant of the threshold because it determines the ease of distinguishing between fundamental and nonfundamental asset-price movements. For example, holding γ constant, increasing p decreases the variance of the bubble relative to the total asset-price variance, increasing the likelihood that any given asset-price movement is fundamental. In these circumstances, the bank should be more cautious about deciding that a given asset-price movement is nonfundamental and adopt a higher threshold number of standard deviations.

The dependence of the optimal threshold number of standard deviations on the probability of the bubble bursting, p, and the growth in the size of the bubble, γ, means that once a central bank has chosen or estimated the parameters of the stochastic process for a_t, it has effectively chosen its threshold for determining whether it should consider an asset-price movement fundamental or not.

In practice, it may be hard for a central bank to know with any accuracy what the value of the parameters p and γ are, so it needs to choose a monetary policy that will be robust to this parameter variation. The following discussion compares the different monetary policy rules across a range of parameter values for p and γ. Ignoring asset prices (i.e., giving them no separate weight in the monetary policy rule) is used as a benchmark because, as discussed in the paper, this is regarded by many in the literature as the optimal response to asset-price movements given the uncertainty about the nature of asset-price movements.

Figure 5 shows the ratio of the welfare loss from using a three-standard-deviation threshold compared with ignoring asset-price movements. Across a wide range of p and γ values, the welfare loss from using a three-standard-deviation rule is significantly less than the loss from ignoring asset prices. In some ranges the loss from the three-standard-deviation rule is only 30–40 percent of the loss incurred by ignoring asset prices. The four dots mark the area where the welfare loss is higher using the three-standard-deviation rule, but even in these cases the loss is only around 2 percent greater than the loss from ignoring asset prices. Furthermore, the absolute welfare gains to be made when the three-standard-deviation rule is superior to ignoring asset prices are far larger than the absolute losses when it is inferior. On average, the absolute welfare gain for the parameter combinations where the three-standard-deviation rule is superior to ignoring asset prices is more than 1,500 times the extra welfare losses for parameter combinations where it is inferior.[11] Over the whole parameter range, the total absolute loss from using the three-standard-deviation rule is 79 percent of the loss from ignoring asset prices.

[11] Welfare-loss averages referred to in the discussion of results throughout the paper are simple averages rather than weighted averages.

Figure 5. Ratio of the Welfare Loss from the Three-Standard-Deviation Rule to the Loss from Ignoring Asset Prices (3sd/ignore)

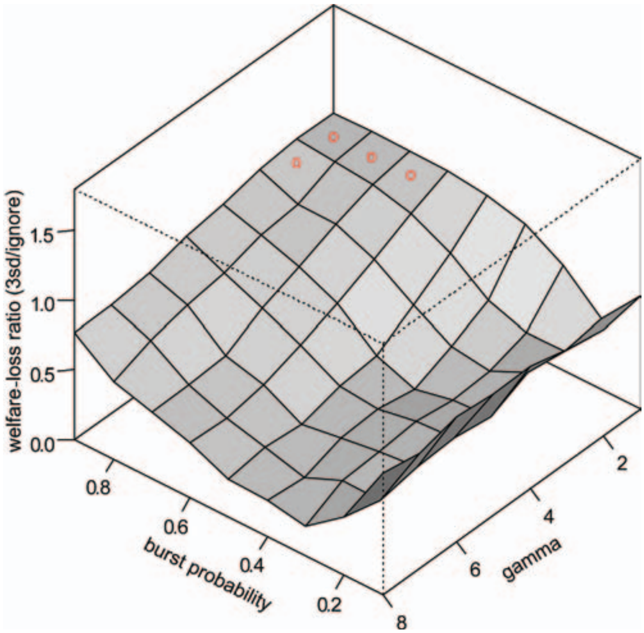
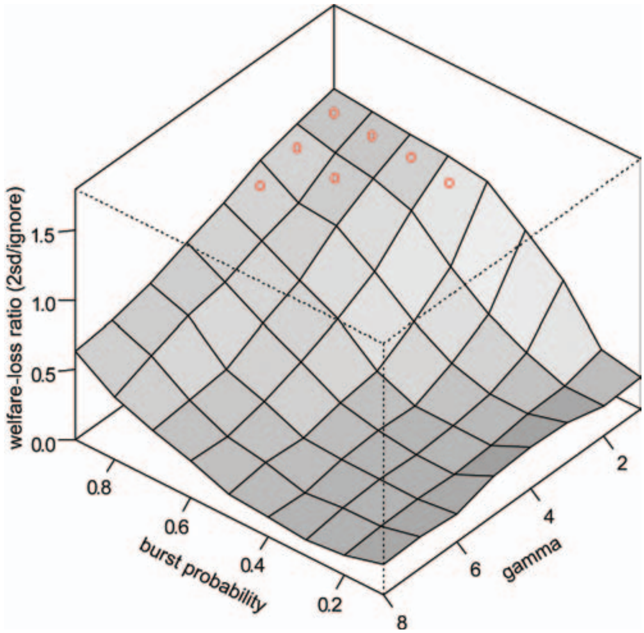


Table 1 and figures 6 and 7 show that as the threshold is lowered further, welfare losses relative to ignoring asset prices will decrease where the threshold rule is superior to ignoring asset prices, but there will be a greater range of bubble parameters where the loss from using the threshold rule will be higher. Figure 6 shows the ratio of welfare losses from using a two-standard-deviation rule. Over the whole parameter range, the total absolute loss from using the two-standard-deviation rule is 23 percent of the loss from ignoring asset prices altogether. However, compared with the three-standard-deviation rule, there is a greater parameter range (marked by the dots in figure 6) over which the welfare loss is greater for the threshold rule than it is for ignoring asset prices. As can be seen in figure 7, reducing the threshold to one standard deviation will further increase the parameter range over which using the threshold will lead to greater losses than ignoring asset prices (marked by the dots

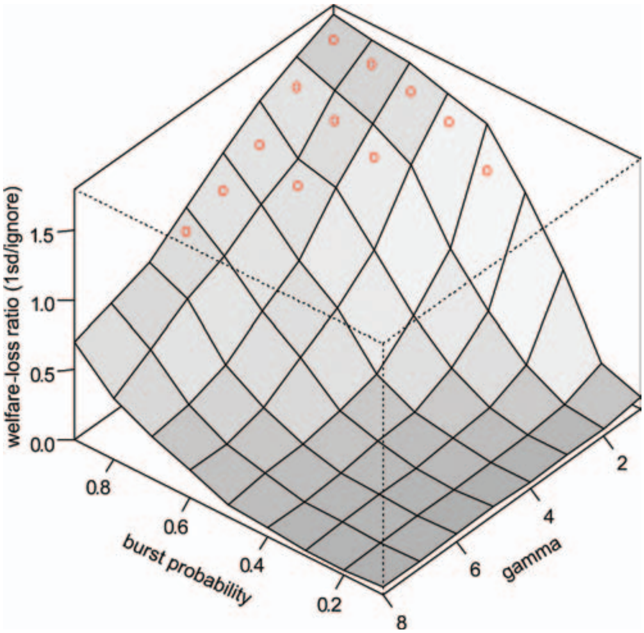
Figure 6. Ratio of the Welfare Loss from the Two-Standard-Deviation Rule to the Loss from Ignoring Asset Prices (2sd/ignore)



in figure 7). The size of these relative losses will also increase. Nevertheless, across the whole parameter range, the total absolute welfare loss will only be 6 percent of the loss from ignoring asset prices.

Overall, a three-standard-deviation rule performs better than ignoring asset prices across almost the entire parameter range for p and γ. Lowering the threshold below three standard deviations will decrease the total welfare loss further but increase the parameter range over which the loss from the threshold rule will be higher than the loss from ignoring asset prices. The results show that once the central bank has chosen or estimated the parameters of the asset-price process, it has effectively chosen its optimal threshold. If the central bank has little confidence in its estimates of the bubble parameters and simply assumes they are all equally likely, the threshold rule it chooses will depend on its degree of risk aversion. The more risk averse the central bank is, the higher the threshold it should use. A conservative approach would be to adopt a

Figure 7. Ratio of the Welfare Loss from the One-Standard-Deviation Rule to the Loss from Ignoring Asset Prices (1sd/ignore)



three-standard-deviation rule, as it will almost always do better than ignoring asset prices and result in only small absolute losses relative to ignoring asset prices in a small parameter range.

3.1 Further Extensions

This subsection examines alterations to a number of assumptions made above that affect the variance of the bubble, a_t, relative to the variance of ap_t, the total deviation in the market asset price from the long-run average, and therefore potentially the appropriate threshold. These include altering q, the probability of a bubble forming; τ, the autoregressive coefficient in the AR(1) equation; and a_t^* (the short-run deviation of the fundamental asset price) from the long-run average market asset price. Another simulation exercise is used to see the effect of changing the error distribution for a_t^*. In the original simulations, $q = 0.1, \tau = 0.9$, and the errors for a_t^* are assumed to be normally distributed. A simulation is also conducted to determine whether the threshold is sensitive to changing the bubble process to allow for asset-price overshooting and a_t falling below zero when the bubble bursts (i.e., a negative bubble).

3.1.1 Altering the Probability of a Bubble Forming, q

The model is simulated again over the range of 0.1 to 0.9 for p, $\gamma = 1$, and $q = 0.5$ and 0.9, i.e., with a 50 percent and 90 percent probability that a bubble will form every year when the economy is in the non-bubble state. All other parameters are the same as the original simulations. These results are compared with the original simulation with $\gamma = 1$ and $q = 0.1$. One of the new values for $q = 0.9$ is close to the top of its range, so this experiment gives the approximate limits of the effect of changing q. Table 2 shows the ratio of welfare losses for each parameter set. The effect of increasing q from 0.1 to 0.5 and 0.9 is to increase the range of p over which lower threshold rules of one and two standard deviations ensure the lowest welfare loss. This is because the increase in q increases the variance of a_t, the bubble, relative to $ap_t = a_t^* + a_t$, the deviation of the market asset price from the long-run average market asset price. This reduces the possibility of mistakenly deciding there is a bubble when there is not.

The general pattern of optimal thresholds remains the same as in the main simulations. In particular, with $q = 0.9$, the three-standard-deviation rule reduces welfare losses compared with ignoring asset prices across a wide range of p, with only small extra losses in a narrow p range. If $q = 0.9$, the range of p where there are greater losses from the three-standard-deviation rule is smaller than if $q = 0.1$, because the variance of the bubble is now larger. As in the main simulations, average welfare losses across the parameter range can be reduced further by decreasing the threshold below three, but this comes at the cost of a greater parameter range over which the threshold rule will lead to greater losses than incurred by ignoring asset prices.

3.1.2 Altering the Autoregressive Coefficient, τ

The model is simulated again across the range of 0.1 to 0.9 for p, $\gamma = 1$, and new lower autoregressive coefficients of $\tau = 0.1$

Table 2. Changing q: Welfare-Loss Ratio (Threshold/Optimal)

p	One Standard Deviation	Two Standard Deviations	Three Standard Deviations	Ignore Asset Price	Best Threshold Out of 1, 2, 3
$\gamma = 1$ and $q = 0.1$					
0.1	4.58	11.85	40.80	50.20	1
0.2	2.02	3.03	5.39	10.43	1
0.3	1.81	1.62	1.90	2.50	2
0.4	1.78	1.36	1.43	1.57	2
0.5	1.80	1.29	1.18	1.20	3
0.6	1.79	1.28	1.16	1.13	3
0.7	1.78	1.23	1.10	1.07	3
0.8	1.77	1.22	1.08	1.05	3
0.9	1.78	1.23	1.05	1.03	3
Mean	2.12	2.68	6.12	7.80	1
$\gamma = 1$ and $q = 0.5$					
0.1	10.39	34.17	90.40	93.30	1
0.2	3.25	7.68	14.79	18.80	1
0.3	2.33	3.20	4.58	5.84	1
0.4	2.01	2.14	2.66	3.18	1
0.5	1.94	1.76	1.88	2.00	2
0.6	1.87	1.54	1.58	1.61	2
0.7	1.85	1.40	1.34	1.35	3
0.8	1.81	1.33	1.22	1.21	3
0.9	1.80	1.29	1.18	1.15	3
Mean	3.03	6.06	13.29	14.27	1
$\gamma = 1$ and $q = 0.9$					
0.1	11.99	39.26	94.68	96.68	1
0.2	3.77	8.20	14.06	16.85	1
0.3	2.59	4.09	5.92	7.26	1
0.4	2.27	2.86	3.80	4.36	1
0.5	2.09	2.13	2.45	2.65	1
0.6	1.97	1.72	1.85	1.92	2
0.7	1.91	1.57	1.55	1.56	2
0.8	1.87	1.45	1.38	1.36	3
0.9	1.85	1.40	1.28	1.26	3
Mean	3.37	6.96	14.11	14.88	1

and 0.5 in the AR(1) that determines a_t^*. τ is set equal to 0.1 to find the approximate limits of the effect of changing this parameter. All other parameters are the same as in the original simulations. The new simulations are compared with the original simulation results, where τ was equal to 0.9. The results are in table 3.

Reducing τ decreases the variance of a_t^*, and so the variance of a_t rises as a proportion of the variance of $ap_t = a_t^* + a_t$. This reduces the probability of mistakenly identifying a given asset-price movement as a bubble when it is not, so lower thresholds produce the lowest welfare losses over a wider parameter range. At the extreme, if $\tau = 0.1$, it is relatively easy to identify bubbles, as most deviations from the long-run average asset price are bubble movements. A threshold of two standard deviations will provide a lower welfare loss than both ignoring asset prices and a three-standard-deviation rule across the entire range for p with $\gamma = 1$ and $\tau = 0.1$. The result that the three-standard-deviation rule is better than ignoring asset prices across the parameter range for p still remains, and it also has lower welfare losses than ignoring asset prices across the entire parameter range for p with $\tau = 0.1$. The advantage of the three-standard-deviation rule over the lower-standard-deviation rules is that variation in τ leads to less variation in its performance, compared with ignoring asset prices, than the other rules; i.e., it is less risky.

3.1.3 Changing the Error Distribution for a_t^*

In the main simulation, it is assumed that a_t^* is determined by an AR(1) with normally distributed errors. The model is simulated again with errors from a t distribution with eight degrees of freedom; i.e., the error distribution for the short-run deviation in the fundamental asset price has fatter tails than a normal distribution. This means that even in non-bubble periods, asset-price deviations from the long-run asset price have fat tails. The simulation is done with $\gamma = 1$ and over the range of 0.1 to 0.9 for p. All other parameter values are the same as in the original simulations. These results are compared with the original simulation in table 4.

The change in the error distribution from the normal to the t distribution raises the variance of a_t^* relative to ap_t. Overall, the

Table 3. Changing τ: Welfare-Loss Ratio (Threshold/Optimal)

p	One Standard Deviation	Two Standard Deviations	Three Standard Deviations	Ignore Asset Price	Best Threshold Out of 1, 2, 3
$\gamma = 1$ and $\tau = 0.9$					
0.1	4.58	11.85	40.80	50.20	1
0.2	2.02	3.03	5.39	10.43	1
0.3	1.81	1.62	1.90	2.50	2
0.4	1.78	1.36	1.43	1.57	2
0.5	1.80	1.29	1.18	1.20	3
0.6	1.79	1.28	1.16	1.13	3
0.7	1.78	1.23	1.10	1.07	3
0.8	1.77	1.22	1.08	1.05	3
0.9	1.78	1.23	1.05	1.03	3
Mean	2.12	2.68	6.12	7.80	1
$\gamma = 1$ and $\tau = 0.5$					
0.1	3.58	13.93	39.47	47.89	1
0.2	1.24	1.94	3.07	6.28	1
0.3	1.13	1.27	1.65	2.82	1
0.4	1.10	1.13	1.25	1.60	1
0.5	1.07	1.06	1.08	1.18	2
0.6	1.08	1.06	1.10	1.16	2
0.7	1.09	1.05	1.06	1.09	2
0.8	1.08	1.04	1.03	1.03	3
0.9	1.09	1.03	1.02	1.02	3
Mean	1.38	2.61	5.64	7.12	1
$\gamma = 1$ and $\tau = 0.1$					
0.1	3.13	11.52	44.64	56.54	1
0.2	1.20	1.91	3.60	7.77	1
0.3	1.08	1.26	1.58	2.87	1
0.4	1.05	1.12	1.23	1.63	1
0.5	1.05	1.07	1.12	1.31	1
0.6	1.06	1.04	1.07	1.13	2
0.7	1.05	1.03	1.05	1.07	2
0.8	1.05	1.02	1.03	1.04	2
0.9	1.05	1.02	1.02	1.02	2
Mean	1.30	2.33	6.26	8.26	1

Table 4. Changing Error Distributions for a_t^*

p	One Standard Deviation	Two Standard Deviations	Three Standard Deviations	Ignore Asset Price	Best Threshold Out of 1, 2, 3
$\gamma = 1$ and Normal Errors for a_t^*					
0.1	4.58	11.85	40.80	50.20	1
0.2	2.02	3.03	5.39	10.43	1
0.3	1.81	1.62	1.90	2.50	2
0.4	1.78	1.36	1.43	1.57	2
0.5	1.80	1.29	1.18	1.20	3
0.6	1.79	1.28	1.16	1.13	3
0.7	1.78	1.23	1.10	1.07	3
0.8	1.77	1.22	1.08	1.05	3
0.9	1.78	1.23	1.05	1.03	3
Mean	2.12	2.68	6.12	7.80	1
$\gamma = 1$ and t Distribution Errors for a_t^*					
0.1	4.06	12.27	33.00	42.77	1
0.2	2.23	2.35	3.94	7.33	1
0.3	2.23	1.70	2.01	2.62	2
0.4	2.23	1.60	1.69	2.01	2
0.5	2.23	1.39	1.25	1.27	3
0.6	2.21	1.29	1.13	1.13	3
0.7	2.20	1.29	1.09	1.06	3
0.8	2.20	1.28	1.07	1.04	3
0.9	2.22	1.28	1.06	1.02	3
Mean	2.42	2.71	5.14	6.69	1

lower-standard-deviation rules result in higher losses than previously because acting early is more likely to result in mistakenly treating an asset-price movement as a bubble when it is not, as it is now more likely the movement will actually be fundamental. The three-standard-deviation rule and ignoring asset prices now result in lower overall losses because it is more likely that the movements these rules fail to treat as a bubble are now fundamental. The pattern of three-standard-deviation rule losses compared with ignoring asset prices remains approximately the same as in the original simulations.

Relative to the original simulations, the above experiments show that a decrease in τ or an increase in q can increase the parameter range over which the one- and two-standard-deviation rules will result in the lowest welfare loss. However, changing the error distribution for a_t^* to a t distribution raised the welfare losses of these lower-standard-deviation rules relative to ignoring asset prices and using a three-standard-deviation rule. The main result, that the three-standard-deviation rule performs better than ignoring asset prices across most of the parameter range for variables that affect the variance of a_t relative to ap_t, remains once the parameter set is extended to q and τ.

3.1.4 Allowing for Asset-Price Overshooting and Negative Bubbles

Central banks assessing the effects of a potential bubble on the economy are often concerned about negative consequences for the real economy should the bubble burst. In the main simulation when the bubble bursts, the temporary positive boost to output from asset prices above fundamentals disappears and the output gap eventually returns to zero. For an economy that is used to much higher than equilibrium output, this is already likely to be regarded as a significant downturn. A more dramatic scenario, though, is that the asset price overshoots its fundamental value, leading to a negative bubble and output below equilibrium. To explore how this overshooting effect would affect welfare outcomes under the various thresholds, the model is simulated again with a bubble, a_t, that can become negative when the positive bubble bursts. In this simulation, if the bubble bursts in period t, in the following period a_{t+1} is given by

$$a_{t+1} = -b_t * a_t,$$

where b_t is a random uniform variable lying between 0 and 1, so that the negative bubble will be equal to between 0 and 100 percent of the positive bubble when the positive bubble bursts. If a negative bubble exists, it reduces by γ in the following periods until $a_t = 0$. While a negative bubble exists, $a_t < 0$, no positive bubble can form. Because the bubble can now become negative, this alters the expected effect of asset prices on the output gap next period and therefore the optimal interest rate rule, which for $a_t \geqslant 0$ becomes

$$r_t = \phi_1 y_t + \phi_2 \pi_t + \phi_3 (v_t \iota ((1-p)(a_t+\gamma) - p b_t a_t) + (1-v_t)(\iota q \gamma)). \quad (26)$$

The main difference between the new rule and the original rule (15) is the term $-pb_t a_t$, which is added because when $v_t = 1$, with probability p_t, the bubble will burst and a_{t+1} will be equal to $-b_t a_t$ instead of 0 in the main simulation without a negative bubble. As in the GPS model, the expected effect on output of the bubble next period is the sum of the positive effect should the bubble continue and the negative effect should the bubble burst. The balance of these effects depends on the size of p and b_t. If the bubble has burst, while $a_t < 0$ interest rates need to offset the negative effect of the bubble on output next period, $\iota(a_t + \gamma)$, and the rule is

$$r_t = \phi_1 y_t + \phi_2 \pi_t + \phi_3(\iota(a_t + \gamma)). \tag{27}$$

If $ap_t \geqslant 0$, the activist central bank's interest rate rule is a modified version of (17):

$$rf_t = \phi_1 yf_t + \phi_2 \pi f_t + \kappa_t \phi_3 \iota((1 - pf_t)(ap_t + \gamma f_t) - pf_t b_t ap_t)), \tag{28}$$

where $\iota pf_t b_t ap_t$ is the expected effect of the bubble bursting on the output gap next period. The size of the negative bubble depends on the random variable b_t, which is set equal to its mean of 0.5. If there is a large downward shift in the asset price and $ap_t < 0$, the activist central bank will assume a bubble has burst and apply the following rule:

$$rf_t = \phi_1 yf_t + \phi_2 \pi f_t + \kappa_t \phi_3 \iota((ap_t + \gamma f_t)). \tag{29}$$

In this case the bank will treat the asset price as being in a negative bubble situation while the market asset-price deviation from the long-run asset price is more than a certain threshold number of standard deviations from its long-run average, and κ_t will be equal to 1 in this case. The results in table 5 show that the three-standard-deviation rule results in a lower welfare loss than ignoring asset prices across a very similar parameter range to the original simulations with no negative bubbles. With negative bubbles for $\gamma \leqslant 3$, p_t needs to be slightly smaller than without negative bubbles before the three-standard-deviation rule results in a lower welfare loss than ignoring asset prices; i.e., the variance of the bubble needs to be slightly larger relative to fundamental asset-price movements before a three-standard-deviation rule will be superior. Overall, the results of the simulation with the possibility of negative bubbles show that

Table 5. Simulation Allowing Negative Bubbles

p	One Standard Deviation	Two Standard Deviations	Three Standard Deviations	Ignore Asset Price	Best Threshold Out of 1, 2, 3
$\gamma = 1$					
0.1	7.48	24.91	53.28	57.81	1
0.2	2.48	2.86	4.01	5.47	1
0.3	2.50	1.67	1.82	2.27	2
0.4	2.62	1.50	1.34	1.35	3
0.5	2.66	1.44	1.17	1.15	3
0.6	2.69	1.40	1.10	1.09	3
0.7	2.70	1.38	1.07	1.04	3
0.8	2.68	1.39	1.05	1.01	3
0.9	2.69	1.39	1.05	1.01	3
Mean	3.17	4.22	7.74	8.02	1
$\gamma = 4$					
0.1	13.51	54.50	121.17	147.76	1
0.2	2.95	9.44	19.33	41.64	1
0.3	1.97	4.39	7.75	13.44	1
0.4	1.86	2.21	3.65	6.73	1
0.5	1.89	1.50	1.83	3.10	2
0.6	2.09	1.37	1.44	1.76	2
0.7	2.26	1.28	1.24	1.43	3
0.8	2.41	1.29	1.16	1.26	3
0.9	2.56	1.29	1.10	1.11	3
Mean	3.50	8.59	17.63	24.25	1
$\gamma = 8$					
0.1	18.51	72.06	212.81	246.32	1
0.2	4.95	24.81	50.46	85.82	1
0.3	1.88	5.42	11.19	27.10	1
0.4	1.64	2.97	5.56	13.94	1
0.5	1.46	1.90	3.00	5.93	1
0.6	1.51	1.55	2.09	4.06	1
0.7	1.61	1.39	1.65	3.07	2
0.8	1.79	1.27	1.31	1.69	3
0.9	1.98	1.25	1.21	1.35	3
Mean	3.93	12.51	32.14	43.25	1

the main result of the paper is also robust to a generalization of the asset-price process that allows for negative bubbles following a bubble bursting.

3.1.5 Other Possible Extensions

Dependence of Fundamental and Nonfundamental Asset-Price Movements. As stated in the paper, it is possible that fundamental and nonfundamental asset-price movements are related. A further extension of the model would be to allow dependence between fundamental and nonfundamental asset prices. This would introduce a covariance term in the variance of ap_t. If there was a positive covariance, then at least part of any asset-price movement would be fundamental and the variance of the bubble would be smaller relative to the total asset-price variance. In this case, the bank should be more cautious in deciding that an asset-price movement was partially nonfundamental and adopt a higher threshold number of standard deviations in its threshold rule. It would not alter the main conclusion that a three-standard-deviation rule is superior to ignoring asset prices. This is because while adding a covariance term would affect the ratio of the variance of the bubble to the total asset price, and therefore the optimal threshold rule, the three-standard-deviation rule has been shown to be robust across a wide range of values for the relative variance by altering other parameters, such as the bubble bursting.

Incorporating Interest Rate Effects on the Asset Price. It is unclear whether asset-price bubbles are affected by interest rates, and, in the model presented in this paper, it is assumed that the asset-price bubble is independent of interest rates. Although policymaker assumptions are often to the contrary, in reality, announced monetary policy and actual changes in interest rates often may not have any effect on investor behavior in a bubble situation. This can be examined from both the point of view of monetary policy expanding the bubble and monetary policy "popping" the bubble.

From the perspective of monetary policy inducing a larger bubble, in the 1990s it was argued that the knowledge that the Federal Reserve would cut interest rates in the event of a crash induced a bias toward gambling on continuing rises in the market.

However, this argument implies that monetary policy is effective very quickly in eliminating the effects of a crash, which market participants almost certainly know is not the case. Without a belief by the market that the central bank can eliminate the effects of a crash through interest rates, central bank announcements of future rate cutting in the event of a crash would not influence the bubble.

Furthermore, from the perspective of monetary policy directly shrinking or "popping" a bubble, if an asset-price bubble is in progress, investors will be earning and expecting large capital gains. These gains (often 20 or 30 percent or more per annum) are likely to more than cover any reasonable increase in interest rates, and expected gains will remain positive. In this situation, investors will continue to purchase the asset despite the monetary policy tightening, and so the bubble will continue.

However, it is also possible that changes in interest rates may indirectly affect asset prices in a bubble situation, particularly via their effect on aggregate demand and output growth. For example, the anticipation of a weaker-than-expected GDP growth outturn following monetary policy tightening may lead to a fall in investors' capital gains expectations.

A further extension of the model would be to incorporate interest rate effects on the asset price. Some potential ways to do this would be via interest rate effects on the probability of the bubble bursting or on the growth of the bubble. In this case, the required optimal interest rate changes would be less than currently produced by the model, as there would be two channels through which interest rates could affect aggregate demand (directly and indirectly via the asset price) and return the economy to potential. However, as above, incorporating interest rate effects on asset prices would not alter the main conclusion that a three-standard-deviation rule is superior to ignoring asset prices. This is because it has already been shown that the rule is robust to variation in the asset-price parameters that would most likely be affected by interest rates, such as the probability of the bubble bursting.

If the bubble is affected by interest rates, then there is no analytical solution such as that used in this paper that is available for the monetary policy rule, and numerical methods would be required to solve the model.

4. Conclusion

The results show that the threshold rule is sensitive to the assumptions about the asset-price change, particularly p, the probability of the bubble bursting, and γ, the growth of the bubble. To be consistent in its approach to monetary policy, the central bank should adjust not only its interest rate policy according to the assumptions it makes about the nature of the bubble process, as shown by Gruen, Plumb, and Stone (2005), but also its approach to determining when it will consider an asset-price movement most likely a bubble.

In practice, central banks face information constraints about the nature of asset-price shocks, and this complicates any active policy approach designed to take account of asset-price information. These results suggest that this problem should not lead to a policy approach where asset-price information is completely ignored prior to a bubble bursting.

A key finding is that it is better to adopt a high-threshold rule for deciding whether to include asset prices in the output-gap and inflation forecasts and monetary policy than to ignore asset prices altogether. Although it is difficult to determine the parameter assumptions for an asset-price bubble, a conservative three-standard-deviation threshold rule will result in lower welfare losses than ignoring asset prices altogether across a wide range of parameter values for the bubble.

Finally, the choice of threshold will depend on the central bank's degree of risk aversion. The more risk averse the bank, the higher the threshold it should adopt. Although a higher threshold will have a higher average absolute loss, it will have a smaller parameter range over which it will make a loss relative to ignoring asset prices.

Appendix 1. Solving for the Optimal Monetary Policy Rule

This appendix gives further detail on the dynamic programming method used to solve the linear-quadratic problem for the optimal monetary policy rule. The method described here closely follows Chow (1973). See also Bertsekas (2000). The economy is described by the system of linear equations, (9) and (10), expressed in matrix form below:

$$z_t = Az_{t-1} + Cx_t + b_t + \varepsilon_t, \tag{30}$$

where z_t is the vector of system variables, y_t and π_t, and x_t is a vector of control variables, r_t. b_t is a known series, in this case the expected effect of a_t on the output gap; ε_t is a vector of white-noise error terms; and A and C are coefficient matrices.

Written in matrix form, the objective function ((11) in the text) is to minimize

$$E_{t-1} \sum_{i=0}^{\infty} \delta^t z_t' K z_t, \tag{31}$$

where δ is the bank's discount factor and K is a diagonal matrix of weights on the system variables, in this case 1 and μ on y_t and π_t, respectively.

The overall problem is to find a rule for the control variables, x_t, that minimizes the objective function (31) subject to the system (30). The problem is solved backwards from the last period, T.

In period T the addition to the welfare function is $\delta^T z_T' K z_T$. Let $W_T = E_{T-1}(\delta^T z_T' K z_T)$. W_T is then written in the form

$$W_T = \delta^T E_{T-1}\big(z_T' H_T z_T + cst\big), \tag{32}$$

where $H_T = K$ and cst contains variables that do not involve x_T.

The term z_T in (32) is then replaced with the expression for z_T in (30) and expectations are taken. $E_{T-1}(\varepsilon_T) = 0$ by assumption, so this term is eliminated.

$$W_T = \delta^T[(Az_{T-1} + Cx_T + b_T)' H_T (Az_{T-1} + Cx_T + b_T) + cst] \tag{33}$$

To minimize W_T with respect to the control vector x_T, differentiate W_T and set it equal to 0:

$$\frac{\delta W_T}{\delta x_T} = 2C' H_T (Az_{T-1} + Cx_T + b_T) = 0. \tag{34}$$

Rearrange (34) to find the optimal policy at T, x_T:

$$C'HCx_T = -C'H_TAz_{T-1} - C^{'}H_Tb_T$$
$$x_T = -(C'H_TC)^{-1}C'H_TAz_{T-1} - (C'H_TC)^{-1}C'H_Tb_T$$
$$= G_Tz_{t-1} + g_T, \tag{35}$$

where

$$G_T = -(C'HC)^{-1}C'H_TA \tag{36}$$
$$g_T = -(C'HC)^{-1}C'H_Tb_T. \tag{37}$$

To find z_T under control, substitute the policy rule (35) into (30) to obtain

$$z_T = (A + CG_T)z_{t-1} + (Cg_T + b_T) + \varepsilon_T. \tag{38}$$

To find welfare, W_T, under control, substitute (38) into (32) to obtain

$$W_T = \delta^T E_{T-1}[((A + CG_T)z_{T-1} + (Cg_T + b_T) + \varepsilon_T)'$$
$$H_T((A + CG_T)z_{T-1} + (Cg_T + b_T) + \varepsilon_T] \tag{39}$$
$$= \delta^T \left[((A + CG_T)z_{T-1})'H_T((A + CG_T)z_{T-1})+\right.$$
$$((A + CG_T)z_{T-1})'H_T(Cg_T + b_T)+$$
$$(Cg_T + b_T)'H_T((A + CG_T)z_{T-1} + (Cg_T + b_T)'$$
$$\left.H_T(Cg_T + b_T) + E_{T-1}\left(\varepsilon_T'\varepsilon_T\right)\right] \tag{40}$$
$$= \delta^T[((A + CG_T)z_{T-1})'H_T((A + CG_T)z_{T-1})$$
$$+ z_{T-1}'(A + CG_T)'H_T(Cg_T + b_T)$$
$$+ (Cg_T + b_T)'H_T(A + CG_T)z_{T-1}] + cst. \tag{41}$$

$z_{T-1}'(A + CG_T)'H_T(Cg_T + b_T)$ is a scalar, so it is equal to its transpose:

$$W_T = \delta^T \left[\left(z_{T-1}'(A + CG_T)'\right)H_T((A + CG_T)z_{T-1})\right.$$
$$\left.+ 2z_{T-1}'(A + CG_T)'H_T(Cg_T + b_T)\right] + cst. \tag{42}$$

Substitute (37) for g_T in (42):

$$W_T = \delta^T \left[\left(z'_{T-1}(A + CG_T)' \right) H_T((A + CG_T)z_{T-1}) \right. \\ \left. + 2z'_{T-1}((A + CG_T)' H_T(C(-(C'H_TC)^{-1}C'H_Tb_T) + b_T) + cst. \right. \tag{43}$$

Rearrange $H_TC(C'H_TC)^{-1}C'$ as $C'H_TC(C'H_TC)^{-1} = I$, where I is the identity matrix.

$$W_T = \delta^T \left[\left(z'_{T-1}(A + CG_T)' \right) H_T((A + CG_T)z_{T-1}) + 2z'_{T-1} \right. \\ \left. ((A + CG_T)'((-C'H_TC(C'H_TC)^{-1}H_Tb_T + H_Tb_T)) + cst \right. \tag{44}$$

$$= \delta^T \left[\left(z'_{T-1}(A + CG_T)' \right) H_T((A + CG_T)z_{T-1}) \right. \\ \left. + 2z'_{T-1}((A + CG_T)'(-H_Tb_T + H_Tb_T) + cst \right. \tag{45}$$

$$= \delta^T \left[\left(z'_{T-1}(A + CG_T)' \right) H_T((A + CG_T)z_{T-1}) + cst, \right. \tag{46}$$

where (46) is the welfare loss at time T with x_T chosen optimally. The next step is to go back one period in time and choose x_{T-1}. We do not need to choose x_T, because it has already been chosen optimally no matter what the value of x_{T-1}. By Bellman's optimality principle we only need to choose x_{T-1} that will minimize the welfare loss at $T-1$ and T, as we know whatever value we choose for x_{T-1}, the value of x_T we have already determined will be optimal. The task is now to minimize W_{T-1} with respect to x_{T-1}:

$$W_{T-1} = \delta^{T-1} z'_{T-1} K z_{T-1} + \delta^T \left[\left(z'_{T-1}(A + CG_T)' \right) \right. \\ \left. H_T((A + CG_T)z_{T-1}) + cst. \right. \tag{47}$$

The first part of (47) is the welfare loss in $T-1$ and the second part is the welfare loss at T with x_T chosen optimally. We cannot improve welfare by changing x_T, but we can affect welfare in both T and $T-1$ with our choice of x_{T-1}.

Equation (47) can be rewritten as

$$W_{T-1} = \delta^{T-1} z'_{T-1} H_{T-1} z_{T-1} + cst, \tag{48}$$

where

$$\begin{aligned} H_{T-1} &= K + \delta(A + CG_T)' H_T (A + CG_T) \\ &= K + \delta A'[H_T - H_T C(C' H_T C)^{-1} C' H_T] A \qquad (49) \\ &\quad \text{(after substitution for } G_T\text{)}. \end{aligned}$$

The minimization problem in (48) is of the same form as the original in (32), except now T has been replaced by $T-1$ so that x_{T-1} is therefore the same as for x_T, except T is now replaced by $T-1$ so that x_{T-1} is a function of G_{T-1}, which is in turn a function of H_{T-1}. The problem can be solved backwards in this iterative manner back to $T = 1$.

Equation (49) is the Ricatti equation. Bertsekas (2000) shows that this difference equation converges to the steady-state solution as $T \to \infty$:

$$H = K + \delta A'[H - HC(C'HC)^{-1} C'H] A \qquad (50)$$

and the optimal policy is

$$x_t = G z_{T-1} + g \qquad (51)$$

$$G = -(CHC)^{-1} C'HA \qquad (52)$$

$$g = -(CHC)^{-1} C'H b_t. \qquad (53)$$

This solution is implemented in Gauss by iterating (49) from initial values equal to the weights of y_t and π_t until it converges to a solution for H. Optimal policy is then calculated using (51)–(53).

Appendix 2. Calculating the Mean and Variance of the Bubble Process

This appendix describes how the mean and the variance of the stochastic bubble process, a_t, are derived.

The Asset-Price Bubble

The asset-price bubble, a_t, evolves according to a Markov chain, v_t:

$$\text{If } v_t = 1, \text{ then } a_t = a_{t-1} + \gamma. \qquad (54)$$

$$\text{If } v_t = 0, \text{ then } a_t = 0. \qquad (55)$$

That is, from an asset-price perspective there are two states of the world: bubble ($v_t = 1$) and no bubble ($v_t = 0$). The transition probability matrix between states is given by

	$v_{t+1} = 0$	$v_{t+1} = 1$
$v_t = 0$	$1 - q$	q
$v_t = 1$	p	$1 - p$

,

where p is the probability of the bubble bursting and q is the probability of the bubble forming.

The unconditional probability of each state is given:

$$pr(v_t = 1) = \frac{1 - (1 - q)}{2 - (1 - q) - (1 - p)}$$

$$pr(v_t = 1) = \frac{q}{q + p} \tag{56}$$

$$pr(v_t = 0) = \frac{p}{p + q}. \tag{57}$$

The Mean of a_t

The mean of the bubble process, $E(a_t)$, is given by summing across the states the unconditional probability of a state occurring, multiplied by the conditional mean of a_t given that state:

$$E(a_t) = pr(v_t = 1)E(a_t|v_t = 1) + pr(v_t = 1)E(a_t|v_t = 0). \tag{58}$$

Given $E(a_t|v_t = 0) = 0$, then

$$E(a_t) = pr(v_t = 1)E(a_t|v_t = 1), \tag{59}$$

where the unconditional probability of the bubble state, $pr(v_t = 1)$, is given above and the conditional mean of the bubble given the bubble state, $E(a_t|v_t = 1)$, is a geometric series given by

$$E(a_t|v_t = 1) = p\gamma \sum_{n=0}^{\infty}(1+n)(1-p)^n \text{ where } |1-p| < 1$$

$$E(a_t|v_t = 1) = \frac{p\gamma}{(1-(1-p))^2} \tag{60}$$

$$E(a_t|v_t = 1) = \frac{\gamma}{p}, \tag{61}$$

so the mean of a_t is given by

$$\begin{aligned} E(a_t) &= pr(v_t = 1)E(a_t|v_t = 1) \\ &= \frac{q\gamma}{(q+p)p}. \end{aligned} \tag{62}$$

The Variance of a_t

The variance of the bubble process, $var(a_t)$, is given by summing across the states the unconditional probability of a state occurring, multiplied by the conditional variance of a_t given that state:

$$\begin{aligned} var(a_t) = pr(v_t = 1)&\left[\left(E(a_t^2)\middle|v_t = 1\right) - (E(a_t|v_t = 1))^2\right] \\ &+ pr(v_t = 0)\left[\left(E(a_t^2)\middle|v_t = 0\right) - (E(a_t|v_t = 0))^2\right], \end{aligned} \tag{63}$$

given $(E(a_t^2)|v_t = 0) = 0$ and $E(a_t|v_t = 0) = 0$, then

$$var(a_t) = pr(v_t = 1)\left[\left(E\left(a_t^2\right)\middle|v_t = 1\right) - (E(a_t|v_t = 1))^2\right], \tag{64}$$

where $(E(a_t^2)|v_t = 1)$ is a geometric series given by

$$\left(E\left(a_t^2\right)\middle|v_t = 1\right) = p\gamma^2 \sum_{n=0}^{\infty}(1+n)^2(1-p)^n \tag{65}$$

$$\left(E\left(a_t^2\right)\middle|v_t = 1\right) = p\gamma^2 \frac{1+(1-p)}{(1-(1-p))^3}$$

$$\left(E\left(a_t^2\right)\middle|v_t = 1\right) = \gamma^2 \frac{2-p}{p^2}, \tag{66}$$

so the variance of a_t is given by

$$\begin{aligned} var(a_t) &= pr(v_t = 1)\left[\left(E\left(a_t^2\right)\middle| v_t = 1\right) - \left(E(a_t | v_t = 1)\right)^2\right] \\ &= \frac{q}{q+p}\left[\gamma^2 \frac{2-p}{p^2} - \left(\frac{\gamma}{p}\right)^2\right] \qquad (67) \\ &= \frac{q}{q+p}\left[\frac{\gamma^2(2-p) - \gamma^2}{p^2}\right]. \qquad (68) \end{aligned}$$

Appendix 3. Full Simulation Results

Table 6. Welfare-Loss Ratio (Threshold/Optimal)

p	**One Standard Deviation**	**Two Standard Deviations**	**Three Standard Deviations**	**Ignore Asset Price**
		$\gamma = 1$		
0.1	4.58	11.85	40.80	50.20
0.2	2.02	3.03	5.39	10.43
0.3	1.81	1.62	1.90	2.50
0.4	1.78	1.36	1.43	1.57
0.5	1.80	1.29	1.18	1.20
0.6	1.79	1.28	1.16	1.13
0.7	1.78	1.23	1.10	1.07
0.8	1.77	1.22	1.08	1.05
0.9	1.78	1.23	1.05	1.03
Mean	2.12	2.68	6.12	7.80
		$\gamma = 2$		
0.1	8.33	35.49	123.90	162.98
0.2	2.64	6.63	14.12	29.82
0.3	1.78	2.41	3.53	7.57
0.4	1.65	1.59	1.92	3.44
0.5	1.67	1.40	1.60	2.12
0.6	1.74	1.29	1.26	1.40
0.7	1.74	1.26	1.20	1.24
0.8	1.75	1.23	1.16	1.16
0.9	1.76	1.23	1.11	1.09
Mean	2.56	5.84	16.64	23.42

(continued)

Table 6. (Continued)

p	**One Standard Deviation**	**Two Standard Deviations**	**Three Standard Deviations**	**Ignore Asset Price**
$\gamma = 3$				
0.1	13.91	70.18	194.38	238.16
0.2	3.01	9.78	23.88	46.68
0.3	1.73	3.32	5.68	15.00
0.4	1.53	1.79	2.46	5.91
0.5	1.52	1.38	1.76	3.07
0.6	1.64	1.34	1.50	1.98
0.7	1.66	1.25	1.28	1.46
0.8	1.67	1.23	1.21	1.38
0.9	1.73	1.23	1.15	1.16
Mean	3.15	10.17	25.92	34.98
$\gamma = 4$				
0.1	21.42	141.66	420.60	474.87
0.2	3.64	12.49	27.65	72.81
0.3	1.73	3.62	7.05	19.79
0.4	1.59	2.20	3.63	12.39
0.5	1.46	1.57	2.14	5.40
0.6	1.59	1.37	1.66	2.63
0.7	1.62	1.33	1.52	2.01
0.8	1.64	1.24	1.30	1.53
0.9	1.68	1.22	1.20	1.29
Mean	4.04	18.52	51.86	65.86
$\gamma = 5$				
0.1	25.87	113.37	356.07	405.26
0.2	4.47	15.64	42.17	101.72
0.3	2.09	5.96	14.49	36.86
0.4	1.58	2.62	4.74	13.60
0.5	1.43	1.75	2.59	7.09
0.6	1.49	1.40	1.80	3.41
0.7	1.53	1.30	1.57	2.42
0.8	1.54	1.22	1.33	1.75
0.9	1.60	1.24	1.30	1.45
Mean	4.62	16.06	47.34	63.73

(continued)

Table 6. (Continued)

p	**One Standard Deviation**	**Two Standard Deviations**	**Three Standard Deviations**	**Ignore Asset Price**
$\gamma = 6$				
0.1	28.22	103.35	452.13	505.21
0.2	5.14	19.13	44.13	111.88
0.3	2.43	7.39	16.02	43.00
0.4	1.59	3.19	6.26	19.13
0.5	1.44	1.99	3.00	10.30
0.6	1.41	1.58	2.19	5.13
0.7	1.44	1.34	1.67	3.31
0.8	1.50	1.26	1.41	1.95
0.9	1.54	1.27	1.40	1.72
Mean	4.97	15.61	58.69	77.96
$\gamma = 7$				
0.1	24.13	103.80	371.88	452.03
0.2	7.09	26.20	56.78	164.20
0.3	2.21	6.46	14.96	43.04
0.4	1.72	3.33	6.61	27.77
0.5	1.42	2.21	3.51	11.24
0.6	1.34	1.77	2.68	7.01
0.7	1.36	1.47	1.91	3.99
0.8	1.40	1.30	1.55	2.36
0.9	1.44	1.23	1.43	1.85
Mean	4.68	16.42	51.26	79.28
$\gamma = 8$				
0.1	25.40	112.87	460.97	524.93
0.2	6.38	24.33	68.51	162.77
0.3	2.68	7.05	13.67	63.11
0.4	1.81	4.20	7.84	29.69
0.5	1.52	2.56	4.45	16.30
0.6	1.35	1.76	2.70	6.79
0.7	1.35	1.55	2.15	4.55
0.8	1.33	1.39	1.73	2.10
0.9	1.42	1.29	1.56	2.03
Mean	4.81	17.44	62.62	90.36

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